

Quantum Algebra and Symmetry

Quantum Algebraic Topology, Quantum Field Theories and Higher Dimensional Algebra

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Quantum Theories

Quantum mechanics

Quantum mechanics, also known as **quantum physics** or **quantum theory**, is a branch of physics providing a mathematical description of much of the dual particle-like and wave-like behavior and interactions of energy and matter. It departs from classical mechanics primarily at the atomic and subatomic scales, the so-called quantum realm. In advanced topics of quantum mechanics, some of these behaviors are macroscopic and only emerge at very low or very high energies or temperatures. The name, coined by Max Planck, derives from the observation that some physical quantities can be changed only by discrete amounts, or quanta, as multiples of the Planck constant, rather than being capable of varying continuously or by any arbitrary amount. For example, the angular momentum, or more generally the action, of an electron bound into an atom or molecule is quantized. While an unbound electron does not exhibit quantized energy levels, an electron bound in an atomic orbital has quantized values of angular momentum. In the context of quantum mechanics, the wave–particle duality of energy and matter and the uncertainty principle provide a unified view of the behavior of photons, electrons and other atomic-scale objects.

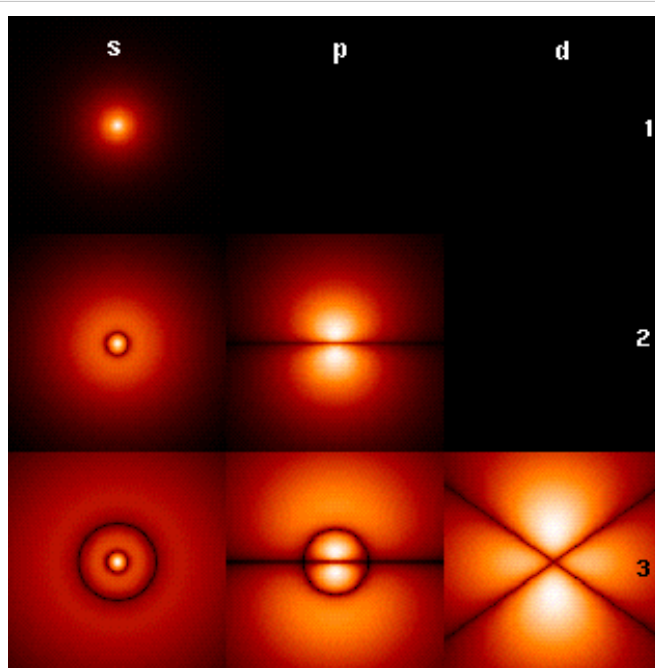


Fig. 1: Probability densities corresponding to the wavefunctions of an electron in a hydrogen atom possessing definite energy levels (increasing from the top of the image to the bottom: $n = 1, 2, 3, \dots$) and angular momentum (increasing across from left to right: $s, p, d, \dots$). Brighter areas correspond to higher probability density in a position measurement. Wavefunctions like these are directly comparable to Chladni's figures of acoustic modes of vibration in classical physics and are indeed modes of oscillation as well: they possess a sharp energy and thus a keen frequency. The angular momentum and energy are quantized, and only take on discrete values like those shown (as is the case for resonant frequencies in acoustics).

The mathematical formulations of quantum mechanics are abstract. Similarly, the implications are often non-intuitive in terms of classic physics. The centerpiece of the mathematical system is the wavefunction. The wavefunction is a mathematical function providing information about the probability amplitude of position and momentum of a particle. Mathematical manipulations of the wavefunction usually involve the bra-ket notation, which requires an understanding of complex numbers and linear functionals. The wavefunction treats the object as a quantum harmonic oscillator and the mathematics is akin to that of acoustic resonance. Many of the results of quantum mechanics do not have models that are easily visualized in terms of classical mechanics; for instance, the ground state in the quantum mechanical model is a non-zero energy state that is the lowest permitted energy state of a system, rather than a more traditional system that is thought of as simply being at rest with zero kinetic energy.

Historically, the earliest versions of quantum mechanics were formulated in the first decade of the 20th century at around the same time as the atomic theory and the corpuscular theory of light as updated by Einstein first came to be widely accepted as scientific fact; these latter theories can be viewed as quantum theories of matter and

electromagnetic radiation. Quantum theory was significantly reformulated in the mid-1920s away from the old quantum theory towards the quantum mechanics formulated by Werner Heisenberg, Max Born, Wolfgang Pauli and their associates, accompanied by the acceptance of the Copenhagen interpretation of Niels Bohr. By 1930, quantum mechanics had been further unified and formalized by the work of Paul Dirac and John von Neumann, with a greater emphasis placed on measurement in quantum mechanics, the statistical nature of our knowledge of reality and philosophical speculation about the role of the observer. Quantum mechanics has since branched out into almost every aspect of 20th century physics and other disciplines such as quantum chemistry, quantum electronics, quantum optics and quantum information science. Much 19th century physics has been re-evaluated as the classical limit of quantum mechanics, and its more advanced developments in terms of quantum field theory, string theory, and speculative quantum gravity theories.

History

The history of quantum mechanics dates back to the 1838 discovery of cathode rays by Michael Faraday. This was followed by the 1859 statement of the black body radiation problem by Gustav Kirchhoff, the 1877 suggestion by Ludwig Boltzmann that the energy states of a physical system can be discrete, and the 1900 quantum hypothesis of Max Planck.^[1] Planck's hypothesis that energy is radiated and absorbed in discrete "quanta", or "energy elements", enabled the correct derivation of the observed patterns of black body radiation. According to Planck, each energy element E is proportional to its frequency ν :

$$E = h\nu$$

where h is Planck's action constant. Planck cautiously insisted that this was simply an aspect of the processes of absorption and emission of radiation and had nothing to do with the physical reality of the radiation itself.^[2] However, in 1905 Albert Einstein interpreted Planck's quantum hypothesis realistically and used it to explain the photoelectric effect, in which shining light on certain materials can eject electrons from the material. Einstein postulated that light itself consists of individual quanta of energy, later called photons.^[3]

The foundations of quantum mechanics were established during the first half of the twentieth century by Niels Bohr, Werner Heisenberg, Max Planck, Louis de Broglie, Albert Einstein, Erwin Schrödinger, Max Born, John von Neumann, Paul Dirac, Wolfgang Pauli, David Hilbert, and others. In the mid-1920s, developments in quantum mechanics quickly led to its becoming the standard formulation for atomic physics. In the summer of 1925, Bohr and Heisenberg published results that closed the "Old Quantum Theory". Out of deference to their dual state as particles, light quanta came to be called photons (1926). From Einstein's simple postulation was born a flurry of debating, theorizing and testing. Thus, the entire field of quantum physics emerged leading to its wider acceptance at the Fifth Solvay Conference in 1927.

The other exemplar that led to quantum mechanics was the study of electromagnetic waves such as light. When it was found in 1900 by Max Planck that the energy of waves could be described as consisting of small packets or quanta, Albert Einstein further developed this idea to show that an electromagnetic wave such as light could be described by a particle called the photon with a discrete energy dependent on its frequency. This led to a theory of unity between subatomic particles and electromagnetic waves called wave–particle duality in which particles and waves were neither one nor the other, but had certain properties of both. While quantum mechanics describes the world of the very small, it also is needed to explain certain macroscopic quantum systems such as superconductors and superfluids.

The word *quantum* derives from Latin meaning "how great" or "how much".^[4] In quantum mechanics, it refers to a discrete unit that quantum theory assigns to certain physical quantities, such as the energy of an atom at rest (see Figure 1). The discovery that particles are discrete packets of energy with wave-like properties led to the branch of physics that deals with atomic and subatomic systems which is today called quantum mechanics. It is the underlying mathematical framework of many fields of physics and chemistry, including condensed matter physics, solid-state physics, atomic physics, molecular physics, computational physics, computational chemistry, quantum chemistry,

particle physics, nuclear chemistry, and nuclear physics.^[5] Some fundamental aspects of the theory are still actively studied.^[6] Quantum mechanics is essential to understand the behavior of systems at atomic length scales and smaller. For example, if classical mechanics governed the workings of an atom, electrons would rapidly travel towards and collide with the nucleus, making stable atoms impossible. However, in the natural world the electrons normally remain in an uncertain, non-deterministic "smeared" (wave–particle wave function) orbital path around or through the nucleus, defying classical electromagnetism.^[7] Quantum mechanics was initially developed to provide a better explanation of the atom, especially the spectra of light emitted by different atomic species. The quantum theory of the atom was developed as an explanation for the electron's staying in its orbital, which could not be explained by Newton's laws of motion and by Maxwell's laws of classical electromagnetism. Broadly speaking, quantum mechanics incorporates four classes of phenomena for which classical physics cannot account:

- The quantization (discretization) of certain physical quantities
- wave–particle duality
- uncertainty principle
- quantum entanglement

Mathematical formulations

In the mathematically rigorous formulation of quantum mechanics developed by Paul Dirac^[8] and John von Neumann,^[9] the possible states of a quantum mechanical system are represented by unit vectors (called "state vectors"). Formally, these reside in a complex separable Hilbert space (variously called the "state space" or the "associated Hilbert space" of the system) well defined up to a complex number of norm 1 (the phase factor). In other words, the possible states are points in the projectivization of a Hilbert space, usually called the complex projective space. The exact nature of this Hilbert space is dependent on the system; for example, the state space for position and momentum states is the space of square-integrable functions, while the state space for the spin of a single proton is just the product of two complex planes. Each observable is represented by a maximally Hermitian (precisely: by a self-adjoint) linear operator acting on the state space. Each eigenstate of an observable corresponds to an eigenvector of the operator, and the associated eigenvalue corresponds to the value of the observable in that eigenstate. If the operator's spectrum is discrete, the observable can only attain those discrete eigenvalues.

In the formalism of quantum mechanics, the state of a system at a given time is described by a complex wave function, also referred to as state vector in a complex vector space.^[10] This abstract mathematical object allows for the calculation of probabilities of outcomes of concrete experiments. For example, it allows one to compute the probability of finding an electron in a particular region around the nucleus at a particular time. Contrary to classical mechanics, one can never make simultaneous predictions of conjugate variables, such as position and momentum, with accuracy. For instance, electrons may be considered to be located somewhere within a region of space, but with their exact positions being unknown. Contours of constant probability, often referred to as "clouds", may be drawn around the nucleus of an atom to conceptualize where the electron might be located with the most probability. Heisenberg's uncertainty principle quantifies the inability to precisely locate the particle given its conjugate momentum.^[11]

As the result of a measurement, the wave function containing the probability information for a system collapses from a given initial state to a particular eigenstate of the observable. The possible results of a measurement are the eigenvalues of the operator representing the observable — which explains the choice of *Hermitian* operators, for which all the eigenvalues are real. We can find the probability distribution of an observable in a given state by computing the spectral decomposition of the corresponding operator. Heisenberg's uncertainty principle is represented by the statement that the operators corresponding to certain observables do not commute.

The probabilistic nature of quantum mechanics thus stems from the act of measurement. This is one of the most difficult aspects of quantum systems to understand. It was the central topic in the famous Bohr-Einstein debates, in which the two scientists attempted to clarify these fundamental principles by way of thought experiments. In the

decades after the formulation of quantum mechanics, the question of what constitutes a "measurement" has been extensively studied. Interpretations of quantum mechanics have been formulated to do away with the concept of "wavefunction collapse"; see, for example, the relative state interpretation. The basic idea is that when a quantum system interacts with a measuring apparatus, their respective wavefunctions become entangled, so that the original quantum system ceases to exist as an independent entity. For details, see the article on measurement in quantum mechanics.^[12] Generally, quantum mechanics does not assign definite values to observables. Instead, it makes predictions using probability distributions; that is, the probability of obtaining possible outcomes from measuring an observable. Often these results are skewed by many causes, such as dense probability clouds^[13] or quantum state nuclear attraction.^{[14] [15]} Naturally, these probabilities will depend on the quantum state at the "instant" of the measurement. Hence, uncertainty is involved in the value. There are, however, certain states that are associated with a definite value of a particular observable. These are known as eigenstates of the observable ("eigen" can be translated from German as inherent or as a characteristic).^[16]

In the everyday world, it is natural and intuitive to think of everything (every observable) as being in an eigenstate. Everything appears to have a definite position, a definite momentum, a definite energy, and a definite time of occurrence. However, quantum mechanics does not pinpoint the exact values of a particle for its position and momentum (since they are conjugate pairs) or its energy and time (since they too are conjugate pairs); rather, it only provides a range of probabilities of where that particle might be given its momentum and momentum probability. Therefore, it is helpful to use different words to describe states having *uncertain* values and states having *definite* values (eigenstate). Usually, a system will not be in an eigenstate of the observable we are interested in. However, if one measures the observable, the wavefunction will instantaneously be an eigenstate (or generalized eigenstate) of that observable. This process is known as wavefunction collapse, a debatable process.^[17] It involves expanding the system under study to include the measurement device. If one knows the corresponding wave function at the instant before the measurement, one will be able to compute the probability of collapsing into each of the possible eigenstates. For example, the free particle in the previous example will usually have a wavefunction that is a wave packet centered around some mean position x_0 , neither an eigenstate of position nor of momentum. When one measures the position of the particle, it is impossible to predict with certainty the result.^[12] It is probable, but not certain, that it will be near x_0 , where the amplitude of the wave function is large. After the measurement is performed, having obtained some result x , the wave function collapses into a position eigenstate centered at x .^[18]

The time evolution of a quantum state is described by the Schrödinger equation, in which the Hamiltonian, the operator corresponding to the total energy of the system, generates time evolution. The time evolution of wave functions is deterministic in the sense that, given a wavefunction at an initial time, it makes a definite prediction of what the wavefunction will be at any later time.^[19]

During a measurement, on the other hand, the change of the wavefunction into another one is not deterministic, but rather unpredictable, i.e., random. A time-evolution simulation can be seen here.^{[20] [21]} Wave functions can change as time progresses. An equation known as the Schrödinger equation describes how wave functions change in time, a role similar to Newton's second law in classical mechanics. The Schrödinger equation, applied to the aforementioned example of the free particle, predicts that the center of a wave packet will move through space at a constant velocity, like a classical particle with no forces acting on it. However, the wave packet will also spread out as time progresses, which means that the position becomes more uncertain. This also has the effect of turning position eigenstates (which can be thought of as infinitely sharp wave packets) into broadened wave packets that are no longer position eigenstates.^[22]

Some wave functions produce probability distributions that are constant, or independent of time, such as when in a stationary state of constant energy, time drops out of the absolute square of the wave function. Many systems that are treated dynamically in classical mechanics are described by such "static" wave functions. For example, a single electron in an unexcited atom is pictured classically as a particle moving in a circular trajectory around the atomic nucleus, whereas in quantum mechanics it is described by a static, spherically symmetric wavefunction surrounding

the nucleus (Fig. 1). (Note that only the lowest angular momentum states, labeled s , are spherically symmetric).^[23]

The Schrödinger equation acts on the entire probability amplitude, not merely its absolute value. Whereas the absolute value of the probability amplitude encodes information about probabilities, its phase encodes information about the interference between quantum states. This gives rise to the wave-like behavior of quantum states. It turns out that analytic solutions of Schrödinger's equation are only available for a small number of model Hamiltonians, of which the quantum harmonic oscillator, the particle in a box, the hydrogen molecular ion and the hydrogen atom are the most important representatives. Even the helium atom, which contains just one more electron than hydrogen, defies all attempts at a fully analytic treatment. There exist several techniques for generating approximate solutions. For instance, in the method known as perturbation theory one uses the analytic results for a simple quantum mechanical model to generate results for a more complicated model related to the simple model by, for example, the addition of a weak potential energy. Another method is the "semi-classical equation of motion" approach, which applies to systems for which quantum mechanics produces weak deviations from classical behavior. The deviations can be calculated based on the classical motion. This approach is important for the field of quantum chaos.

There are numerous mathematically equivalent formulations of quantum mechanics. One of the oldest and most commonly used formulations is the transformation theory proposed by Cambridge theoretical physicist Paul Dirac, which unifies and generalizes the two earliest formulations of quantum mechanics, matrix mechanics (invented by Werner Heisenberg)^[24] ^[25] and wave mechanics (invented by Erwin Schrödinger).^[26] In this formulation, the instantaneous state of a quantum system encodes the probabilities of its measurable properties, or "observables". Examples of observables include energy, position, momentum, and angular momentum. Observables can be either continuous (e.g., the position of a particle) or discrete (e.g., the energy of an electron bound to a hydrogen atom).^[27] An alternative formulation of quantum mechanics is Feynman's path integral formulation, in which a quantum-mechanical amplitude is considered as a sum over histories between initial and final states; this is the quantum-mechanical counterpart of action principles in classical mechanics.

Interactions with other scientific theories

The fundamental rules of quantum mechanics are very deep. They assert that the state space of a system is a Hilbert space and the observables are Hermitian operators acting on that space, but do not tell us which Hilbert space or which operators, or if it even exists. These must be chosen appropriately in order to obtain a quantitative description of a quantum system. An important guide for making these choices is the correspondence principle, which states that the predictions of quantum mechanics reduce to those of classical physics when a system moves to higher energies or equivalently, larger quantum numbers. In other words, classical mechanics is simply a quantum mechanics of large systems. This "high energy" limit is known as the *classical* or *correspondence limit*. One can therefore start from an established classical model of a particular system, and attempt to guess the underlying quantum model that gives rise to the classical model in the correspondence limit.

When quantum mechanics was originally formulated, it was applied to models whose correspondence limit was non-relativistic classical mechanics. For instance, the well-known model of the quantum harmonic oscillator uses an explicitly non-relativistic expression for the kinetic energy of the oscillator, and is thus a quantum version of the classical harmonic oscillator.

Early attempts to merge quantum mechanics with special relativity involved the replacement of the Schrödinger equation with a covariant equation such as the Klein-Gordon equation or the Dirac equation. While these theories were successful in explaining many experimental results, they had certain unsatisfactory qualities stemming from their neglect of the relativistic creation and annihilation of particles. A fully relativistic quantum theory required the development of quantum field theory, which applies quantization to a field rather than a fixed set of particles. The first complete quantum field theory, quantum electrodynamics, provides a fully quantum description of the electromagnetic interaction. The full apparatus of quantum field theory is often unnecessary for describing electrodynamic systems. A simpler approach, one employed since the inception of quantum mechanics, is to treat

charged particles as quantum mechanical objects being acted on by a classical electromagnetic field. For example, the elementary quantum model of the hydrogen atom describes the electric field of the hydrogen atom using a classical $-\frac{e^2}{4\pi\epsilon_0 r}$ Coulomb potential. This "semi-classical" approach fails if quantum fluctuations in the electromagnetic field play an important role, such as in the emission of photons by charged particles. Quantum field theories for the strong nuclear force and the weak nuclear force have been developed. The quantum field theory of the strong nuclear force is called quantum chromodynamics, and describes the interactions of the subnuclear particles: quarks and gluons. The weak nuclear force and the electromagnetic force were unified, in their quantized forms, into a single quantum field theory known as electroweak theory, by the physicists Abdus Salam, Sheldon Glashow and Steven Weinberg. These three men shared the Nobel Prize in Physics in 1979 for this work.^[28]

It has proven difficult to construct quantum models of gravity, the remaining fundamental force. Semi-classical approximations are workable, and have led to predictions such as Hawking radiation. However, the formulation of a complete theory of quantum gravity is hindered by apparent incompatibilities between general relativity, the most accurate theory of gravity currently known, and some of the fundamental assumptions of quantum theory. The resolution of these incompatibilities is an area of active research, and theories such as string theory are among the possible candidates for a future theory of quantum gravity. Classical mechanics has been extended into the complex domain and complex classical mechanics exhibits behaviours similar to quantum mechanics.^[29]

Quantum mechanics and classical physics

Predictions of quantum mechanics have been verified experimentally to a very high degree of accuracy. According to the correspondence principle between classical and quantum mechanics, all objects obey the laws of quantum mechanics, and classical mechanics is just an approximation for large systems (or a statistical quantum mechanics of a large collection of particles). The laws of classical mechanics thus follow from the laws of quantum mechanics as a statistical average at the limit of large systems or large quantum numbers.^[30] However, chaotic systems do not have good quantum numbers, and quantum chaos studies the relationship between classical and quantum descriptions in these systems.

Quantum coherence is an essential difference between classical and quantum theories, and is illustrated by the Einstein-Podolsky-Rosen paradox. Quantum interference involves the addition of *probability amplitudes*, whereas when classical waves interfere there is an addition of *intensities*. For microscopic bodies, the extension of the system is much smaller than the coherence length, which gives rise to long-range entanglement and other nonlocal phenomena characteristic of quantum systems.^[31] Quantum coherence is not typically evident at macroscopic scales, although an exception to this rule can occur at extremely low temperatures, when quantum behavior can manifest itself on more macroscopic scales (see Bose-Einstein condensate). This is in accordance with the following observations:

- Many macroscopic properties of a classical system are a direct consequences of the quantum behavior of its parts. For example, the stability of bulk matter (which consists of atoms and molecules which would quickly collapse under electric forces alone), the rigidity of solids, and the mechanical, thermal, chemical, optical and magnetic properties of matter are all results of the interaction of electric charges under the rules of quantum mechanics.^[32]
- While the seemingly exotic behavior of matter posited by quantum mechanics and relativity theory become more apparent when dealing with extremely fast-moving or extremely tiny particles, the laws of classical Newtonian physics remain accurate in predicting the behavior of large objects—of the order of the size of large molecules and bigger—at velocities much smaller than the velocity of light.^[33]

Relativity and quantum mechanics

Main articles: Quantum gravity and Theory of everything

Even with the defining postulates of both Einstein's theory of general relativity and quantum theory being indisputably supported by rigorous and repeated empirical evidence and while they do not directly contradict each other theoretically (at least with regard to primary claims), they are resistant to being incorporated within one cohesive model.^[34]

Einstein himself is well known for rejecting some of the claims of quantum mechanics. While clearly contributing to the field, he did not accept the more philosophical consequences and interpretations of quantum mechanics, such as the lack of deterministic causality and the assertion that a single subatomic particle can occupy numerous areas of space at one time. He also was the first to notice some of the apparently exotic consequences of entanglement and used them to formulate the Einstein-Podolsky-Rosen paradox, in the hope of showing that quantum mechanics had unacceptable implications. This was 1935, but in 1964 it was shown by John Bell (see Bell inequality) that, although Einstein was correct in identifying seemingly paradoxical implications of quantum mechanical nonlocality, these implications could be experimentally tested. Alain Aspect's initial experiments in 1982, and many subsequent experiments since, have verified quantum entanglement.

According to the paper of J. Bell and the Copenhagen interpretation (the common interpretation of quantum mechanics by physicists since 1927), and contrary to Einstein's ideas, quantum mechanics was not at the same time

- a "realistic" theory
- and a *local* theory.

The Einstein-Podolsky-Rosen paradox shows in any case that there exist experiments by which one can measure the state of one particle and instantaneously change the state of its entangled partner, although the two particles can be an arbitrary distance apart; however, this effect does not violate causality, since no transfer of information happens. Quantum entanglement is at the basis of quantum cryptography, with high-security commercial applications in banking and government.

Gravity is negligible in many areas of particle physics, so that unification between general relativity and quantum mechanics is not an urgent issue in those applications. However, the lack of a correct theory of quantum gravity is an important issue in cosmology and physicists' search for an elegant "theory of everything". Thus, resolving the inconsistencies between both theories has been a major goal of twentieth- and twenty-first-century physics. Many prominent physicists, including Stephen Hawking, have labored in the attempt to discover a theory underlying *everything*, combining not only different models of subatomic physics, but also deriving the universe's four forces —the strong force, electromagnetism, weak force, and gravity— from a single force or phenomenon. One of the leaders in this field is Edward Witten, a theoretical physicist who formulated the groundbreaking M-theory, which is an attempt at describing the supersymmetrical based string theory.

Attempts at a unified field theory

As of 2010 the quest for unifying the fundamental forces through quantum mechanics is still ongoing. Quantum electrodynamics (or "quantum electromagnetism"), which is currently (in the perturbative regime at least) the most accurately tested physical theory,^[35] has been successfully merged with the weak nuclear force into the electroweak force and work is currently being done to merge the electroweak and strong force into the electrostrong force. Current predictions state that at around 10^{14} GeV the three aforementioned forces are fused into a single unified field,^[36] Beyond this "grand unification," it is speculated that it may be possible to merge gravity with the other three gauge symmetries, expected to occur at roughly 10^{19} GeV. However— and while special relativity is parsimoniously incorporated into quantum electrodynamics — the expanded general relativity, currently the best theory describing the gravitation force, has not been fully incorporated into quantum theory.

Philosophical implications

Since its inception, the many counter-intuitive results of quantum mechanics have provoked strong philosophical debate and many interpretations. Even fundamental issues such as Max Born's basic rules concerning probability amplitudes and probability distributions took decades to be appreciated.

Richard Feynman said, "I think I can safely say that nobody understands quantum mechanics."^[37]

The Copenhagen interpretation, due largely to the Danish theoretical physicist Niels Bohr, is the interpretation of the quantum mechanical formalism most widely accepted amongst physicists. According to it, the probabilistic nature of quantum mechanics is not a temporary feature which will eventually be replaced by a deterministic theory, but instead must be considered to be a final renunciation of the classical ideal of causality. In this interpretation, it is believed that any well-defined application of the quantum mechanical formalism must always make reference to the experimental arrangement, due to the complementarity nature of evidence obtained under different experimental situations.

Albert Einstein, himself one of the founders of quantum theory, disliked this loss of determinism in measurement. (This dislike is the source of his famous quote, "God does not play dice with the universe.") Einstein held that there should be a local hidden variable theory underlying quantum mechanics and that, consequently, the present theory was incomplete. He produced a series of objections to the theory, the most famous of which has become known as the Einstein-Podolsky-Rosen paradox. John Bell showed that the EPR paradox led to experimentally testable differences between quantum mechanics and local realistic theories. Experiments have been performed confirming the accuracy of quantum mechanics, thus demonstrating that the physical world cannot be described by local realistic theories.^[38] The *Bohr-Einstein debates* provide a vibrant critique of the Copenhagen Interpretation from an epistemological point of view.

The Everett many-worlds interpretation, formulated in 1956, holds that all the possibilities described by quantum theory simultaneously occur in a multiverse composed of mostly independent parallel universes.^[39] This is not accomplished by introducing some new axiom to quantum mechanics, but on the contrary by *removing* the axiom of the collapse of the wave packet: All the possible consistent states of the measured system and the measuring apparatus (including the observer) are present in a *real* physical (not just formally mathematical, as in other interpretations) quantum superposition. Such a superposition of consistent state combinations of different systems is called an entangled state. While the multiverse is deterministic, we perceive non-deterministic behavior governed by probabilities, because we can observe only the universe, i.e. the consistent state contribution to the mentioned superposition, we inhabit. Everett's interpretation is perfectly consistent with John Bell's experiments and makes them intuitively understandable. However, according to the theory of quantum decoherence, the parallel universes will never be accessible to us. This inaccessibility can be understood as follows: Once a measurement is done, the measured system becomes entangled with both the physicist who measured it and a huge number of other particles, some of which are photons flying away towards the other end of the universe; in order to prove that the wave function did not collapse one would have to bring all these particles back and measure them again, together with the system that was measured originally. This is completely impractical, but even if one could theoretically do this, it would destroy any evidence that the original measurement took place (including the physicist's memory).

Applications

Quantum mechanics had enormous success in explaining many of the features of our world. The individual behaviour of the subatomic particles that make up all forms of matter—electrons, protons, neutrons, photons and others—can often only be satisfactorily described using quantum mechanics. Quantum mechanics has strongly influenced string theory, a candidate for a theory of everything (see reductionism) and the multiverse hypothesis.

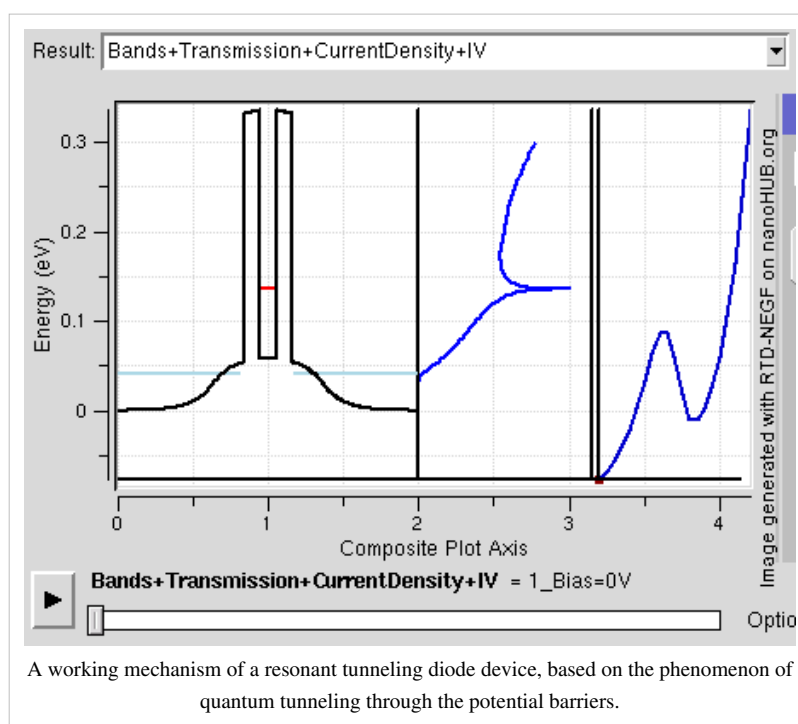
Quantum mechanics is important for understanding how individual atoms combine covalently to form chemicals or molecules. The application of quantum mechanics to chemistry is known as quantum chemistry. (Relativistic) quantum mechanics can in principle mathematically describe most of chemistry. Quantum mechanics can provide quantitative insight into ionic and covalent bonding processes by explicitly showing which molecules are energetically favorable to which others, and by approximately how much.^[40] Most of the calculations performed in computational chemistry rely on quantum mechanics.^[41]

Much of modern technology operates at a scale where quantum effects are significant. Examples include the laser, the transistor (and thus the microchip), the electron microscope, and magnetic resonance imaging. The study of semiconductors led to the invention of the diode and the transistor, which are indispensable for modern electronics.

Researchers are currently seeking robust methods of directly manipulating quantum states. Efforts are being made to develop quantum cryptography, which will allow guaranteed secure transmission of information. A more distant goal is the development of quantum computers, which are expected to perform certain computational tasks exponentially faster than classical computers. Another active research topic is quantum teleportation, which deals with techniques to transmit quantum information over arbitrary distances.

Quantum tunneling is vital in many devices, even in the simple light switch, as otherwise the electrons in the electric current could not penetrate the potential barrier made up of a layer of oxide. Flash memory chips found in USB drives use quantum tunneling to erase their memory cells.

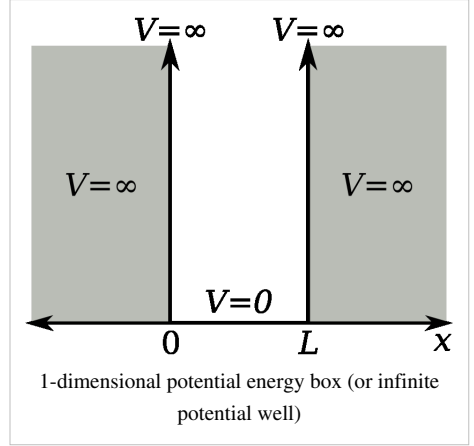
Quantum mechanics primarily applies to the atomic regimes of matter and energy, but some systems exhibit quantum mechanical effects on a large scale; superfluidity (the frictionless flow of a liquid at temperatures near absolute zero) is one well-known example. Quantum theory also provides accurate descriptions for many previously unexplained phenomena such as black body radiation and the stability of electron orbitals. It has also given insight into the workings of many different biological systems, including smell receptors and protein structures.^[42] Recent work on photosynthesis has provided evidence that quantum correlations play an essential role in this most fundamental process of the plant kingdom.^[43] Even so, classical physics often can be a good approximation to results otherwise obtained by **quantum physics**, typically in circumstances with large numbers of particles or large quantum numbers. (However, some open questions remain in the field of quantum chaos.)



Examples

Particle in a box

The particle in a 1-dimensional potential energy box is the most simple example where restraints lead to the quantization of energy levels. The box is defined as having zero potential energy inside a certain region and infinite potential energy everywhere outside that region. For the 1-dimensional case in the x direction, the time-independent Schrödinger equation can be written as:^[44]



$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi.$$

Writing the differential operator

$$\hat{p}_x = -i\hbar \frac{d}{dx}$$

the previous equation can be seen to be evocative of the classic analogue

$$\frac{1}{2m} \hat{p}_x^2 = E$$

with E as the energy for the state ψ , in this case coinciding with the kinetic energy of the particle.

The general solutions of the Schrödinger equation for the particle in a box are:

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad E = \frac{\hbar^2 k^2}{2m}$$

or, from Euler's formula,

$$\psi(x) = C \sin kx + D \cos kx.$$

The presence of the walls of the box determines the values of C , D , and k . At each wall ($x = 0$ and $x = L$), $\psi = 0$.

Thus when $x = 0$,

$$\psi(0) = 0 = C \sin 0 + D \cos 0 = D$$

and so $D = 0$. When $x = L$,

$$\psi(L) = 0 = C \sin kL.$$

C cannot be zero, since this would conflict with the Born interpretation. Therefore $\sin kL = 0$, and so it must be that kL is an integer multiple of π . Therefore,

$$k = \frac{n\pi}{L} \quad n = 1, 2, 3, \dots$$

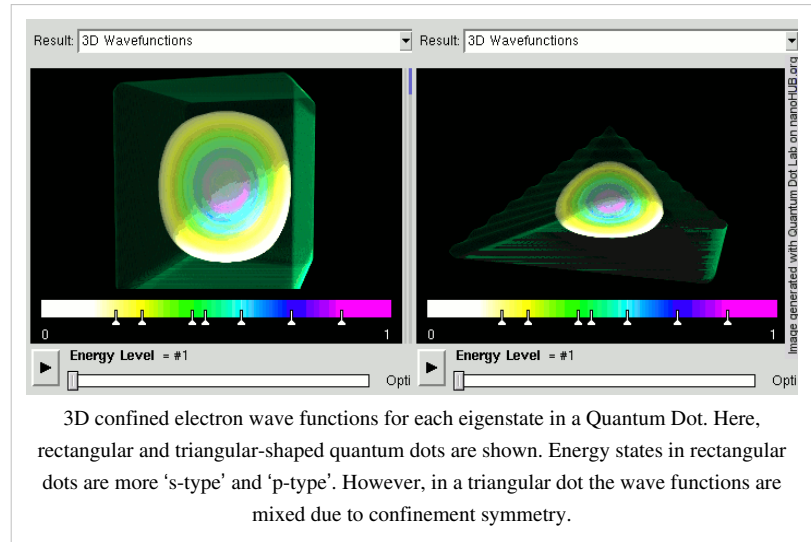
The quantization of energy levels follows from this constraint on k , since

$$E = \frac{\hbar^2 \pi^2 n^2}{2mL^2} = \frac{n^2 h^2}{8mL^2}.$$

Free particle

For example, consider a free particle. In quantum mechanics, there is wave-particle duality so the properties of the particle can be described as the properties of a wave. Therefore, its quantum state can be represented as a wave of arbitrary shape and extending over space as a wave function. The position and momentum of the particle are observables. The Uncertainty Principle states that both the position and the momentum cannot simultaneously be measured with full precision at the same time. However, one can measure the position alone of a

moving free particle creating an eigenstate of position with a wavefunction that is very large (a Dirac delta) at a particular position x and zero everywhere else. If one performs a position measurement on such a wavefunction, the result x will be obtained with 100% probability (full certainty). This is called an eigenstate of position (mathematically more precise: a *generalized position eigenstate (eigendistribution)*). If the particle is in an eigenstate of position then its momentum is completely unknown. On the other hand, if the particle is in an eigenstate of momentum then its position is completely unknown.^[45] In an eigenstate of momentum having a plane wave form, it can be shown that the wavelength is equal to h/p , where h is Planck's constant and p is the momentum of the eigenstate.^[46]



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- Quantum Physics Made Relatively Simple (<http://bethe.cornell.edu/>): three video lectures by Hans Bethe
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Course material

- Doron Cohen: Lecture notes in Quantum Mechanics (comprehensive, with advanced topics). (<http://arxiv.org/abs/quant-ph/0605180>)
- MIT OpenCourseWare: Chemistry (<http://ocw.mit.edu/OcwWeb/Chemistry/index.htm>).
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- AQME (<http://www.nanohub.org/topics/AQME>) : Advancing Quantum Mechanics for Engineers — by T.Barzso, D.Vasileska and G.Klimeck online learning resource with simulation tools on nanohub
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FAQs

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Media

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Schrödinger equation

In physics, specifically quantum mechanics, the **Schrödinger equation**, formulated in 1926 by Austrian physicist Erwin Schrödinger, is an equation that describes how the quantum state of a physical system changes in time. It is as central to quantum mechanics as Newton's laws are to classical mechanics.

$E\Psi = \hat{H}\Psi$
$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$
Two forms of the Schrödinger equation

In the standard interpretation of quantum mechanics, the quantum state, also called a wavefunction or state vector, is the most complete description that can be given to a physical system. Solutions to Schrödinger's equation describe not only molecular, atomic and subatomic systems, but also macroscopic systems, possibly even the whole universe.
[1]

The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time. For systems in a stationary state, the time-independent Schrödinger equation is sufficient. Approximate solutions to the time-independent Schrödinger equation are commonly used to calculate the energy levels and other properties of atoms and molecules.

Schrödinger's equation can be mathematically transformed into Werner Heisenberg's matrix mechanics, and into Richard Feynman's path integral formulation. The Schrödinger equation describes time in a way that is inconvenient for relativistic theories, a problem which is not as severe in matrix mechanics and completely absent in the path integral formulation.

The Schrödinger equation

The Schrödinger equation takes several different forms, depending on the physical situation. This section presents the equation for the general case and for the simple case encountered in many textbooks.

General quantum system

For a general quantum system: ^[2]

$$i\hbar \frac{\partial}{\partial t} \Psi = \hat{H} \Psi$$

where

- Ψ is the wave function; the probability amplitude for different configurations of the system at different times,
- $i\hbar \frac{\partial}{\partial t}$ is the energy operator (i is the imaginary unit and $\hbar$ is the reduced Planck constant),
- $\hat{H}$ is the Hamiltonian operator.

Single particle in a potential

For a single particle with potential energy V in position space, the Schrödinger equation takes the form: ^[3]

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \hat{H} \Psi = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right) \Psi(\mathbf{r}, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\mathbf{r}, t) + V(\mathbf{r}) \Psi(\mathbf{r}, t)$$

where

- $-\frac{\hbar^2}{2m} \nabla^2$ is the kinetic energy operator, where m is the mass of the particle.
- ∇^2 is the Laplace operator. In three dimensions, the Laplace operator is $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, where $x, y,$ and z are the Cartesian coordinates of space.
- $V(\mathbf{r})$ is the time-independent potential energy at the position $\mathbf{r}$.
- $\Psi(\mathbf{r}, t)$ is the probability amplitude for the particle to be found at position $\mathbf{r}$ at time t .
- $\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r})$ is the Hamiltonian operator for a single particle in a potential.

Time independent or stationary equation

The time independent equation, again for a single particle with potential energy V takes the form: ^[4]

$$E\psi(r) = -\frac{\hbar^2}{2m} \nabla^2 \psi(r) + V(r)\psi(r).$$

This equation describes the standing wave solutions of the time-dependent equation, which are the states with definite energy.

Historical background and development

Following Max Planck's quantization of light (see black body radiation), Albert Einstein interpreted Planck's quantum to be photons, particles of light, and proposed that the energy of a photon is proportional to its frequency, one of the first signs of wave–particle duality. Since energy and momentum are related in the same way as frequency and wavenumber in special relativity, it followed that the momentum p of a photon is proportional to its wavenumber k .

$$p = \frac{h}{\lambda} = \hbar k$$

Louis de Broglie hypothesized that this is true for all particles, even particles such as electrons. Assuming that the waves travel roughly along classical paths, he showed that they form standing waves for certain discrete frequencies. These correspond to discrete energy levels, which reproduced the old quantum condition.^[5]

Following up on these ideas, Schrödinger decided to find a proper wave equation for the electron. He was guided by William R. Hamilton's analogy between mechanics and optics, encoded in the observation that the zero-wavelength limit of optics resembles a mechanical system—the trajectories of light rays become sharp tracks which obey Fermat's principle, an analog of the principle of least action.^[6] A modern version of his reasoning is reproduced in the next section. The equation he found is:

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(x, t) + V(x) \Psi(x, t).$$

Using this equation, Schrödinger computed the hydrogen spectral series by treating a hydrogen atom's electron as a wave $\Psi(x, t)$, moving in a potential well V , created by the proton. This computation accurately reproduced the energy levels of the Bohr model.

However, by that time, Arnold Sommerfeld had refined the Bohr model with relativistic corrections.^[7] ^[8] Schrödinger used the relativistic energy momentum relation to find what is now known as the Klein–Gordon equation in a Coulomb potential (in natural units):

$$\left(E + \frac{e^2}{r}\right)^2 \psi(x) = -\nabla^2 \psi(x) + m^2 \psi(x).$$

He found the standing waves of this relativistic equation, but the relativistic corrections disagreed with Sommerfeld's formula. Discouraged, he put away his calculations and secluded himself in an isolated mountain cabin with a lover.^[9]

While at the cabin, Schrödinger decided that his earlier non-relativistic calculations were novel enough to publish, and decided to leave off the problem of relativistic corrections for the future. He put together his wave equation and the spectral analysis of hydrogen in a paper in 1926.^[10] The paper was enthusiastically endorsed by Einstein, who saw the matter-waves as an intuitive depiction of nature, as opposed to Heisenberg's matrix mechanics, which he considered overly formal.^[11]

The Schrödinger equation details the behaviour of ψ but says nothing of its *nature*. Schrödinger tried to interpret it as a charge density in his fourth paper, but he was unsuccessful.^[12] In 1926, just a few days after Schrödinger's fourth and final paper was published, Max Born successfully interpreted ψ as a probability amplitude.^[13] Schrödinger, though, always opposed a statistical or probabilistic approach, with its associated discontinuities—much like Einstein, who believed that quantum mechanics was a statistical approximation to an underlying deterministic theory— and never reconciled with the Copenhagen interpretation.^[14]

Derivation

Short heuristic derivation

Schrödinger's equation can be derived in the following short heuristic way.

Assumptions

1. The total energy E of a particle is

$$E = T + V = \frac{p^2}{2m} + V.$$

This is the classical expression for a particle with mass m where the total energy E is the sum of the kinetic energy T , and the potential energy V (which can vary with position, and time). p and m are respectively the momentum and the mass of the particle.

2. Einstein's light quanta hypothesis of 1905, which asserts that the energy E of a photon is proportional to the frequency ν (or angular frequency, $\omega = 2\pi\nu$) of the corresponding electromagnetic wave:

$$E = h\nu = \hbar\omega ,$$

3. The de Broglie hypothesis of 1924, which states that any particle can be associated with a wave, and that the momentum p of the particle is related to the wavelength λ (or wavenumber k) of such a wave by:

$$p = \frac{h}{\lambda} = \hbar k ,$$

Expressing $\mathbf{p}$ and $\mathbf{k}$ as vectors, we have

$$\mathbf{p} = \hbar\mathbf{k} .$$

4. The three assumptions above allow one to derive the equation for plane waves only. To conclude that it is true in general requires the superposition principle, and thus, one must separately postulate that the Schrödinger equation is linear.

Expressing the wave function as a complex plane wave

Schrödinger's idea was to express the phase of a plane wave as a complex phase factor:

$$\Psi(\mathbf{x}, t) = Ae^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

and to realize that since

$$\frac{\partial}{\partial t}\Psi = -i\omega\Psi$$

then

$$E\Psi = \hbar\omega\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

and similarly since

$$\frac{\partial}{\partial x}\Psi = ik_x\Psi$$

and

$$\frac{\partial^2}{\partial x^2}\Psi = -k_x^2\Psi$$

we find:

$$p_x^2\Psi = (\hbar k_x)^2\Psi = -\hbar^2\frac{\partial^2}{\partial x^2}\Psi$$

so that, again for a plane wave, he obtained:

$$p^2\Psi = (p_x^2 + p_y^2 + p_z^2)\Psi = -\hbar^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Psi = -\hbar^2 \nabla^2 \Psi$$

And, by inserting these expressions for the energy and momentum into the classical formula we started with, we get Schrödinger's famed equation, for a single particle in the 3-dimensional case in the presence of a potential V :

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi$$

Versions

There are several equations that go by Schrödinger's name:

Time dependent equation

This is the equation of motion for the quantum state. In the most general form, it is written:^[15]

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \hat{H} \Psi(x, t)$$

where $\hat{H}$ is a linear operator acting on the wavefunction Ψ . For the specific case of a single particle in one dimension moving under the influence of a potential V .^[15]

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x, t) + V(x) \Psi(x, t)$$

and the operator $\hat{H}$ can be read off:

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x).$$

For a particle in three dimensions, the only difference is more derivatives:

$$i\hbar \frac{\partial}{\partial t} \Psi(x, y, z, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(x, y, z, t) + V(x, y, z) \Psi(x, y, z, t)$$

and for N particles, the difference is that the wavefunction is in $3N$ -dimensional configuration space, the space of all possible particle positions.^[16]

$$i\hbar \frac{\partial}{\partial t} \Psi(x_1, \dots, x_n, t) = \hbar^2 \left(-\frac{\nabla_1^2}{2m_1} - \frac{\nabla_2^2}{2m_2} \dots - \frac{\nabla_N^2}{2m_N} \right) \Psi(x_1, \dots, x_n, t) + V(x_1, \dots, x_n, t) \Psi(x_1, \dots, x_n, t).$$

This last equation is in a very high dimension, so that the solutions are not easy to visualize.

Time independent equation

This is the equation for the standing waves, the eigenvalue equation for $\hat{H}$. In abstract form, for a general quantum system, it is written:^[15]

$$\hat{H}\psi = E\psi.$$

For a particle in one dimension,

$$E\psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi.$$

But there is a further restriction—the solution must not grow at infinity, so that it has either a finite L^2 -norm (if it is a bound state) or a slowly diverging norm (if it is part of a continuum):^[17]

$$\|\psi\|^2 = \int |\psi(x)|^2 dx.$$

For example, when there is no potential, the equation reads:^[18]

$$-E\psi = \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}$$

which has oscillatory solutions for $E > 0$ (the C_n are arbitrary constants):

$$\psi_E(x) = C_1 e^{i\sqrt{2mE/\hbar^2} x} + C_2 e^{-i\sqrt{2mE/\hbar^2} x}$$

and exponential solutions for $E < 0$

$$\psi_{-|E|}(x) = C_1 e^{\sqrt{2m|E|/\hbar^2} x} + C_2 e^{-\sqrt{2m|E|/\hbar^2} x}.$$

The exponentially growing solutions have an infinite norm, and are not physical. They are not allowed in a finite volume with periodic or fixed boundary conditions.

For a constant potential V the solution is oscillatory for $E > V$ and exponential for $E < V$, corresponding to energies which are allowed or disallowed in classical mechanics. Oscillatory solutions have a classically allowed energy and correspond to actual classical motions, while the exponential solutions have a disallowed energy and describe a small amount of quantum bleeding into the classically disallowed region, to quantum tunneling. If the potential V grows at infinity, the motion is classically confined to a finite region, which means that in quantum mechanics every solution becomes an exponential far enough away. The condition that the exponential is decreasing restricts the energy levels to a discrete set, called the allowed energies.

Nonlinear equation

The nonlinear Schrödinger equation is the partial differential equation (in dimensionless form)^[19]

$$i\partial_t \psi = -\frac{1}{2}\partial_x^2 \psi + \kappa |\psi|^2 \psi$$

for the complex field $\psi(x, t)$.

This equation arises from the Hamiltonian^[19]

$$H = \int dx \left[\frac{1}{2} |\partial_x \psi|^2 + \frac{\kappa}{2} |\psi|^4 \right]$$

with the Poisson brackets

$$\begin{aligned} \{\psi(x), \psi(y)\} &= \{\psi^*(x), \psi^*(y)\} = 0 \\ \{\psi^*(x), \psi(y)\} &= i\delta(x - y). \end{aligned}$$

It must be noted that this is a classical field equation. Unlike its linear counterpart, it never describes the time evolution of a quantum state.

Properties

The Schrödinger equation has certain properties.

Local conservation of probability

The probability density of a particle is $\Psi^*(x, t)\Psi(x, t)$. The probability flux is defined as [in units of (probability)/(area $\times$ time)]:

$$\mathbf{j} = -\frac{i\hbar}{2m} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) = \frac{\hbar}{m} \text{Im} (\Psi^* \nabla \Psi).$$

The probability flux satisfies the continuity equation:

$$\frac{\partial}{\partial t} P(x, t) + \nabla \cdot \mathbf{j} = 0$$

where $P(x, t)$ is the probability density [measured in units of (probability)/(volume)]. This equation is the mathematical equivalent of the probability conservation law.

For a plane wave:

$$\Psi(x, t) = A e^{i(\mathbf{k}x - \omega t)}$$

$$j(x, t) = |A|^2 \frac{\hbar k}{m}.$$

So that not only is the probability of finding the particle the same everywhere, but the probability flux is as expected from an object moving at the classical velocity p/m . The reason that the Schrödinger equation admits a probability flux is because all the hopping is local and forward in time.

Relativity

The Schrödinger equation does not take into account relativistic effects; as a wave equation, it is invariant under a Galilean transformation, but not under a Lorentz transformation. But in order to include relativity, the physical picture must be altered.

The Klein–Gordon equation uses the relativistic mass-energy relation:

$$E^2 = p^2 c^2 + m^2 c^4$$

to produce the differential equation:

$$-\frac{1}{c} \frac{\partial^2}{\partial t^2} \psi = -\nabla^2 \psi + \frac{m^2 c^2}{\hbar^2} \psi$$

which is relativistically invariant.

Solutions

Some general techniques are:

- Perturbation theory
- The variational method
- Quantum Monte Carlo methods
- Density functional theory
- The WKB approximation and semi-classical expansion

In some special cases, special methods can be used:

- List of quantum-mechanical systems with analytical solutions
- Hartree-Fock method and post Hartree-Fock methods
- Discrete delta-potential method

Notes

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External links

- Quantum Physics (http://www.lightandmatter.com/html_books/0sn/ch13/ch13.html) - a textbook with a treatment of the time-independent Schrödinger equation
- Linear Schrödinger Equation (<http://eqworld.ipmnet.ru/en/solutions/lpde/lpde108.pdf>) at EqWorld: The World of Mathematical Equations.
- Nonlinear Schrödinger Equation (<http://eqworld.ipmnet.ru/en/solutions/npde/npde1403.pdf>) at EqWorld: The World of Mathematical Equations.
- The Schrödinger Equation in One Dimension (<http://www.colorado.edu/UCB/AcademicAffairs/ArtsSciences/physics/TZD/PageProofs1/TAYL07-203-247.I.pdf>) as well as the directory of the book (<http://www.colorado.edu/UCB/AcademicAffairs/ArtsSciences/physics/TZD/PageProofs1/>).
- All about 3D Schrödinger Equation (<http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html>)
- Mathematical aspects of Schrödinger equations are discussed on the Dispersive PDE Wiki (http://tosio.math.toronto.edu/wiki/index.php/Main_Page).
- Web-Schrödinger: Interactive solution of the 2D time dependent Schrödinger equation (<http://www.nanotechnology.hu/online/web-schroedinger/index.html>)
- An alternate derivation of the Schrödinger Equation (<http://behindtheguesses.blogspot.com/2009/06/schrodinger-equation-corrections.html>)
- Online software- Periodic Potential Lab (<http://nanohub.org/resources/3847>) Solves the time independent Schrödinger equation for arbitrary periodic potentials.

Dirac equation

The **Dirac equation** is a relativistic quantum mechanical wave equation formulated by British physicist Paul Dirac in 1928. It provides a description of elementary spin-1/2 particles, such as electrons, consistent with both the principles of quantum mechanics and the theory of special relativity. The equation demands the existence of antiparticles and actually predated their experimental discovery. This made the discovery of the positron, the antiparticle of the electron, one of the greatest triumphs of modern theoretical physics.

Mathematical formulation

The Dirac equation in the form originally proposed by Dirac is:

$$\left(\beta mc^2 + \sum_{k=1}^3 \alpha_k p_k c \right) \psi(\mathbf{x}, t) = i\hbar \frac{\partial \psi(\mathbf{x}, t)}{\partial t}$$

where

m is the rest mass of the electron,

c is the speed of light,

p is the momentum operator,

$\mathbf{x}$ and t are the space and time coordinates,

$\hbar = h/2\pi$ is the reduced Planck constant, also known as Dirac's constant.

The new elements in this equation are the 4×4 matrices α_k and β , and the four-component wavefunction ψ . The matrices are all Hermitian and have squares equal to the identity matrix:

$$\alpha_i^2 = \beta^2 = I_4$$

and they all mutually anticommute: $\{\alpha_i, \alpha_j\} = 0$ and $\{\alpha_i, \beta\} = 0$. Explicitly,

$$\alpha_i \alpha_j = -\alpha_j \alpha_i,$$

$$\alpha_i \beta = -\beta \alpha_i,$$

where i and j are distinct and range from 1 to 3. These matrices, and the form of the wavefunction, have a deep mathematical significance. The algebraic structure represented by the Dirac matrices had been created some 50 years earlier by the English mathematician W. K. Clifford. In turn, Clifford's formulation had been based on the mid-19th century work of the German mathematician Hermann Grassmann in his "Lineare Ausdehnungslehre" (Theory of Linear Extensions). The latter had been regarded as well-nigh incomprehensible by most of his contemporaries. The appearance of something so seemingly abstract, at such a late date, in such a direct physical manner, is one of the most remarkable chapters in the history of physics.

The commutation rules are designed so that a solution of Dirac's equation will automatically also be a solution of

$$((mc^2)^2 + \sum_{k=1}^3 (p_k c)^2) \psi = E^2 \psi$$

which is the relativistic energy-momentum equation.

Comparison with the Schrödinger equation

The Dirac equation is superficially similar to the Schrödinger equation for a free particle:

$$-\frac{\hbar^2}{2m}\nabla^2\phi = i\hbar\frac{\partial}{\partial t}\phi.$$

The left side represents the square of the momentum operator divided by twice the mass, which is the nonrelativistic kinetic energy. A relativistic generalization of this equation requires that space and time derivatives must enter symmetrically, as they do in the relativistic Maxwell equations—the derivatives must be of the *same order* in space and time. In relativity, the momentum and the energy are the space and time parts of a geometrical space-time vector, the 4-momentum, and they are related by the relativistically invariant relation

$$\frac{E^2}{c^2} - p^2 = m^2 c^2$$

which says that the length of this vector is the rest mass m . Replacing E and p by $i\hbar\frac{\partial}{\partial t}$ and $-i\hbar\nabla$ as Schrödinger theory requires, we get a relativistic equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = \frac{m^2 c^2}{\hbar^2}\phi$$

and the wave function ϕ is a relativistic scalar: a complex number which has the same numerical value in all frames. Because the equation is second order in the time derivative, one must specify the initial values of not only ϕ , but also of $\partial_t\phi$. This is normal for classical waves, where the initial conditions are the position and velocity.

However, in quantum mechanics, the wavefunction is supposed to be the complete description. That is, just knowing the wavefunction should determine the future.

In the Schrödinger theory, the probability density is given by the positive definite expression

$$\rho = \phi^*\phi$$

and its current by

$$J = -\frac{i\hbar}{2m}(\phi^*\nabla\phi - \phi\nabla\phi^*)$$

and the conservation of probability density has a local form:

$$\nabla \cdot J + \frac{\partial\rho}{\partial t} = 0.$$

In a relativistic theory, the form of the probability density and the current must form a four vector, so the form of the probability density can be found from the current just by replacing ∇ by ∂_t :

$$\rho = \frac{i\hbar}{2m}(\phi^*\partial_t\phi - \phi\partial_t\phi^*).$$

Everything is relativistic now, but *the probability density is not positive definite*, because the initial values of both ϕ and $\partial_t\phi$ can be freely chosen. This expression reduces to Schrödinger's density and current for superpositions of positive frequency waves whose wavelength is long compared to the Compton wavelength, that is, for nonrelativistic motions. It reduces to a negative definite quantity for superpositions of negative frequency waves only. It mixes up both signs when forces which have an appreciable amplitude to produce relativistic motions are involved, at which point scattering can produce particles and antiparticles.

Although it was not a successful description of a single particle, this equation is resurrected in quantum field theory, where it is known as the Klein–Gordon equation, and describes a relativistic spin-0 complex field. The non-positive probability density and current are the charge-density and current, while the particles are described by a mode-expansion.

To interpret the Klein–Gordon equation as an equation for the probability amplitude for a single particle at a given position, negative frequency solutions must be interpreted as describing the particle travelling backwards in time,

propagating into the past.

The equation with this interpretation does not predict the future from the present except in the nonrelativistic limit, rather it places a global constraint on the amplitudes. This can be used to construct a perturbation expansion with particles zipping backwards and forwards in time, the Feynman diagrams, but it does not allow a straightforward wavefunction description, since each particle has its own separate proper time. Ultimately, the entity the Klein-Gordon equation (and Dirac equation) acts on is a field, not a wavefunction. The fields are physical fields whose values are observables. They are not probability amplitudes.

Dirac's coup

Thinking in terms of wavefunctions rather than fields, Dirac reasoned that the necessary equation is first-order in both space and time. One could formally take the relativistic expression for the energy $E = c\sqrt{p^2 + m^2c^2}$, replace p by its operator equivalent, expand the square root in an infinite series of derivative operators, set up an eigenvalue problem, then solve the equation formally by iterations. Most physicists had little faith in such a process, even if it were technically possible.

As the story goes, Dirac was staring into the fireplace at Cambridge, pondering this problem, when he hit upon the idea of taking the square root of the wave operator thus:

$$\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = (A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t)(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t).$$

On multiplying out the right side we see that, to get all the cross-terms such as $\partial_x\partial_y$ to vanish, we must assume

$$AB + BA = 0, \dots$$

with

$$A^2 = B^2 = \dots = 1.$$

Dirac, who had just then been intensely involved with working out the foundations of Heisenberg's matrix mechanics, immediately understood that these conditions could be met if $A, B, \dots$ are *matrices*, with the implication that the wave function has *multiple components*. This immediately explained the appearance of two-component wave functions in Pauli's phenomenological theory of spin, something that up until then had been regarded as mysterious, even to Pauli himself. However, one needs at least 4×4 matrices to set up a system with the properties desired—so the wave function had *four* components, not two, as in the Pauli theory.

Given the factorization in terms of these matrices, one can now write down immediately an equation

$$(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t)\psi = \kappa\psi$$

with κ to be determined. Applying again the matrix operator on either side yields

$$(\nabla^2 - \frac{1}{c^2}\partial_t^2)\psi = \kappa^2\psi.$$

On taking $\kappa = mc/\hbar$ we find that all the components of the wave function *individually* satisfy the relativistic energy–momentum relation. Thus the sought-for equation that is first-order in both space and time is

$$(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t - \frac{mc}{\hbar})\psi = 0.$$

With $(A, B, C) = i\beta\alpha_k$ and $D = \beta$, we get the Dirac equation.

Comparison with the Pauli theory

The necessity of introducing half-integral spin goes back experimentally to the results of the Stern–Gerlach experiment. A beam of atoms is run through a strong inhomogeneous magnetic field, which then splits into N parts depending on the intrinsic angular momentum of the atoms. It was found that for silver atoms, the beam was split into two—the ground state therefore could not be integral, because even if the intrinsic angular momentum of the atoms were as small as possible, 1, the beam would be split into 3 parts, corresponding to atoms with $L_z = -1, 0$, and $+1$. The conclusion is that silver atoms have net intrinsic angular momentum of $\frac{1}{2}$. Pauli set up a theory which explained this splitting by introducing a two-component wave function and a corresponding correction term in the Hamiltonian, representing a semi-classical coupling of this wave function to an applied magnetic field, as so:

$$H = \frac{1}{2m} \left(\sigma \cdot \left(p - \frac{e}{c} A \right) \right)^2 + eA^0.$$

Here A^μ is the applied electromagnetic field, and the three sigmas are Pauli matrices. e is the charge of the particle, e.g. $e = -e_0$ for the electron. On squaring out the first term, a residual interaction with the magnetic field is found, along with the usual Hamiltonian of a charged particle interacting with an applied field:

$$H = \frac{1}{2m} \left(p - \frac{e}{c} A \right)^2 + eA^0 - \frac{e\hbar}{2mc} \sigma \cdot B.$$

This Hamiltonian is now a 2×2 matrix, so the Schrödinger equation based on it,

$$H\phi = i\hbar \frac{\partial \phi}{\partial t}$$

must use a two-component wave function. Pauli had introduced the sigma matrices

$$\sigma_k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as pure *phenomenology*—Dirac now had a *theoretical argument* that implied that spin was somehow the consequence of the marriage of quantum theory to relativity.

The Pauli matrices share the same properties as the Dirac matrices—they are all Hermitian, square to 1, and anticommute. This allows one to immediately find a representation of the Dirac matrices in terms of the Pauli matrices:

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \\ \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

The Dirac equation now may be written as an equation coupling two-component spinors:

$$\begin{pmatrix} mc^2 & c\sigma \cdot p \\ c\sigma \cdot p & -mc^2 \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}.$$

Notice that on the diagonal we find the rest energy of the particle. If we set the momentum to zero—that is, bring the particle to rest—then we have

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \begin{pmatrix} mc^2 & 0 \\ 0 & -mc^2 \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}.$$

The equations for the individual two-spinors are now decoupled, and we see that the "top" and "bottom" two-spinors are individually eigenfunctions of the energy with eigenvalues equal to plus and minus the rest energy, respectively. The appearance of this *negative energy* eigenvalue is completely consistent with relativity.

It should be strongly emphasized that this separation in the rest frame is *not* an invariant statement—the "bottom" two-spinor does not represent antimatter as such in general. The entire four-component spinor represents an *irreducible* whole—in general, states will have an admixture of positive *and* negative energy components. If we

couple the Dirac equation to an electromagnetic field, as in the Pauli theory, then the positive and negative energy parts will be mixed together, even if they are originally decoupled. Dirac's main problem was to find a consistent interpretation of this mixing. As we shall see below, it brings a new phenomenon into physics—matter/antimatter creation and annihilation.

Covariant form and relativistic invariance

The explicitly covariant form of the Dirac equation is (employing the Einstein summation convention):

$$-i\hbar\gamma^\mu\partial_\mu\psi + mc\psi = 0.$$

In the above, γ^μ are the Dirac matrices. γ^0 is Hermitian, and the γ^k are anti-Hermitian, with the definition

$$\begin{aligned}\gamma^0 &= \beta \\ \gamma^k &= \gamma^0\alpha^k.\end{aligned}$$

Thus

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

This may be summarized using the Minkowski metric on spacetime in the form

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

where the bracket expression $\{a, b\}$ means $ab + ba$, the anticommutator. These are the defining relations of a Clifford algebra over a pseudo-orthogonal 4-d space with metric signature $(+ - - -)$. (Note that one may also employ the metric form $(- + + +)$ by multiplying all the gammas by a factor of i .) The specific Clifford algebra employed in the Dirac equation is known as the Dirac algebra.

The Dirac equation may be interpreted as an eigenvalue expression, where the rest mass is proportional to an eigenvalue of the 4-momentum operator, the proportion being the speed of light in vacuo:

$$P_{op}\psi = mc\psi.$$

In practice, physicists often use units of measure such that $\hbar$ and c are equal to 1, known as "natural" units. The equation then takes the simple form

$$(-i\gamma^\mu\partial_\mu + m)\psi = 0$$

or, if Feynman slash notation is employed,

$$(-i\not{\partial} + m)\psi = 0.$$

A fundamental theorem states that if two distinct sets of matrices are given that both satisfy the Clifford relations, then they are connected to each other by a similarity transformation:

$$\gamma'^\mu = S^{-1}\gamma^\mu S.$$

If in addition the matrices are all unitary, as are the Dirac set, then S itself is unitary;

$$\gamma'^\mu = U^\dagger\gamma^\mu U.$$

The transformation U is unique up to a multiplicative factor of absolute value 1. Let us now imagine a Lorentz transformation to have been performed on the derivative operators, which form a covariant vector. For the operator $\gamma^\mu\partial_\mu$ to remain invariant, the gammas must transform among themselves as a contravariant vector with respect to their spacetime index. These new gammas will themselves satisfy the Clifford relations, because of the orthogonality of the Lorentz transformation. By the fundamental theorem, we may replace the new set by the old set subject to a unitary transformation. In the new frame, remembering that the rest mass is a relativistic scalar, the Dirac equation will then take the form

$$(-iU^\dagger \gamma^\mu U \partial'_\mu + m)\psi(x', t') = 0$$

$$U^\dagger (-i\gamma^\mu \partial'_\mu + m)U\psi(x', t') = 0.$$

If we now define the transformed spinor

$$\psi' = U\psi$$

then we have the transformed Dirac equation

$$(-i\gamma^\mu \partial'_\mu + m)\psi'(x', t') = 0.$$

Thus, once we settle on a unitary representation of the gammas, it is final provided we transform the spinor according the unitary transformation that corresponds to the given Lorentz transformation.

These considerations reveal the origin of the gammas in *geometry*, hearkening back to Grassmann's original motivation - they represent a fixed basis of unit vectors in spacetime. Similarly, products of the gammas such as $\gamma_\mu \gamma_\nu$ represent *oriented surface elements*, and so on. With this in mind, we can find the form the unit volume element on spacetime in terms of the gammas as follows. By definition, it is

$$V = \frac{1}{4!} \epsilon_{\mu\nu\alpha\beta} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta.$$

For this to be an invariant, the epsilon symbol must be a tensor, and so must contain a factor of $\sqrt{g}$, where g is the determinant of the metric tensor. Since this is negative, that factor is *imaginary*. Thus

$$V = i\gamma^0 \gamma^1 \gamma^2 \gamma^3.$$

This matrix is given the special symbol γ_5 , owing to its importance when one is considering improper transformations of spacetime, that is, those that change the orientation of the basis vectors. In the representation we are using for the gammas, it is

$$\gamma_5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

Also note that we could as easily have taken the negative square root of the determinant of g - the choice amounts to an initial handedness convention.

Lorentz Invariance of the Dirac equation

The Lorentz invariance of the Dirac equation follows from its covariant nature.

Comparison with the Klein-Gordon equation

Using the Feynman slash notation, the Klein-Gordon equation can be factored:

$$0 = (\partial^2 + m^2)\psi = (\not{\partial}^2 + m^2)\psi = (i\not{\partial} + m)(-i\not{\partial} + m)\psi.$$

The last factor is simply the Dirac equation. Hence any solution to the Dirac equation is automatically a solution to the Klein-Gordon equation:

$$(-i\not{\partial} + m)\psi = 0 \rightarrow (\partial^2 + m^2)\psi = 0.$$

But the converse is not true; not all solutions to the Klein-Gordon equation solve the Dirac equation.

Adjoint equation and Dirac current

By defining the adjoint spinor

$$\bar{\psi} = \psi^\dagger \gamma^0$$

where $\psi^\dagger$ is the conjugate transpose of ψ , and noticing that

$$(\gamma^\mu)^\dagger \gamma^0 = \gamma^0 \gamma^\mu,$$

we obtain, by taking the Hermitian conjugate of the Dirac equation and multiplying from the right by γ^0 , the adjoint equation:

$$\bar{\psi}(i\gamma^\mu \partial_\mu + m) = 0$$

where ∂_μ is understood to act to the left. Multiplying the Dirac equation by $\bar{\psi}$ from the left, and the adjoint equation by ψ from the right, and adding, produces the law of conservation of the Dirac current in covariant form:

$$\partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0.$$

Now we see the great advantage of the first-order equation over the one Schrödinger had tried - this is the conserved current density required by relativistic invariance, only now its 0-component is *positive definite*:

$$J^0 = \bar{\psi} \gamma^0 \psi = \psi^\dagger \psi.$$

The Dirac equation and its adjoint are the Euler–Lagrange equations of the 4-d invariant action integral

$$S = \int L d^4x,$$

where $d^4x \equiv dt dx dy dz$, and the scalar L is the Dirac Lagrangian

$$L = mc\bar{\psi}\psi - \frac{i\hbar}{2}(\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi)$$

and for the purposes of variation, ψ and $\bar{\psi}$ are regarded as independent fields. The relativistic invariance also follows immediately from the variational principle.

Coupling to an electromagnetic field

To consider problems in which an applied electromagnetic field interacts with the particles described by the Dirac equation, one uses the correspondence principle, and takes over into the theory the corresponding expression from classical mechanics, whereby the total momentum of a charged particle in an external field is modified as so:

$$p \rightarrow p - \frac{q}{c}A,$$

(where q is the charge of the particle; for example, $q = -e < 0$ for an electron). In natural units, the Dirac equation then takes the form

$$[-i\gamma^\mu(\partial_\mu + iqA_\mu) + m]\psi = 0.$$

The validity of this prescription has been confirmed experimentally with great precision. It is known as *minimal coupling*, and is found throughout particle physics. Indeed, while the introduction of the electromagnetic field in this way is essentially phenomenological in this context, it arises from a fundamental principle in quantum field theory.

Now as stated above, the transformation U is defined only up to a phase factor $e^{i\theta}$. Also, the fundamental observable of the Dirac theory, the current, is unchanged if we multiply the field (which is not a wavefunction) by an arbitrary phase. Because the field is not a wavefunction, this phase invariance has a different physical meaning from the phase invariance of probability amplitudes. We may exploit this to get the form of the mutual interaction of a Dirac particle and the electromagnetic field, as opposed to simply considering a Dirac particle in an applied field, by assuming this arbitrary phase factor to depend continuously on position:

$$\psi \rightarrow \psi' = e^{i\theta(x,t)}\psi.$$

Notice now that

$$-i\gamma^\mu(\partial_\mu + iqA_\mu)\psi' = e^{i\theta}\gamma^\mu(-i\partial_\mu + q(A_\mu + \frac{1}{q}\partial_\mu\theta))\psi = e^{i\theta}\gamma^\mu(-i\partial_\mu + qA'_\mu)\psi.$$

To preserve minimal coupling, we must add to the potential a term proportional to the gradient of the phase. But we know from electrodynamics that this leaves the electromagnetic field itself invariant. The value of the phase is arbitrary, but *not* how it changes from place to place. This is the starting point of gauge theory, which is the main principle on which quantum field theory is based. The simplest such theory, and the one most thoroughly understood, is known as quantum electrodynamics. The equations of field theory thus have invariance under both Lorentz transformations and gauge transformations.

Curved spacetime Dirac equation

The Dirac equation can be written in curved spacetime using vierbein fields. Vierbeins describe a local frame that enables to define Dirac matrices at every point. Contracting these matrices with vierbeins give the right transformation properties. This way Dirac's equation takes the following form in curved spacetime ^[1]:

$$-i\gamma^a e_a^\mu D_\mu \Psi + m\Psi = 0.$$

Here e_a^μ is the vierbein and D_μ is the covariant derivative for fermion fields, defined as follows

$$D_\mu = \partial_\mu - \frac{i}{4}\eta_{ac}\omega_{b\mu}^c\sigma^{ab}$$

where η_{ac} is the Lorentzian metric, σ^{ab} is the commutator of Dirac matrices:

$$\sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]$$

and $\omega_{b\mu}^c$ is the spin connection:

$$\omega_{b\mu}^c = e_\nu^c \partial_\mu e_b^\nu + e_\nu^c e_b^\sigma \Gamma_{\sigma\mu}^\nu$$

where $\Gamma_{\sigma\mu}^\nu$ is the Christoffel symbol. Note that here, Latin letters denote the "Lorentzian" indices and Greek ones denote "Riemannian" indices.

Physical interpretation

The Dirac theory, while providing a wealth of information that is accurately confirmed by experiments, nevertheless introduces a new physical paradigm that appears at first difficult to interpret and even paradoxical. Some of these issues of interpretation must be regarded as open questions. Here we will see how the Dirac theory brilliantly answered some of the outstanding issues in physics at the time it was put forward, while posing others that are still the subject of debate.

Identification of observables

The critical physical question in a quantum theory is - what are the physically observable quantities defined by the theory? According to general principles, such quantities are defined by Hermitian operators that act on the Hilbert space of possible states of a system. The eigenvalues of these operators are then the possible results of measuring the corresponding physical quantity. In the Schrödinger theory, the simplest such object is the overall Hamiltonian, which represents the total energy of the system. If we wish to maintain this interpretation on passing to the Dirac theory, we must take the Hamiltonian to be

$$H = \gamma^0 \left(mc^2 + c \sum_{k=1}^3 \gamma^k (p_k - \frac{q}{c} A_k) c \right) + qA^0.$$

This looks promising, because we see by inspection the rest energy of the particle and, in case $A = 0$, the energy of a charge placed in an electric potential qA^0 . What about the term involving the vector potential? In classical electrodynamics, the energy of a charge moving in an applied potential is

$$H = c\sqrt{(p - \frac{q}{c}A)^2 + m^2c^2} + qA^0.$$

Thus the Dirac Hamiltonian is *fundamentally distinguished* from its classical counterpart, and we must take great care to correctly identify what is an observable in this theory. Much of the apparent paradoxical behavior implied by the Dirac equation amounts to a misidentification of these observables. Let us now describe one such effect. (cont'd)

History

Since the Dirac equation was originally invented to describe the electron, we will generally speak of "electrons" in this article. The equation also applies to quarks, which are also elementary spin- $\frac{1}{2}$ particles. A modified Dirac equation can be used to approximately describe protons and neutrons, which are not elementary particles (they are made up of quarks), but have a net spin of $\frac{1}{2}$. Another modification of the Dirac equation, called the Majorana equation, is thought to describe neutrinos — also spin- $\frac{1}{2}$ particles.

The Dirac equation describes the probability amplitudes for a *single* electron. This is a single-particle theory; in other words, it does not account for the creation and destruction of the particles, and for the ultimate need to switch from the Dirac equation for wavefunctions to the physically distinct Dirac equation for fields. It gives a good prediction of the magnetic moment of the electron and explains much of the fine structure observed in atomic spectral lines. It also explains the spin of the electron. Two of the four solutions of the equation correspond to the two spin states of the electron. The other two solutions make the peculiar prediction that there exist an infinite set of quantum states in which the electron possesses negative energy. This strange result led Dirac to predict, via a remarkable hypothesis known as "hole theory," the existence of particles behaving like positively-charged electrons. Dirac thought at first these particles might be protons. He was chagrined when the strict prediction of his equation (which actually specifies particles of the same mass as the electron) was verified by the discovery of the positron in 1932. When asked later why he hadn't actually boldly predicted the yet unfound positron with its correct mass, Dirac answered "Pure cowardice!" He shared the Nobel Prize anyway, in 1933.

A similar equation for spin $3/2$ particles is called the Rarita-Schwinger equation.

Hole theory

The negative E solutions found in the preceding section are problematic, for it was assumed that the particle has a positive energy. Mathematically speaking, however, there seems to be no reason for us to reject the negative-energy solutions. Since they exist, we cannot simply ignore them, for once we include the interaction between the electron and the electromagnetic field, any electron placed in a positive-energy eigenstate would decay into negative-energy eigenstates of successively lower energy by emitting excess energy in the form of photons. Real electrons obviously do not behave in this way.

To cope with this problem, Dirac introduced the hypothesis, known as **hole theory**, that the vacuum is the many-body quantum state in which all the negative-energy electron eigenstates are occupied. This description of the vacuum as a "sea" of electrons is called the Dirac sea. Since the Pauli exclusion principle forbids electrons from occupying the same state, any additional electron would be forced to occupy a positive-energy eigenstate, and positive-energy electrons would be forbidden from decaying into negative-energy eigenstates.

Dirac further reasoned that if the negative-energy eigenstates are incompletely filled, each unoccupied eigenstate — called a **hole** — would behave like a positively charged particle. The hole possesses a *positive* energy, since energy is required to create a particle-hole pair from the vacuum. As noted above, Dirac initially thought that the hole might be the proton, but Hermann Weyl pointed out that the hole should behave as if it had the same mass as an electron, whereas the proton is over 1800 times heavier. The hole was eventually identified as the positron, experimentally discovered by Carl Anderson in 1932.

It is not entirely satisfactory to describe the "vacuum" using an infinite sea of negative-energy electrons. The infinitely negative contributions from the sea of negative-energy electrons has to be canceled by an infinite positive "bare" energy and the contribution to the charge density and current coming from the sea of negative-energy electrons is exactly canceled by an infinite positive "jellium" background so that the net electric charge density of the vacuum is zero. In quantum field theory, a Bogoliubov transformation on the creation and annihilation operators (turning an occupied negative-energy electron state into an unoccupied positive energy positron state and an unoccupied negative-energy electron state into an occupied positive energy positron state) allows us to bypass the Dirac sea formalism even though, formally, it is equivalent to it.

In certain applications of condensed matter physics, however, the underlying concepts of "hole theory" are valid. The sea of conduction electrons in an electrical conductor, called a Fermi sea, contains electrons with energies up to the chemical potential of the system. An unfilled state in the Fermi sea behaves like a positively-charged electron, though it is referred to as a "hole" rather than a "positron". The negative charge of the Fermi sea is balanced by the positively-charged ionic lattice of the material.

Dirac bilinears

There are five different (neutral) Dirac bilinear terms not involving any derivatives:

- (S)calar: $\bar{\psi}\psi$ (scalar, P-even)
- (P)seudoscalar: $\bar{\psi}\gamma^5\psi$ (scalar, P-odd)
- (V)ector: $\bar{\psi}\gamma^\mu\psi$ (vector, P-odd)
- (A)xial: $\bar{\psi}\gamma^\mu\gamma^5\psi$ (vector, P-even)
- (T)ensor: $\bar{\psi}\sigma^{\mu\nu}\psi$ (antisymmetric tensor, P-even),

where $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ and $\gamma^5 = \gamma_5 = \frac{i}{4!} \epsilon_{\mu\nu\rho\lambda} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\lambda = i\gamma^0\gamma^1\gamma^2\gamma^3$.

A Dirac mass term is an S coupling. A Yukawa coupling may be S or P. The electromagnetic coupling is V. The weak interactions are V-A.

See also

- Bohr–Sommerfeld theory
- Breit equation
- Dirac field
- Einstein–Maxwell–Dirac equations
- Feynman checkerboard
- Foldy–Wouthuysen transformation
- Klein–Gordon equation
- Quantum electrodynamics
- Rarita–Schwinger equation
- Theoretical and experimental justification for the Schrödinger equation
- The Dirac Equation appears on the floor of Westminster Abbey. It appears on the plaque commemorating Paul Dirac's life which was inaugurated on November 13, 1995^[2].

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External links

- The Dirac Equation (<http://www.mathpages.com/home/kmath654/kmath654.htm>) at MathPages
- The Nature of the Dirac Equation, its solutions and Spin (http://www.mc.maricopa.edu/~kevinlg/i256/Nature_Dirac.pdf)
- Dirac equation for a spin $\frac{1}{2}$ particle (<http://electron6.phys.utk.edu/qm2/modules/m9/dirac.htm>)
- Unitary Principle and Mechanical Non-solution Proving of Neomorphic Dirac Equation (http://commons.wikimedia.org/wiki/File:Unitary_Principle_and_Mechanical_Non-solution_Proving_of_Neomorphic_Dirac_Equation.pdf)
- Pedagogic Aids to Quantum Field Theory (<http://www.quantumfieldtheory.info>) click on Chap. 4 for a step-by-small-step introduction to the Dirac equation, spinors, and relativistic spin/helicity operators.

Klein-Gordon equation

The **Klein–Gordon equation** (**Klein–Fock–Gordon equation** or sometimes *Klein–Gordon–Fock equation*) is a relativistic version of the Schrödinger equation.

It is the equation of motion of a quantum scalar or pseudoscalar field, a field whose quanta are spinless particles. It cannot be straightforwardly interpreted as a Schrödinger equation for a quantum state, because it is second order in time and because it does not admit a positive definite conserved probability density. Still, with the appropriate interpretation, it does describe the quantum amplitude for finding a point particle in various places, the relativistic wavefunction, but the particle propagates both forwards and backwards in time. Any solution to the Dirac equation is automatically a solution to the Klein–Gordon equation, but the converse is not true.

Statement

The Klein–Gordon equation is

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi - \nabla^2 \psi + \frac{m^2 c^2}{\hbar^2} \psi = 0.$$

It is most often written in natural units:

$$-\partial_t^2 \psi + \nabla^2 \psi = m^2 \psi$$

The form is determined by requiring that plane wave solutions of the equation:

$$\psi = e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} = e^{ik_\mu x^\mu}$$

obey the energy momentum relation of special relativity:

$$-p_\mu p^\mu = E^2 - P^2 = \omega^2 - k^2 = -k_\mu k^\mu = m^2$$

Unlike the Schrödinger equation, there are two values of ω for each $\mathbf{k}$, one positive and one negative. Only by separating out the positive and negative frequency parts does the equation describe a relativistic wavefunction. For the time-independent case, the Klein–Gordon equation becomes

$$\left[\nabla^2 - \frac{m^2 c^2}{\hbar^2} \right] \psi(\mathbf{r}) = 0$$

which is the homogeneous screened Poisson equation.

History

The equation was named after the physicists Oskar Klein and Walter Gordon, who in 1927 proposed that it describes relativistic electrons. Although it turned out that the Dirac equation describes the spinning electron, the Klein Gordon equation correctly describes the spinless pion. The pion is a composite particle; no spinless elementary particles have yet been found, although the Higgs boson is theorized to exist as a spin-zero boson, according to the Standard Model.

The Klein–Gordon equation was first considered as a quantum wave equation by Schrödinger in his search for an equation describing de Broglie waves. The equation is found in his notebooks from late 1925, and he appears to have prepared a manuscript applying it to the hydrogen atom. Yet, without taking into account the electron's spin, the Klein–Gordon equation predicts the hydrogen atom's fine structure incorrectly, including overestimating the overall magnitude of the splitting pattern by a factor of $4n/(2n-1)$ for the n -th energy level. In January 1926, Schrödinger submitted for publication instead *his* equation, a non-relativistic approximation that predicts the Bohr energy levels of hydrogen without fine structure.

In 1926, soon after the Schrödinger equation was introduced, Vladimir Fock wrote an article about its generalization for the case of magnetic fields, where forces were dependent on velocity, and independently derived this equation.

Both Klein and Fock used Kaluza and Klein's method. Fock also determined the gauge theory for the wave equation. The Klein–Gordon equation for a free particle has a simple plane wave solution.

Derivation

The non-relativistic equation for the energy of a free particle is

$$\frac{\mathbf{p}^2}{2m} = E.$$

By quantizing this, we get the non-relativistic Schrödinger equation for a free particle,

$$\frac{\mathbf{p}^2}{2m}\psi = i\hbar\frac{\partial}{\partial t}\psi$$

where

$$\mathbf{p} = -i\hbar\nabla$$

is the momentum operator (∇ being the del operator).

The Schrödinger equation suffers from not being relativistically covariant, meaning it does not take into account Einstein's special relativity.

It is natural to try to use the identity from special relativity

$$\sqrt{\mathbf{p}^2c^2 + m^2c^4} = E$$

for the energy; then, just inserting the quantum mechanical momentum operator, yields the equation

$$\sqrt{(-i\hbar\nabla)^2c^2 + m^2c^4}\psi = i\hbar\frac{\partial}{\partial t}\psi.$$

This, however, is a cumbersome expression to work with because the differential operator cannot be evaluated while under the square root sign. In addition, this equation, as it stands, is nonlocal.

Klein and Gordon instead began with the square of the above identity, i.e.

$$\mathbf{p}^2c^2 + m^2c^4 = E^2$$

which, when quantized, gives

$$((-i\hbar\nabla)^2c^2 + m^2c^4)\psi = (i\hbar\frac{\partial}{\partial t})^2\psi$$

which simplifies to

$$-\hbar^2c^2\nabla^2\psi + m^2c^4\psi = -\hbar^2\frac{\partial^2}{(\partial t)^2}\psi.$$

Rearranging terms yields

$$\frac{1}{c^2}\frac{\partial^2}{(\partial t)^2}\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0.$$

Since all reference to imaginary numbers has been eliminated from this equation, it can be applied to fields that are real valued as well as those that have complex values.

Using the reciprocal of the Minkowski metric $\text{diag}(-c^2, 1, 1, 1)$, we get

$$-\eta^{\mu\nu}\partial_\mu\partial_\nu\psi + \frac{m^2c^2}{\hbar^2}\psi = 0$$

in covariant notation. This is often abbreviated as

$$(\square + \mu^2)\psi = 0,$$

where

$$\mu = \frac{mc}{\hbar}$$

and

$$\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2.$$

This operator is called the d'Alembert operator. Today this form is interpreted as the relativistic field equation for a scalar (i.e. spin-0) particle. Furthermore, any solution to the Dirac equation (for a spin-one-half particle) is automatically a solution to the Klein–Gordon equation, though not all solutions of the Klein–Gordon equation are solutions of the Dirac equation. It is noteworthy that the Klein–Gordon equation is very similar to the Proca equation.

Relativistic free particle solution

The Klein–Gordon equation for a free particle can be written as

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi = \frac{m^2 c^2}{\hbar^2} \psi$$

with the same solution as in the non-relativistic case:

$$\psi(\mathbf{r}, t) = e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

except with the constraint

$$-k^2 + \frac{\omega^2}{c^2} = \frac{m^2 c^2}{\hbar^2}.$$

Just as with the non-relativistic particle, we have for energy and momentum:

$$\langle \mathbf{p} \rangle = \langle \psi | -i\hbar \nabla | \psi \rangle = \hbar \mathbf{k},$$

$$\langle E \rangle = \langle \psi | i\hbar \frac{\partial}{\partial t} | \psi \rangle = \hbar \omega.$$

Except that now when we solve for k and ω and substitute into the constraint equation, we recover the relationship between energy and momentum for relativistic massive particles:

$$\langle E \rangle^2 = m^2 c^4 + \langle \mathbf{p} \rangle^2 c^2.$$

For massless particles, we may set $m = 0$ in the above equations. We then recover the relationship between energy and momentum for massless particles:

$$\langle E \rangle = \langle |\mathbf{p}| \rangle c.$$

Action

The Klein–Gordon equation can also be derived from the following action

$$\mathcal{S} = \int \left(-\frac{\hbar^2}{m} \eta^{\mu\nu} \partial_\mu \bar{\psi} \partial_\nu \psi - mc^2 \bar{\psi} \psi \right) d^4x$$

where ψ is the Klein–Gordon field and m is its mass. The complex conjugate of ψ is written $\bar{\psi}$. If the scalar field is taken to be real-valued, then $\bar{\psi} = \psi$.

From this we can derive the stress-energy tensor of the scalar field. It is

$$T^{\mu\nu} = \frac{\hbar^2}{m} (\eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha} - \eta^{\mu\nu} \eta^{\alpha\beta}) \partial_\alpha \bar{\psi} \partial_\beta \psi - \eta^{\mu\nu} mc^2 \bar{\psi} \psi.$$

Electromagnetic interaction

There is a simple way to make any field interact with electromagnetism in a gauge invariant way: replace the derivative operators with the gauge covariant derivative operators. The Klein Gordon equation becomes:

$$D_\mu D^\mu \phi = -(\partial_t - ieA_0)^2 \phi + (\partial_i - ieA_i)^2 \phi = m^2 \phi$$

in natural units, where A is the vector potential. While it is possible to add many higher order terms, for example,

$$D_\mu D^\mu \phi + AF^{\mu\nu} D_\mu \phi D_\nu (D_\alpha D^\alpha \phi) = 0$$

these terms are not renormalizable in 3+1 dimensions.

The field equation for a charged scalar field multiplies by i , which means the field must be complex. In order for a field to be charged, it must have two components that can rotate into each other, the real and imaginary parts.

The action for a charged scalar is the covariant version of the uncharged action:

$$S = \int_x (\partial_\mu \phi^* + ieA_\mu \phi^*)(\partial_\nu \phi - ieA_\nu \phi) \eta^{\mu\nu} = \int_x |D\phi|^2$$

Gravitational interaction

In general relativity, we include the effect of gravity and the Klein–Gordon equation becomes

$$\frac{-1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu \psi) + \frac{m^2 c^2}{\hbar^2} \psi = 0$$

or equivalently

$$\begin{aligned} 0 &= -g^{\mu\nu} \nabla_\mu \nabla_\nu \psi + \frac{m^2 c^2}{\hbar^2} \psi \\ &= -g^{\mu\nu} \partial_\mu \partial_\nu \psi + g^{\mu\nu} \Gamma^\sigma_{\mu\nu} \partial_\sigma \psi + \frac{m^2 c^2}{\hbar^2} \psi \end{aligned}$$

where $g^{\alpha\beta}$ is the reciprocal of the metric tensor that is the gravitational potential field, g is the determinant of the metric tensor, ∇_μ is the covariant derivative and $\Gamma^\sigma_{\mu\nu}$ is the Christoffel symbol that is the gravitational force field.

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- Linear Klein–Gordon Equation^[2] at EqWorld: The World of Mathematical Equations.
- Nonlinear Klein–Gordon Equation^[3] at EqWorld: The World of Mathematical Equations.

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 [3] <http://eqworld.ipmnet.ru/en/solutions/npde/npde2107.pdf>

Einstein-Maxwell-Dirac equations

Einstein-Maxwell-Dirac equations (EMD) are related to quantum field theory. The current Big Bang Model is a quantum field theory in a curved spacetime. Unfortunately, no such theory is mathematically well-defined; in spite of this, theoreticians claim to extract information from this hypothetical theory. On the other hand, the super-classical limit of the not mathematically well-defined QED in a curved spacetime is the mathematically well-defined Einstein-Maxwell-Dirac system. (One could get a similar system for the standard model.) As a super theory, EMD violates the positivity condition in the Penrose-Hawking Singularity Theorem. Thus, it is possible that there would be complete solutions without any singularities-Yau has in fact constructed some. Furthermore, it is known that the Einstein-Maxwell-Dirac system admits of solitonic solutions, i.e., classical electrons and photons. This is the kind of theory Einstein was hoping for. EMD is also a totally geometrized theory as a non-commutative geometry; here, the charge e and the mass m of the electron are geometric invariants of the non-commutative geometry analogous to π .

One way of trying to construct a rigorous QED and beyond is to attempt to apply the deformation quantization program to MD, and more generally, EMD. This would involve the following.

Program for SCESM

The Super-Classical Einstein-Standard Model:

- 1. Extend Asymptotic Completeness, Global Existence and the Infrared Problem for the Maxwell-Dirac Equations to SCESM (*Memoirs of the American Mathematical Society*), by M. Flato, Jacques C. H. Simon, Erik Taflin (http://www.amazon.com/Asymptotic-Completeness-Existence-Maxwell-Dirac-Mathematical/dp/0821806831/ref=sr_1_1?ie=UTF8&s=books&qid=1240926988&sr=1-1).
- 2. Show that the positivity condition in the Penrose-Hawking singularity theorem is violated for the SCESM. Construct smooth solutions to SCESM having Dark Stars. See here: *The Large Scale Structure of Space-Time* by Stephen W. Hawking, G. F. R. Ellis (http://www.amazon.com/Structure-Space-Time-Cambridge-Monographs-Mathematical/dp/0521099064/ref=pd_bbs_sr_1?ie=UTF8&s=books&qid=1240927769&sr=8-1)
- 3. Follow three substeps
 - i. Derive approximate history of the universe from SCESM - both analytically and *via* computer simulation.
 - ii. Compare with ESM (the QSM in a curved space-time).
 - iii. Compare with observation. See: *Cosmology* by Steven Weinberg (http://www.amazon.com/Cosmology-StevenWeinberg/dp/0198526822/ref=sr_1_1?ie=UTF8&s=books&qid=1240928371&sr=1-1)
- 4. Show that the solution space to SCESM, F , is a reasonable infinite dimensional super-symplectic manifold. See: *Supersymmetry for Mathematicians: An Introduction* (http://www.amazon.com/Supersymmetry-Mathematicians-Introduction-Courant-Lecture/dp/0821835742/ref=sr_1_5?ie=UTF8&s=books&qid=1240894011&sr=1-5)
- 5. Apply deformation quantization to F to obtain mathematically rigorous definition of SQESM (quantum version of SCESM). See:

Deformation Theory and Symplectic Geometry by Daniel Sternheimer, John Rawnsley (http://www.amazon.com/Deformation-Symplectic-Geometry-Mathematical-Physics/dp/0792345258/ref=sr_1_1?ie=UTF8&s=books&qid=1240930131&sr=1-1)

- 6. Derive history of the universe from SQESM and compare with observation.

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Rigged Hilbert space

In mathematics, a **rigged Hilbert space** (**Gelfand triple**, **nested Hilbert space**, **equipped Hilbert space**) is a construction designed to link the distribution and square-integrable aspects of functional analysis. Such spaces were introduced to study spectral theory in the broad sense. They can bring together the 'bound state' (eigenvector) and 'continuous spectrum', in one place.

Motivation

Since a function such as

$$x \mapsto e^{ix},$$

which is in an obvious sense an eigenvector of the differential operator

$$-i \frac{d}{dx}$$

on the real line $\mathbf{R}$, is not square-integrable for the usual Borel measure on $\mathbf{R}$, this requires some way of stepping outside the strict confines of the Hilbert space theory. This was supplied by the apparatus of Schwartz distributions, and a *generalized eigenfunction* theory was developed in the years after 1950.

Functional analysis approach

The concept of rigged Hilbert space places this idea in abstract functional-analytic framework. Formally, a rigged Hilbert space consists of a Hilbert space H , together with a subspace Φ which carries a finer topology, that is one for which the natural inclusion

$$\Phi \subseteq H$$

is continuous. It is no loss to assume that Φ is dense in H for the Hilbert norm. We consider the inclusion of dual spaces H^* in Φ^* . The latter, dual to Φ in its 'test function' topology, is realised as a space of distributions or generalised functions of some sort, and the linear functionals on the subspace Φ of type

$$\phi \mapsto \langle v, \phi \rangle$$

for v in H are faithfully represented as distributions (because we assume Φ dense).

Now by applying the Riesz representation theorem we can identify H^* with H . Therefore the definition of *rigged Hilbert space* is in terms of a sandwich:

$$\Phi \subseteq H \subseteq \Phi^*.$$

The most significant examples are for which Φ is a nuclear space; this comment is an abstract expression of the idea that Φ consists of test functions and Φ^* of the corresponding distributions.

Formal definition (Gelfand triple)

A **rigged Hilbert space** is a pair (H, Φ) with H a Hilbert space, Φ a dense subspace, such that Φ is given a topological vector space structure for which the inclusion map i is continuous. Identifying H with its dual space H^* , the adjoint to i is the map $i^* : H = H^* \rightarrow \Phi^*$. The duality pairing between Φ and Φ^* has to be compatible with the inner product on H : $\langle u, v \rangle_{\Phi \times \Phi^*} = (u, v)_H$ whenever $u \in \Phi \subset H$ and $v \in H = H^* \subset \Phi^*$.

The specific triple $\{\Phi, H, \Phi^*\}$ is often named the "Gelfand triple" (after the mathematician Israel Gelfand).

Note that even though Φ is isomorphic to Φ^* if Φ is a Hilbert space in its own right, this isomorphism is *not* the same as the composition of the inclusion i with its adjoint i^*

$$i^*i : \Phi \subset H = H^* \rightarrow \Phi^*.$$

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Quantum inverse scattering method

Quantum inverse scattering method relates two different approaches: 1) Inverse scattering transform is a method of solving classical integrable differential equations of evolutionary type. Important concept is Lax representation. 2) Bethe ansatz is a method of solving quantum models in one space and one time dimension. Quantum inverse scattering method starts by quantization of Lax representation and reproduce results of Bethe ansatz. Actually it permits to rewrite Bethe ansatz in a new form: algebraic Bethe ansatz. This led to further progress in understanding of Heisenberg model (quantum), quantum Nonlinear Schrödinger equation and Hubbard model.

In mathematics, the **quantum inverse scattering method** is a method for solving integrable models in 1+1 dimensions introduced by L. D. Faddeev in about 1979.

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Quasi-Hopf algebra

A **quasi-Hopf algebra** is a generalization of a Hopf algebra, which was defined by the Russian mathematician Vladimir Drinfeld in 1989.

A *quasi-Hopf algebra* is a quasi-bialgebra $\mathcal{B}_{\mathcal{A}} = (\mathcal{A}, \Delta, \varepsilon, \Phi)$ for which there exist $\alpha, \beta \in \mathcal{A}$ and a bijective antihomomorphism S (antipode) of $\mathcal{A}$ such that

$$\sum_i S(b_i) \alpha c_i = \varepsilon(a) \alpha$$

$$\sum_i b_i \beta S(c_i) = \varepsilon(a) \beta$$

for all $a \in \mathcal{A}$ and where

$$\Delta(a) = \sum_i b_i \otimes c_i$$

and

$$\sum_i X_i \beta S(Y_i) \alpha Z_i = \mathbb{I},$$

$$\sum_j S(P_j) \alpha Q_j \beta S(R_j) = \mathbb{I}.$$

where the expansions for the quantities Φ and Φ^{-1} are given by

$$\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$$

and

$$\Phi^{-1} = \sum_j P_j \otimes Q_j \otimes R_j.$$

As for a quasi-bialgebra, the property of being quasi-Hopf is preserved under twisting.

Usage

Quasi-Hopf algebras form the basis of the study of Drinfeld twists and the representations in terms of F-matrices associated with finite-dimensional irreducible representations of quantum affine algebra. F-matrices can be used to factorize the corresponding R-matrix. This leads to applications in Statistical mechanics, as quantum affine algebras, and their representations give rise to solutions of the Yang-Baxter equation, a solvability condition for various statistical models, allowing characteristics of the model to be deduced from its corresponding quantum affine algebra. The study of F-matrices has been applied to models such as the Heisenberg XXZ model in the framework of the algebraic Bethe ansatz. It provides a framework for solving two-dimensional integrable models by using the Quantum inverse scattering method.

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Quasitriangular Hopf algebra

In mathematics, a Hopf algebra, H , is **quasitriangular**^[1] if there exists an invertible element, R , of $H \otimes H$ such that

- $R \Delta(x) = (T \circ \Delta)(x) R$ for all $x \in H$, where Δ is the coproduct on H , and the linear map $T : H \otimes H \rightarrow H \otimes H$ is given by $T(x \otimes y) = y \otimes x$,
- $(\Delta \otimes 1)(R) = R_{13} R_{23}$,
- $(1 \otimes \Delta)(R) = R_{13} R_{12}$,

where $R_{12} = \phi_{12}(R)$, $R_{13} = \phi_{13}(R)$, and $R_{23} = \phi_{23}(R)$, where $\phi_{12} : H \otimes H \rightarrow H \otimes H \otimes H$, $\phi_{13} : H \otimes H \rightarrow H \otimes H \otimes H$, and $\phi_{23} : H \otimes H \rightarrow H \otimes H \otimes H$, are algebra morphisms determined by

$$\begin{aligned}\phi_{12}(a \otimes b) &= a \otimes b \otimes 1, \\ \phi_{13}(a \otimes b) &= a \otimes 1 \otimes b, \\ \phi_{23}(a \otimes b) &= 1 \otimes a \otimes b.\end{aligned}$$

R is called the R-matrix.

As a consequence of the properties of quasitriangularity, the R-matrix, R , is a solution of the Yang-Baxter equation (and so a module V of H can be used to determine quasi-invariants of braids, knots and links). Also as a consequence of the properties of quasitriangularity, $(\epsilon \otimes 1)R = (1 \otimes \epsilon)R = 1 \in H$; moreover $R^{-1} = (S \otimes 1)(R)$, $R = (1 \otimes S)(R^{-1})$, and $(S \otimes S)(R) = R$. One may further show that the antipode S must be a linear isomorphism, and thus S^2 is an automorphism. In fact, S^2 is given by conjugating by an invertible element: $S(x) = u x u^{-1}$ where $u = m(S \otimes 1)R^{21}$ (cf. Ribbon Hopf algebras). It is possible to construct a quasitriangular Hopf algebra from a Hopf algebra and its dual, using the Drinfel'd quantum double construction.

Twisting

The property of being a quasi-triangular Hopf algebra is preserved by twisting via an invertible element $F = \sum_i f^i \otimes f_i \in \mathcal{A} \otimes \mathcal{A}$ such that $(\varepsilon \otimes id)F = (id \otimes \varepsilon)F = 1$ and satisfying the cocycle condition

$$(F \otimes 1) \circ (\Delta \otimes id)F = (1 \otimes F) \circ (id \otimes \Delta)F$$

Furthermore, $u = \sum_i f^i S(f_i)$ is invertible and the twisted antipode is given by $S'(a) = uS(a)u^{-1}$, with the twisted comultiplication, R-matrix and co-unit change according to those defined for the quasi-triangular Quasi-Hopf algebra. Such a twist is known as an admissible (or Drinfel'd) twist.

Notes

[1] Montgomery & Schneider (2002), p. 72 ([http://books.google.com/books?id=I3IK9U5Co_0C&pg=PA72&dq="Quasitriangular"](http://books.google.com/books?id=I3IK9U5Co_0C&pg=PA72&dq=)).

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Ribbon Hopf algebra

A **ribbon Hopf algebra** $(A, m, \Delta, u, \varepsilon, S, \mathcal{R}, \nu)$ is a quasitriangular Hopf algebra which possess an invertible central element ν more commonly known as the ribbon element, such that the following conditions hold:

$$\nu^2 = uS(u), \quad S(\nu) = \nu, \quad \varepsilon(\nu) = 1$$

$$\Delta(\nu) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}(\nu \otimes \nu)$$

where $u = m(S \otimes id)(\mathcal{R}_{21})$. Note that the element u exists for any quasitriangular Hopf algebra, and $uS(u)$ must always be central and satisfies $S(uS(u)) = uS(u)$, $\varepsilon(uS(u)) = 1$, $\Delta(uS(u)) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-2}(uS(u) \otimes uS(u))$, so that all that is required is that it have a central square root with the above properties. Here

A is a vector space

m is the multiplication map $m : A \otimes A \rightarrow A$

Δ is the co-product map $\Delta : A \rightarrow A \otimes A$

u is the unit operator $u : \mathbb{C} \rightarrow A$

ε is the co-unit operator $\varepsilon : A \rightarrow \mathbb{C}$

S is the antipode $S : A \rightarrow A$

$\mathcal{R}$ is a universal R matrix

We assume that the underlying field K is $\mathbb{C}$

See also

- Quasitriangular Hopf algebra
- Quasi-triangular Quasi-Hopf algebra

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Quasi-triangular Quasi-Hopf algebra

A **quasi-triangular quasi-Hopf algebra** is a specialized form of a quasi-Hopf algebra defined by the Ukrainian mathematician Vladimir Drinfeld in 1989. It is also a generalized form of a quasi-triangular Hopf algebra.

A **quasi-triangular quasi-Hopf algebra** is a set $\mathcal{H}_{\mathcal{A}} = (\mathcal{A}, R, \Delta, \varepsilon, \Phi)$ where $\mathcal{B}_{\mathcal{A}} = (\mathcal{A}, \Delta, \varepsilon, \Phi)$ is a quasi-Hopf algebra and $R \in \mathcal{A} \otimes \mathcal{A}$ known as the R-matrix, is an invertible element such that

$$R\Delta(a) = \sigma \circ \Delta(a)R, a \in \mathcal{A}$$

$$\sigma : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$$

$$x \otimes y \rightarrow y \otimes x$$

so that σ is the switch map and

$$(\Delta \otimes id)R = \Phi_{321}R_{13}\Phi_{132}^{-1}R_{23}\Phi_{123}$$

$$(id \otimes \Delta)R = \Phi_{231}^{-1}R_{13}\Phi_{213}R_{12}\Phi_{123}^{-1}$$

where $\Phi_{abc} = x_a \otimes x_b \otimes x_c$ and $\Phi_{123} = \Phi = x_1 \otimes x_2 \otimes x_3 \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$.

The quasi-Hopf algebra becomes *triangular* if in addition, $R_{21}R_{12} = 1$.

The twisting of $\mathcal{H}_{\mathcal{A}}$ by $F \in \mathcal{A} \otimes \mathcal{A}$ is the same as for a quasi-Hopf algebra, with the additional definition of the twisted R-matrix

A quasi-triangular (resp. triangular) quasi-Hopf algebra with $\Phi = 1$ is a quasi-triangular (resp. triangular) Hopf algebra as the latter two conditions in the definition reduce the conditions of quasi-triangularity of a Hopf algebra.

Similarly to the twisting properties of the quasi-Hopf algebra, the property of being quasi-triangular or triangular quasi-Hopf algebra is preserved by twisting.

See also

- Quasitriangular Hopf algebra
- Ribbon Hopf algebra

References

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Grassmann algebra

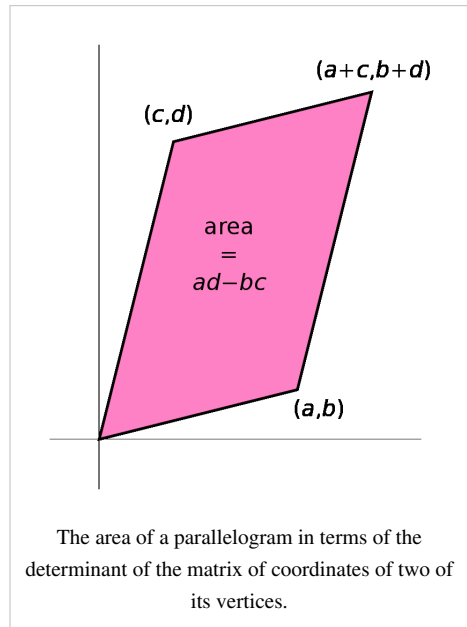
In mathematics, the **exterior product** or **wedge product** of vectors is an algebraic construction generalizing certain features of the cross product to higher dimensions. Like the cross product, and the scalar triple product, the exterior product of vectors is used in Euclidean geometry to study areas, volumes, and their higher-dimensional analogs. Also, like the cross product, the exterior product is alternating, meaning that $u \wedge u = 0$ for all vectors u , or equivalently^[1] $u \wedge v = -v \wedge u$ for all vectors u and v . In linear algebra, the exterior product provides an abstract algebraic manner for describing the determinant and the minors of a linear transformation that is basis-independent, and is fundamentally related to ideas of rank and linear independence.

The **exterior algebra** (also known as the **Grassmann algebra**, after Hermann Grassmann^[2]) of a given vector space V over a field K is the unital associative algebra $\Lambda(V)$ generated by the exterior product. It is widely used in contemporary geometry, especially differential geometry and algebraic geometry through the algebra of differential forms, as well as in multilinear algebra and related fields. In terms of category theory, the exterior algebra is a type of functor on vector spaces, given by a universal construction. The universal construction allows the exterior algebra to be defined, not just for vector spaces over a field, but also for modules over a commutative ring, and for other structures of interest. The exterior algebra is one example of a bialgebra, meaning that its dual space also possesses a product, and this dual product is compatible with the wedge product. This dual algebra is precisely the algebra of alternating multilinear forms on V , and the pairing between the exterior algebra and its dual is given by the interior product.

Motivating examples

Areas in the plane

The Cartesian plane $\mathbf{R}^2$ is a vector space equipped with a basis consisting of a pair of unit vectors



$$\mathbf{e}_1 = (1, 0), \quad \mathbf{e}_2 = (0, 1).$$

Suppose that

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2, \quad \mathbf{w} = w_1 \mathbf{e}_1 + w_2 \mathbf{e}_2$$

are a pair of given vectors in $\mathbf{R}^2$, written in components. There is a unique parallelogram having $\mathbf{v}$ and $\mathbf{w}$ as two of its sides. The *area* of this parallelogram is given by the standard determinant formula:

$$A = |\det [\mathbf{v} \ \mathbf{w}]| = |v_1 w_2 - v_2 w_1|.$$

Consider now the exterior product of $\mathbf{v}$ and $\mathbf{w}$:

$$\begin{aligned} \mathbf{v} \wedge \mathbf{w} &= (v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2) \wedge (w_1 \mathbf{e}_1 + w_2 \mathbf{e}_2) \\ &= v_1 w_1 \mathbf{e}_1 \wedge \mathbf{e}_1 + v_1 w_2 \mathbf{e}_1 \wedge \mathbf{e}_2 + v_2 w_1 \mathbf{e}_2 \wedge \mathbf{e}_1 + v_2 w_2 \mathbf{e}_2 \wedge \mathbf{e}_2 \\ &= (v_1 w_2 - v_2 w_1) \mathbf{e}_1 \wedge \mathbf{e}_2 \end{aligned}$$

where the first step uses the distributive law for the wedge product, and the last uses the fact that the wedge product is alternating, and in particular $\mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_1 \wedge \mathbf{e}_2$. Note that the coefficient in this last expression is precisely the determinant of the matrix $[\mathbf{v} \ \mathbf{w}]$. The fact that this may be positive or negative has the intuitive meaning that $\mathbf{v}$ and $\mathbf{w}$ may be oriented in a counterclockwise or clockwise sense as the vertices of the parallelogram they define. Such an area is called the **signed area** of the parallelogram: the absolute value of the signed area is the ordinary area, and the sign determines its orientation.

The fact that this coefficient is the signed area is not an accident. In fact, it is relatively easy to see that the exterior product should be related to the signed area if one tries to axiomatize this area as an algebraic construct. In detail, if $A(\mathbf{v}, \mathbf{w})$ denotes the signed area of the parallelogram determined by the pair of vectors $\mathbf{v}$ and $\mathbf{w}$, then A must satisfy the following properties:

1. $A(a\mathbf{v}, b\mathbf{w}) = a b A(\mathbf{v}, \mathbf{w})$ for any real numbers a and b , since rescaling either of the sides rescales the area by the same amount (and reversing the direction of one of the sides reverses the orientation of the parallelogram).
2. $A(\mathbf{v}, \mathbf{v}) = 0$, since the area of the degenerate parallelogram determined by $\mathbf{v}$ (i.e., a line segment) is zero.
3. $A(\mathbf{w}, \mathbf{v}) = -A(\mathbf{v}, \mathbf{w})$, since interchanging the roles of $\mathbf{v}$ and $\mathbf{w}$ reverses the orientation of the parallelogram.
4. $A(\mathbf{v} + a\mathbf{w}, \mathbf{w}) = A(\mathbf{v}, \mathbf{w})$, since adding a multiple of $\mathbf{w}$ to $\mathbf{v}$ affects neither the base nor the height of the parallelogram and consequently preserves its area.
5. $A(\mathbf{e}_1, \mathbf{e}_2) = 1$, since the area of the unit square is one.

With the exception of the last property, the wedge product satisfies the same formal properties as the area. In a certain sense, the wedge product generalizes the final property by allowing the area of a parallelogram to be compared to that of any "standard" chosen parallelogram. In other words, the exterior product in two-dimensions is a *basis-independent* formulation of area.^[3]

Cross and triple products

For vectors in $\mathbf{R}^3$, the exterior algebra is closely related to the cross product and triple product. Using the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the wedge product of a pair of vectors

$$\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3$$

and

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3$$

is

$$\mathbf{u} \wedge \mathbf{v} = (u_1 v_2 - u_2 v_1)(\mathbf{e}_1 \wedge \mathbf{e}_2) + (u_3 v_1 - u_1 v_3)(\mathbf{e}_3 \wedge \mathbf{e}_1) + (u_2 v_3 - u_3 v_2)(\mathbf{e}_2 \wedge \mathbf{e}_3)$$

where $\{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_3 \wedge \mathbf{e}_1, \mathbf{e}_2 \wedge \mathbf{e}_3\}$ is the basis for the three-dimensional space $\Lambda^2(\mathbf{R}^3)$. This imitates the usual definition of the cross product of vectors in three dimensions.

Bringing in a third vector

$$\mathbf{w} = w_1 \mathbf{e}_1 + w_2 \mathbf{e}_2 + w_3 \mathbf{e}_3,$$

the wedge product of three vectors is

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = (u_1 v_2 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_1 v_3 w_2 - u_2 v_1 w_3 - u_3 v_2 w_1)(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$$

where $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ is the basis vector for the one-dimensional space $\Lambda^3(\mathbf{R}^3)$. This imitates the usual definition of the triple product.

The cross product and triple product in three dimensions each admit both geometric and algebraic interpretations. The cross product $\mathbf{u} \times \mathbf{v}$ can be interpreted as a vector which is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ and whose magnitude is equal to the area of the parallelogram determined by the two vectors. It can also be interpreted as the vector consisting of the minors of the matrix with columns $\mathbf{u}$ and $\mathbf{v}$. The triple product of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is geometrically a (signed) volume. Algebraically, it is the determinant of the matrix with columns $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. The exterior product in three-dimensions allows for similar interpretations. In fact, in the presence of a positively oriented orthonormal basis, the exterior product generalizes these notions to higher dimensions.

Formal definitions and algebraic properties

The exterior algebra $\Lambda(V)$ over a vector space V is defined as the quotient algebra of the tensor algebra by the two-sided ideal I generated by all elements of the form $x \otimes x$ such that $x \in V$.^[4] Symbolically,

$$\Lambda(V) := T(V)/I.$$

The wedge product $\wedge$ of two elements of $\Lambda(V)$ is defined by

$$\alpha \wedge \beta = \alpha \otimes \beta \pmod{I}.$$

Anticommutativity of the wedge product

The wedge product is *alternating* on elements of V , which means that $x \wedge x = 0$ for all $x \in V$. It follows that the product is also anticommutative on elements of V , for supposing that $x, y \in V$,

$$0 = (x + y) \wedge (x + y) = x \wedge x + x \wedge y + y \wedge x + y \wedge y = x \wedge y + y \wedge x$$

whence

$$x \wedge y = -y \wedge x.$$

More generally, if $x_1, x_2, \dots, x_k$ are elements of V , and σ is a permutation of the integers $[1, \dots, k]$, then

$$x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \dots \wedge x_{\sigma(k)} = \text{sgn}(\sigma) x_1 \wedge x_2 \wedge \dots \wedge x_k,$$

where $\text{sgn}(\sigma)$ is the signature of the permutation σ .^[5]

The exterior power

The k th **exterior power** of V , denoted $\Lambda^k(V)$, is the vector subspace of $\Lambda(V)$ spanned by elements of the form

$$x_1 \wedge x_2 \wedge \dots \wedge x_k, \quad x_i \in V, i = 1, 2, \dots, k.$$

If $\alpha \in \Lambda^k(V)$, then α is said to be a **k -multivector**. If, furthermore, α can be expressed as a wedge product of k elements of V , then α is said to be **decomposable**. Although decomposable multivectors span $\Lambda^k(V)$, not every element of $\Lambda^k(V)$ is decomposable. For example, in $\mathbf{R}^4$, the following 2-multivector is not decomposable:

$$\alpha = e_1 \wedge e_2 + e_3 \wedge e_4.$$

(This is in fact a symplectic form, since $\alpha \wedge \alpha \neq 0$.^[6])

Basis and dimension

If the dimension of V is n and $\{e_1, \dots, e_n\}$ is a basis of V , then the set

$$\{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$$

is a basis for $\Lambda^k(V)$. The reason is the following: given any wedge product of the form

$$v_1 \wedge \dots \wedge v_k$$

then every vector v_j can be written as a linear combination of the basis vectors e_i ; using the bilinearity of the wedge product, this can be expanded to a linear combination of wedge products of those basis vectors. Any wedge product in which the same basis vector appears more than once is zero; any wedge product in which the basis vectors do not appear in the proper order can be reordered, changing the sign whenever two basis vectors change places. In general, the resulting coefficients of the basis k -vectors can be computed as the minors of the matrix that describes the vectors v_j in terms of the basis e_i .

By counting the basis elements, the dimension of $\Lambda^k(V)$ is the binomial coefficient $C(n, k)$. In particular, $\Lambda^k(V) = \{0\}$ for $k > n$.

Any element of the exterior algebra can be written as a sum of multivectors. Hence, as a vector space the exterior algebra is a direct sum

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \dots \oplus \Lambda^n(V)$$

(where by convention $\Lambda^0(V) = K$ and $\Lambda^1(V) = V$), and therefore its dimension is equal to the sum of the binomial coefficients, which is 2^n .

Rank of a multivector

If $\alpha \in \Lambda^k(V)$, then it is possible to express α as a linear combination of decomposable multivectors:

$$\alpha = \alpha^{(1)} + \alpha^{(2)} + \dots + \alpha^{(s)}$$

where each $\alpha^{(i)}$ is decomposable, say

$$\alpha^{(i)} = \alpha_1^{(i)} \wedge \dots \wedge \alpha_k^{(i)}, \quad i = 1, 2, \dots, s.$$

The **rank** of the multivector α is the minimal number of decomposable multivectors in such an expansion of α . This is similar to the notion of tensor rank.

Rank is particularly important in the study of 2-multivectors (Sternberg 1974, §III.6) (Bryant et al. 1991). The rank of a 2-multivector α can be identified with half the rank of the matrix of coefficients of α in a basis. Thus if e_i is a basis for V , then α can be expressed uniquely as

$$\alpha = \sum_{i,j} a_{ij} e_i \wedge e_j$$

where $a_{ij} = -a_{ji}$ (the matrix of coefficients is skew-symmetric). The rank of the matrix a_{ij} is therefore even, and is twice the rank of the form α .

In characteristic 0, the 2-multivector α has rank p if and only if

$$\underbrace{\alpha \wedge \dots \wedge \alpha}_p \neq 0$$

and

$$\underbrace{\alpha \wedge \dots \wedge \alpha}_{p+1} = 0.$$

Graded structure

The wedge product of a k -multivector with a p -multivector is a $(k+p)$ -multivector, once again invoking bilinearity. As a consequence, the direct sum decomposition of the preceding section

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \cdots \oplus \Lambda^n(V)$$

gives the exterior algebra the additional structure of a graded algebra. Symbolically,

$$(\Lambda^k(V)) \wedge (\Lambda^p(V)) \subset \Lambda^{k+p}(V).$$

Moreover, the wedge product is graded anticommutative, meaning that if $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^p(V)$, then

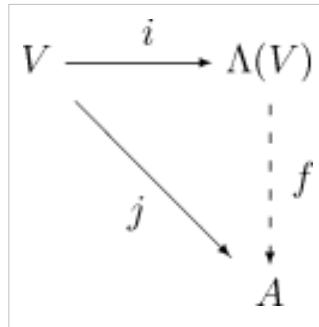
$$\alpha \wedge \beta = (-1)^{kp} \beta \wedge \alpha.$$

In addition to studying the graded structure on the exterior algebra, Bourbaki (1989) studies additional graded structures on exterior algebras, such as those on the exterior algebra of a graded module (a module that already carries its own gradation).

Universal property

Let V be a vector space over the field K . Informally, multiplication in $\Lambda(V)$ is performed by manipulating symbols and imposing a distributive law, an associative law, and using the identity $v \wedge v = 0$ for $v \in V$. Formally, $\Lambda(V)$ is the "most general" algebra in which these rules hold for the multiplication, in the sense that any unital associative K -algebra containing V with alternating multiplication on V must contain a homomorphic image of $\Lambda(V)$. In other words, the exterior algebra has the following universal property:^[7]

Given any unital associative K -algebra A and any K -linear map $j : V \rightarrow A$ such that $j(v)j(v) = 0$ for every v in V , then there exists *precisely one* unital algebra homomorphism $f : \Lambda(V) \rightarrow A$ such that $f(v) = j(i(v))$ for all v in V .



To construct the most general algebra that contains V and whose multiplication is alternating on V , it is natural to start with the most general algebra that contains V , the tensor algebra $T(V)$, and then enforce the alternating property by taking a suitable quotient. We thus take the two-sided ideal I in $T(V)$ generated by all elements of the form $v \otimes v$ for v in V , and define $\Lambda(V)$ as the quotient

$$\Lambda(V) = T(V)/I$$

(and use $\wedge$ as the symbol for multiplication in $\Lambda(V)$). It is then straightforward to show that $\Lambda(V)$ contains V and satisfies the above universal property.

As a consequence of this construction, the operation of assigning to a vector space V its exterior algebra $\Lambda(V)$ is a functor from the category of vector spaces to the category of algebras.

Rather than defining $\Lambda(V)$ first and then identifying the exterior powers $\Lambda^k(V)$ as certain subspaces, one may alternatively define the spaces $\Lambda^k(V)$ first and then combine them to form the algebra $\Lambda(V)$. This approach is often used in differential geometry and is described in the next section.

Generalizations

Given a commutative ring R and an R -module M , we can define the exterior algebra $\Lambda(M)$ just as above, as a suitable quotient of the tensor algebra $\mathbf{T}(M)$. It will satisfy the analogous universal property. Many of the properties of $\Lambda(M)$ also require that M be a projective module. Where finite-dimensionality is used, the properties further require that M be finitely generated and projective. Generalizations to the most common situations can be found in (Bourbaki 1989).

Exterior algebras of vector bundles are frequently considered in geometry and topology. There are no essential differences between the algebraic properties of the exterior algebra of finite-dimensional vector bundles and those of the exterior algebra of finitely-generated projective modules, by the Serre-Swan theorem. More general exterior algebras can be defined for sheaves of modules.

Duality

Alternating operators

Given two vector spaces V and X , an **alternating operator** (or *anti-symmetric operator*) from V^k to X is a multilinear map

$$f : V^k \rightarrow X$$

such that whenever $v_1, \dots, v_k$ are linearly dependent vectors in V , then

$$f(v_1, \dots, v_k) = 0$$

A well-known example is the determinant, an alternating operator from $(K^n)^n$ to K .

The map

$$w : V^k \rightarrow \wedge^k(V)$$

which associates to k vectors from V their wedge product, i.e. their corresponding k -vector, is also alternating. In fact, this map is the "most general" alternating operator defined on V^k : given any other alternating operator $f : V^k \rightarrow X$, there exists a unique linear map $\varphi : \wedge^k(V) \rightarrow X$ with $f = \varphi \circ w$. This universal property characterizes the space $\wedge^k(V)$ and can serve as its definition.

Alternating multilinear forms

The above discussion specializes to the case when $X = K$, the base field. In this case an alternating multilinear function

$$f : V^k \rightarrow K$$

is called an **alternating multilinear form**. The set of all alternating multilinear forms is a vector space, as the sum of two such maps, or the product of such a map with a scalar, is again alternating. By the universal property of the exterior power, the space of alternating forms of degree k on V is naturally isomorphic with the dual vector space $(\wedge^k V)^*$. If V is finite-dimensional, then the latter is naturally isomorphic to $\wedge^k(V^*)$. In particular, the dimension of the space of anti-symmetric maps from V^k to K is the binomial coefficient n choose k .

Under this identification, the wedge product takes a concrete form: it produces a new anti-symmetric map from two given ones. Suppose $\omega : V^k \rightarrow K$ and $\eta : V^m \rightarrow K$ are two anti-symmetric maps. As in the case of tensor products of multilinear maps, the number of variables of their wedge product is the sum of the numbers of their variables. It is defined as follows:

$$\omega \wedge \eta = \frac{(k+m)!}{k!m!} \text{Alt}(\omega \otimes \eta)$$

where the alternation Alt of a multilinear map is defined to be the signed average of the values over all the permutations of its variables:

$$\text{Alt}(\omega)(x_1, \dots, x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \omega(x_{\sigma(1)}, \dots, x_{\sigma(k)}).$$

This definition of the wedge product is well-defined even if the field K has finite characteristic, if one considers an equivalent version of the above that does not use factorials or any constants:

$$\omega \wedge \eta(x_1, \dots, x_{k+m}) = \sum_{\sigma \in Sh_{k,m}} \text{sgn}(\sigma) \omega(x_{\sigma(1)}, \dots, x_{\sigma(k)}) \eta(x_{\sigma(k+1)}, \dots, x_{\sigma(k+m)}),$$

where here $Sh_{k,m} \subset S_{k+m}$ is the subset of (k,m) shuffles: permutations σ of the set $\{1, 2, \dots, k+m\}$ such that $\sigma(1) < \sigma(2) < \dots < \sigma(k)$, and $\sigma(k+1) < \sigma(k+2) < \dots < \sigma(k+m)$.^[8]

Bialgebra structure

In formal terms, there is a correspondence between the graded dual of the graded algebra $\Lambda(V)$ and alternating multilinear forms on V . The wedge product of multilinear forms defined above is dual to a coproduct defined on $\Lambda(V)$, giving the structure of a coalgebra.

The **coproduct** is a linear function $\Delta : \Lambda(V) \rightarrow \Lambda(V) \otimes \Lambda(V)$ given on decomposable elements by

$$\Delta(x_1 \wedge \dots \wedge x_k) = \sum_{p=0}^k \sum_{\sigma \in Sh_{p, k-p}} \text{sgn}(\sigma) (x_{\sigma(1)} \wedge \dots \wedge x_{\sigma(p)}) \otimes (x_{\sigma(p+1)} \wedge \dots \wedge x_{\sigma(k)}).$$

For example,

$$\Delta(x_1) = 1 \otimes x_1 + x_1 \otimes 1,$$

$$\Delta(x_1 \wedge x_2) = 1 \otimes (x_1 \wedge x_2) + x_1 \otimes x_2 - x_2 \otimes x_1 + (x_1 \wedge x_2) \otimes 1.$$

This extends by linearity to an operation defined on the whole exterior algebra. In terms of the coproduct, the wedge product on the dual space is just the graded dual of the coproduct:

$$(\alpha \wedge \beta)(x_1 \wedge \dots \wedge x_k) = (\alpha \otimes \beta)(\Delta(x_1 \wedge \dots \wedge x_k))$$

where the tensor product on the right-hand side is of multilinear linear maps (extended by zero on elements of incompatible homogeneous degree: more precisely, $\alpha \wedge \beta = \varepsilon \circ (\alpha \otimes \beta) \circ \Delta$, where ε is the counit, as defined presently).

The **counit** is the homomorphism $\varepsilon : \Lambda(V) \rightarrow K$ which returns the 0-graded component of its argument. The coproduct and counit, along with the wedge product, define the structure of a bialgebra on the exterior algebra.

With an **antipode** defined on homogeneous elements by $S(x) = (-1)^{\deg x} x$, the exterior algebra is furthermore a Hopf algebra.^[9]

Interior product

Suppose that V is finite-dimensional. If V^* denotes the dual space to the vector space V , then for each $\alpha \in V^*$, it is possible to define an antiderivation on the algebra $\Lambda(V)$,

$$i_\alpha : \Lambda^k V \rightarrow \Lambda^{k-1} V.$$

This derivation is called the **interior product** with α , or sometimes the **insertion operator**, or **contraction** by α .

Suppose that $\mathbf{w} \in \Lambda^k V$. Then $\mathbf{w}$ is a multilinear mapping of V^* to K , so it is defined by its values on the k -fold Cartesian product $V^* \times V^* \times \dots \times V^*$. If $u_1, u_2, \dots, u_{k-1}$ are $k-1$ elements of V^* , then define

$$(i_\alpha \mathbf{w})(u_1, u_2, \dots, u_{k-1}) = \mathbf{w}(\alpha, u_1, u_2, \dots, u_{k-1}).$$

Additionally, let $i_\alpha f = 0$ whenever f is a pure scalar (i.e., belonging to $\Lambda^0 V$).

Axiomatic characterization and properties

The interior product satisfies the following properties:

1. For each k and each $\alpha \in V^*$,

$$i_\alpha : \Lambda^k V \rightarrow \Lambda^{k-1} V.$$

(By convention, $\Lambda^{-1} = 0$.)

2. If v is an element of $V (= \Lambda^1 V)$, then $i_\alpha v = \alpha(v)$ is the dual pairing between elements of V and elements of V^* .
3. For each $\alpha \in V^*$, i_α is a graded derivation of degree -1 :

$$i_\alpha(a \wedge b) = (i_\alpha a) \wedge b + (-1)^{\deg a} a \wedge (i_\alpha b).$$

In fact, these three properties are sufficient to characterize the interior product as well as define it in the general infinite-dimensional case.

Further properties of the interior product include:

- $i_\alpha \circ i_\alpha = 0$.
- $i_\alpha \circ i_\beta = -i_\beta \circ i_\alpha$.

Hodge duality

Suppose that V has finite dimension n . Then the interior product induces a canonical isomorphism of vector spaces

$$\Lambda^k(V^*) \otimes \Lambda^n(V) \rightarrow \Lambda^{n-k}(V).$$

In the geometrical setting, a non-zero element of the top exterior power $\Lambda^n(V)$ (which is a one-dimensional vector space) is sometimes called a **volume form** (or **orientation form**, although this term may sometimes lead to ambiguity.) Relative to a given volume form σ , the isomorphism is given explicitly by

$$\alpha \in \Lambda^k(V^*) \mapsto i_\alpha \sigma \in \Lambda^{n-k}(V).$$

If, in addition to a volume form, the vector space V is equipped with an inner product identifying V with V^* , then the resulting isomorphism is called the **Hodge dual** (or more commonly the **Hodge star operator**)

$$* : \Lambda^k(V) \rightarrow \Lambda^{n-k}(V).$$

The composite of $*$ with itself maps $\Lambda^k(V) \rightarrow \Lambda^k(V)$ and is always a scalar multiple of the identity map. In most applications, the volume form is compatible with the inner product in the sense that it is a wedge product of an orthonormal basis of V . In this case,

$$* \circ * : \Lambda^k(V) \rightarrow \Lambda^k(V) = (-1)^{k(n-k)+q} I$$

where I is the identity, and the inner product has metric signature (p, q) — p plusses and q minuses.

Inner product

For V a finite-dimensional space, an inner product on V defines an isomorphism of V with V^* , and so also an isomorphism of $\Lambda^k V$ with $(\Lambda^k V)^*$. The pairing between these two spaces also takes the form of an inner product. On decomposable k -multivectors,

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle),$$

the determinant of the matrix of inner products. In the special case $v_i = w_i$, the inner product is the square norm of the multivector, given by the determinant of the Gramian matrix $(\langle v_i, v_j \rangle)$. This is then extended bilinearly (or sesquilinearly in the complex case) to a non-degenerate inner product on $\Lambda^k V$. If e_i , $i=1, 2, \dots, n$, form an orthonormal basis of V , then the vectors of the form

$$e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad i_1 < \cdots < i_k,$$

constitute an orthonormal basis for $\Lambda^k(V)$.

With respect to the inner product, exterior multiplication and the interior product are mutually adjoint. Specifically, for $\mathbf{v} \in \Lambda^{k-1}(V)$, $\mathbf{w} \in \Lambda^k(V)$, and $x \in V$,

$$\langle x \wedge \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, i_{x^\flat} \mathbf{w} \rangle$$

where $x^\flat \in V^*$ is the linear functional defined by

$$x^\flat(y) = \langle x, y \rangle$$

for all $y \in V$. This property completely characterizes the inner product on the exterior algebra.

Functoriality

Suppose that V and W are a pair of vector spaces and $f : V \rightarrow W$ is a linear transformation. Then, by the universal construction, there exists a unique homomorphism of graded algebras

$$\Lambda(f) : \Lambda(V) \rightarrow \Lambda(W)$$

such that

$$\Lambda(f)|_{\Lambda^1(V)} = f : V = \Lambda^1(V) \rightarrow W = \Lambda^1(W).$$

In particular, $\Lambda(f)$ preserves homogeneous degree. The k -graded components of $\Lambda(f)$ are given on decomposable elements by

$$\Lambda(f)(x_1 \wedge \dots \wedge x_k) = f(x_1) \wedge \dots \wedge f(x_k).$$

Let

$$\Lambda^k(f) = \Lambda(f)_{\Lambda^k(V)} : \Lambda^k(V) \rightarrow \Lambda^k(W).$$

The components of the transformation $\Lambda(k)$ relative to a basis of V and W is the matrix of $k \times k$ minors of f . In particular, if $V = W$ and V is of finite dimension n , then $\Lambda^n(f)$ is a mapping of a one-dimensional vector space Λ^n to itself, and is therefore given by a scalar: the determinant of f .

Exactness

If

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$$

is a short exact sequence of vector spaces, then

$$0 \rightarrow \Lambda^1(U) \wedge \Lambda(V) \rightarrow \Lambda(V) \rightarrow \Lambda(W) \rightarrow 0$$

is an exact sequence of graded vector spaces^[10] as is

$$0 \rightarrow \Lambda(U) \rightarrow \Lambda(V).^{[11]}$$

Direct sums

In particular, the exterior algebra of a direct sum is isomorphic to the tensor product of the exterior algebras:

$$\Lambda(V \oplus W) = \Lambda(V) \otimes \Lambda(W).$$

This is a graded isomorphism; i.e.,

$$\Lambda^k(V \oplus W) = \bigoplus_{p+q=k} \Lambda^p(V) \otimes \Lambda^q(W).$$

Slightly more generally, if

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$$

is a short exact sequence of vector spaces then $\Lambda^k(V)$ has a filtration

$$0 = F^0 \subseteq F^1 \subseteq \dots \subseteq F^k \subseteq F^{k+1} = \Lambda^k(V)$$

with quotients : $F^{p+1}/F^p = \Lambda^{k-p}(U) \otimes \Lambda^p(W)$. In particular, if U is 1-dimensional then

$$0 \rightarrow U \otimes \Lambda^{k-1}(W) \rightarrow \Lambda^k(V) \rightarrow \Lambda^k(W) \rightarrow 0$$

is exact, and if W is 1-dimensional then

$$0 \rightarrow \Lambda^k(U) \rightarrow \Lambda^k(V) \rightarrow \Lambda^{k-1}(U) \otimes W \rightarrow 0$$

is exact.^[12]

The alternating tensor algebra

If K is a field of characteristic 0,^[13] then the exterior algebra of a vector space V can be canonically identified with the vector subspace of $T(V)$ consisting of antisymmetric tensors. Recall that the exterior algebra is the quotient of $T(V)$ by the ideal I generated by $x \otimes x$.

Let $T^r(V)$ be the space of homogeneous tensors of degree r . This is spanned by decomposable tensors

$$v_1 \otimes \dots \otimes v_r, \quad v_i \in V.$$

The **antisymmetrization** (or sometimes the **skew-symmetrization**) of a decomposable tensor is defined by

$$\text{Alt}(v_1 \otimes \dots \otimes v_r) = \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} \text{sgn}(\sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(r)}$$

where the sum is taken over the symmetric group of permutations on the symbols $\{1, \dots, r\}$. This extends by linearity and homogeneity to an operation, also denoted by Alt , on the full tensor algebra $T(V)$. The image $\text{Alt}(T(V))$ is the **alternating tensor algebra**, denoted $A(V)$. This is a vector subspace of $T(V)$, and it inherits the structure of a graded vector space from that on $T(V)$. It carries an associative graded product $\hat{\otimes}$ defined by

$$t \hat{\otimes} s = \text{Alt}(t \otimes s).$$

Although this product differs from the tensor product, the kernel of Alt is precisely the ideal I (again, assuming that K has characteristic 0), and there is a canonical isomorphism

$$A(V) \cong \Lambda(V).$$

Index notation

Suppose that V has finite dimension n , and that a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of V is given. then any alternating tensor $t \in A^r(V) \subset T^r(V)$ can be written in index notation as

$$t = t^{i_1 i_2 \dots i_r} \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \dots \otimes \mathbf{e}_{i_r}$$

where $t^{i_1 \dots i_r}$ is completely antisymmetric in its indices.

The wedge product of two alternating tensors t and s of ranks r and p is given by

$$t \hat{\otimes} s = \frac{1}{(r+p)!} \sum_{\sigma \in \mathfrak{S}_{r+p}} \text{sgn}(\sigma) t^{i_{\sigma(1)} \dots i_{\sigma(r)}} s^{i_{\sigma(r+1)} \dots i_{\sigma(r+p)}} \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \dots \otimes \mathbf{e}_{i_{r+p}}.$$

The components of this tensor are precisely the skew part of the components of the tensor product $s \otimes t$, denoted by square brackets on the indices:

$$(t \hat{\otimes} s)^{i_1 \dots i_{r+p}} = t^{[i_1 \dots i_r} s^{i_{r+1} \dots i_{r+p}]}$$

The interior product may also be described in index notation as follows. Let $t = t^{i_0 i_1 \dots i_{r-1}}$ be an antisymmetric tensor of rank r . Then, for $\alpha \in V^*$, $i_\alpha t$ is an alternating tensor of rank $r-1$, given by

$$(i_\alpha t)^{i_1 \dots i_{r-1}} = r \sum_{j=0}^n \alpha_j t^{j i_1 \dots i_{r-1}}.$$

where n is the dimension of V .

Applications

Linear geometry

The decomposable k -vectors have geometric interpretations: the bivector $u \wedge v$ represents the plane spanned by the vectors, "weighted" with a number, given by the area of the oriented parallelogram with sides u and v . Analogously, the 3-vector $u \wedge v \wedge w$ represents the spanned 3-space weighted by the volume of the oriented parallelepiped with edges u , v , and w .

Projective geometry

Decomposable k -vectors in $\Lambda^k V$ correspond to weighted k -dimensional subspaces of V . In particular, the Grassmannian of k -dimensional subspaces of V , denoted $Gr_k(V)$, can be naturally identified with an algebraic subvariety of the projective space $\mathbf{P}(\Lambda^k V)$. This is called the Plücker embedding.

Differential geometry

The exterior algebra has notable applications in differential geometry, where it is used to define differential forms. A differential form at a point of a differentiable manifold is an alternating multilinear form on the tangent space at the point. Equivalently, a differential form of degree k is a linear functional on the k -th exterior power of the tangent space. As a consequence, the wedge product of multilinear forms defines a natural wedge product for differential forms. Differential forms play a major role in diverse areas of differential geometry.

In particular, the exterior derivative gives the exterior algebra of differential forms on a manifold the structure of a differential algebra. The exterior derivative commutes with pullback along smooth mappings between manifolds, and it is therefore a natural differential operator. The exterior algebra of differential forms, equipped with the exterior derivative, is a differential complex whose cohomology is called the de Rham cohomology of the underlying manifold and plays a vital role in the algebraic topology of differentiable manifolds.

Representation theory

In representation theory, the exterior algebra is one of the two fundamental Schur functors on the category of vector spaces, the other being the symmetric algebra. Together, these constructions are used to generate the irreducible representations of the general linear group; see fundamental representation.

Physics

The exterior algebra is an archetypal example of a superalgebra, which plays a fundamental role in physical theories pertaining to fermions and supersymmetry. For a physical discussion, see Grassmann number. For various other applications of related ideas to physics, see superspace and supergroup (physics).

Lie algebra homology

Let L be a Lie algebra over a field k , then it is possible to define the structure of a chain complex on the exterior algebra of L . This is a k -linear mapping

$$\partial : \Lambda^{p+1} L \rightarrow \Lambda^p L$$

defined on decomposable elements by

$$\partial(x_1 \wedge \cdots \wedge x_{p+1}) = \frac{1}{p+1} \sum_{j < \ell} (-1)^{j+\ell+1} [x_j, x_\ell] \wedge x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge \hat{x}_\ell \wedge \cdots \wedge x_{p+1}.$$

The Jacobi identity holds if and only if $\partial\partial = 0$, and so this is a necessary and sufficient condition for an anticommutative nonassociative algebra L to be a Lie algebra. Moreover, in that case ΛL is a chain complex with boundary operator ∂ . The homology associated to this complex is the Lie algebra homology.

Homological algebra

The exterior algebra is the main ingredient in the construction of the Koszul complex, a fundamental object in homological algebra.

History

The exterior algebra was first introduced by Hermann Grassmann in 1844 under the blanket term of *Ausdehnungslehre*, or *Theory of Extension*.^[14] This referred more generally to an algebraic (or axiomatic) theory of extended quantities and was one of the early precursors to the modern notion of a vector space. Saint-Venant also published similar ideas of exterior calculus for which he claimed priority over Grassmann.^[15]

The algebra itself was built from a set of rules, or axioms, capturing the formal aspects of Cayley and Sylvester's theory of multivectors. It was thus a *calculus*, much like the propositional calculus, except focused exclusively on the task of formal reasoning in geometrical terms.^[16] In particular, this new development allowed for an *axiomatic* characterization of dimension, a property that had previously only been examined from the coordinate point of view.

The import of this new theory of vectors and multivectors was lost to mid 19th century mathematicians,^[17] until being thoroughly vetted by Giuseppe Peano in 1888. Peano's work also remained somewhat obscure until the turn of the century, when the subject was unified by members of the French geometry school (notably Henri Poincaré, Élie Cartan, and Gaston Darboux) who applied Grassmann's ideas to the calculus of differential forms.

A short while later, Alfred North Whitehead, borrowing from the ideas of Peano and Grassmann, introduced his universal algebra. This then paved the way for the 20th century developments of abstract algebra by placing the axiomatic notion of an algebraic system on a firm logical footing.

Notes

[1] Provided the characteristic is different from 2.

[2] Grassmann (1844) introduced these as *extended* algebras (cf. Clifford 1878). He used the word *äußere* (literally translated as *outer*, or *exterior*) only to indicate the *produkt* he defined, which is nowadays conventionally called *exterior product*, probably to distinguish it from the *outer product* as defined in modern linear algebra.

[3] This axiomatization of areas is due to Leopold Kronecker and Karl Weierstrass; see Bourbaki (1989, Historical Note). For a modern treatment, see MacLane & Birkhoff (1999, Theorem IX.2.2). For an elementary treatment, see Strang (1993, Chapter 5).

[4] This definition is a standard one. See, for instance, MacLane & Birkhoff (1999).

[5] A proof of this can be found in more generality in Bourbaki (1989).

[6] See Sternberg (1964, §III.6).

[7] See Bourbaki (1989, III.7.1), and MacLane & Birkhoff (1999, Theorem XVI.6.8). More detail on universal properties in general can be found in MacLane & Birkhoff (1999, Chapter VI), and throughout the works of Bourbaki.

[8] Some conventions, particularly in physics, define the wedge product as

$$\omega \wedge \eta = \text{Alt}(\omega \otimes \eta).$$

This convention is not adopted here, but is discussed in connection with alternating tensors.

[9] Indeed, the exterior algebra of V is the enveloping algebra of the abelian Lie superalgebra structure on V .

[10] This part of the statement also holds in greater generality if V and W are modules over a commutative ring: That Λ converts epimorphisms to epimorphisms. See Bourbaki (1989, Proposition 3, III.7.2).

[11] This statement generalizes only to the case where V and W are projective modules over a commutative ring. Otherwise, it is generally not the case that Λ converts monomorphisms to monomorphisms. See Bourbaki (1989, Corollary to Proposition 12, III.7.9).

[12] Such a filtration also holds for vector bundles, and projective modules over a commutative ring. This is thus more general than the result quoted above for direct sums, since not every short exact sequence splits in other abelian categories.

[13] See Bourbaki (1989, III.7.5) for generalizations.

[14] Kannenberg (2000) published a translation of Grassmann's work in English; he translated *Ausdehnungslehre* as *Extension Theory*.

[15] J Itard, Biography in Dictionary of Scientific Biography (New York 1970-1990).

[16] Authors have in the past referred to this calculus variously as the *calculus of extension* (Whitehead 1898; Forder 1941), or *extensive algebra* (Clifford 1878), and recently as *extended vector algebra* (Browne 2007).

[17] Bourbaki 1989, p. 661.

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Supergroup

The concept of **supergroup** is a generalization of that of group. In other words, every group is a supergroup but not every supergroup is a group. A supergroup is like a Lie group in that there is a well defined notion of smooth function defined on them. However the functions may have even and odd parts. Moreover a supergroup has a super Lie algebra which plays a role similar to that of a Lie algebra for Lie groups in that they determine most of the representation theory and which is the starting point for classification.

More formally, a **Lie supergroup** is a supermanifold G together with a multiplication morphism $\mu : G \times G \rightarrow G$, an inversion morphism $i : G \rightarrow G$ and a unit morphism $e : 1 \rightarrow G$ which makes G a group object in the category of supermanifolds. This means that, formulated as commutative diagrams, the usual associativity and inversion axioms of a group continue to hold. Since every manifold is a super manifold, a Lie supergroup generalises the notion of a Lie group.

There are many possible supergroups. The ones of most interest in theoretical physics are the ones which extend the Poincaré group or the conformal group. Of particular interest are the **orthosymplectic groups** $Osp(N/M)$ and the **superconformal groups** $SU(N/M)$.

An equivalent algebraic approach starts from the observation that a super manifold is determined by its ring of supercommutative smooth functions, and that a morphism of super manifolds corresponds one to one with an algebra homomorphism between their functions in the opposite direction, i.e that the category of supermanifolds is opposite to the category of algebras of smooth graded commutative functions. Reversing all the arrows in the commutative diagrams that define a Lie supergroup then shows that functions over the supergroup have the structure of a $\mathbb{Z}_2$ -graded Hopf algebra. Likewise the representations of this Hopf algebra turn out to be $\mathbb{Z}_2$ -graded comodules. This Hopf algebra gives the global properties of the supergroup.

There is another related Hopf algebra which is the dual of the previous Hopf algebra. It can be identified with the Hopf algebra of graded differential operators at the origin. It only gives the local properties of the symmetries i.e., it only gives information about infinitesimal supersymmetry transformations. The representations of this Hopf algebra are modules. Like in the non graded case, this Hopf algebra can be described purely algebraically as the universal enveloping algebra of the Lie superalgebra.

In a similar way one can define an affine algebraic supergroup as a group object in the category of super algebraic affine varieties. An affine algebraic supergroup has a similar one to one relation to its Hopf algebra of super Polynomials. Using the language of schemes, which combines the geometric and algebraic point of view, algebraic supergroup schemes can be defined including super Abelian varieties.

Superalgebra

In mathematics and theoretical physics, a **superalgebra** is a $\mathbf{Z}_2$ -graded algebra.^[1] That is, it is an algebra over a commutative ring or field with a decomposition into "even" and "odd" pieces and a multiplication operator that respects the grading.

The prefix *super-* comes from the theory of supersymmetry in theoretical physics. Superalgebras and their representations, supermodules, provide an algebraic framework for formulating supersymmetry. The study of such objects is sometimes called super linear algebra. Superalgebras also play an important role in related field of supergeometry where they enter into the definitions of graded manifolds, supermanifolds and superschemes.

Formal definition

Let K be a fixed commutative ring. In most applications, K is a field such as $\mathbf{R}$ or $\mathbf{C}$.

A **superalgebra** over K is a K -module A with a direct sum decomposition

$$A = A_0 \oplus A_1$$

together with a bilinear multiplication $A \times A \rightarrow A$ such that

$$A_i A_j \subseteq A_{i+j}$$

where the subscripts are read modulo 2.

A **superring**, or $\mathbf{Z}_2$ -graded ring, is a superalgebra over the ring of integers $\mathbf{Z}$.

The elements of A_i are said to be **homogeneous**. The **parity** of a homogeneous element x , denoted by $|x|$, is 0 or 1 according to whether it is in A_0 or A_1 . Elements of parity 0 are said to be **even** and those of parity 1 to be **odd**. If x and y are both homogeneous then so is the product xy and $|xy| = |x| + |y|$.

An **associative superalgebra** is one whose multiplication is associative and a **unital superalgebra** is one with a multiplicative identity element. The identity element in a unital superalgebra is necessarily even. Unless otherwise specified, all superalgebras in this article are assumed to be associative and unital.

A **commutative superalgebra** is one which satisfies a graded version of commutativity. Specifically, A is commutative if

$$yx = (-1)^{|x||y|}xy$$

for all homogeneous elements x and y of A .

Examples

- Any algebra over a commutative ring K may be regarded as a purely even superalgebra over K ; that is, by taking A_1 to be trivial.
- Any $\mathbf{Z}$ or $\mathbf{N}$ -graded algebra may be regarded as superalgebra by reading the grading modulo 2. This includes examples such as tensor algebras and polynomial rings over K .
- In particular, any exterior algebra over K is a superalgebra. The exterior algebra is the standard example of a supercommutative algebra.
- The symmetric polynomials and alternating polynomials together form a superalgebra, being the even and odd parts, respectively. Note that this is a different grading from the grading by degree.
- Clifford algebras are superalgebras. They are generally noncommutative.
- The set of all endomorphisms (both even and odd) of a super vector space forms a superalgebra under composition.
- The set of all square supermatrices with entries in K forms a superalgebra denoted by $M_{plq}(K)$. This algebra may be identified with the algebra of endomorphisms of a free supermodule over K of rank plq .

- Lie superalgebras are a graded analog of Lie algebras. Lie superalgebras are nonunital and nonassociative; however, one may construct the analog of a universal enveloping algebra of a Lie superalgebra which is a unital, associative superalgebra.

Further definitions and constructions

A superalgebra is an algebra with a $\mathbb{Z}_2$ grading (“even” and “odd” elements) such that (i) the bracket of two generators is always antisymmetric except for two odd elements where it is symmetric and (ii) the Jacobi identities are satisfied.^[2]

$$\begin{aligned} [E_i, \{0_k, 0_b\}] &= \{[E_i, 0_k], 0_b\} + \{[E_i, 0_b], 0_k\} \\ [0_k, \{0_b, 0_a\}] &= [\{0_k, 0_b\}, 0_a] + [\{0_k, 0_a\}, 0_b] \end{aligned}$$

The first of these three identities says that the 0 form a representation of the ordinary Lie algebra spanned by E (Consider the 0 as vectors on which the E act.) The second is equivalent to the first if the Killing form is nonsingular. The last identity restricts the possible representations 0 of the ordinary Lie algebra. This relation is the reason that not every ordinary Lie algebra can be extended to a superalgebra.

Even subalgebra

Let A be a superalgebra over a commutative ring K . The submodule A_0 , consisting of all even elements, is closed under multiplication and contains the identity of A and therefore forms a subalgebra of A , naturally called the **even subalgebra**. It forms an ordinary algebra over K .

The set of all odd elements A_1 is an A_0 -bimodule whose scalar multiplication is just multiplication in A . The product in A equips A_1 with a bilinear form

$$\mu : A_1 \otimes_{A_0} A_1 \rightarrow A_0$$

such that

$$\mu(x \otimes y) \cdot z = x \cdot \mu(y \otimes z)$$

for all x, y , and z in A_1 . This follows from the associativity of the product in A .

Grade involution

There is a canonical involutive automorphism on any superalgebra called the **grade involution**. It is given on homogeneous elements by

$$\hat{x} = (-1)^{|x|} x$$

and on arbitrary elements by

$$\hat{x} = x_0 - x_1$$

where x_i are the homogeneous parts of x . If A has no 2-torsion (in particular, if 2 is invertible) then the grade involution can be used to distinguish the even and odd parts of A :

$$A_i = \{x \in A : \hat{x} = (-1)^i x\}.$$

Supercommutativity

The **supercommutator** on A is the binary operator given by

$$[x, y] = xy - (-1)^{|x||y|}yx$$

on homogeneous elements. This can be extended to all of A by linearity. Elements x and y of A are said to **supercommute** if $[x, y] = 0$.

The **supercenter** of A is the set of all elements of A which supercommute with all elements of A :

$$Z(A) = \{a \in A : [a, x] = 0 \text{ for all } x \in A\}.$$

The supercenter of A is, in general, different than the center of A as an ungraded algebra. A commutative superalgebra is one whose supercenter is all of A .

Super tensor product

The graded tensor product of two superalgebras may be regarded as a superalgebra with a multiplication rule determined by:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|}(a_1 a_2 \otimes b_1 b_2).$$

Generalizations and categorical definition

One can easily generalize the definition of superalgebras to include superalgebras over a commutative superring. The definition given above is then a specialization to the case where the base ring is purely even.

Let R be a commutative superring. A **superalgebra** over R is a R -supermodule A with a R -bilinear multiplication $A \times A \rightarrow A$ that respects the grading. Bilinearity here means that

$$r \cdot (xy) = (r \cdot x)y = (-1)^{|r||x|}x(r \cdot y)$$

for all homogeneous elements $r \in R$ and $x, y \in A$.

Equivalently, one may define a superalgebra over R as a superring A together with an superring homomorphism $R \rightarrow A$ whose image lies in the supercenter of A .

One may also define superalgebras categorically. The category of all R -supermodules forms a monoidal category under the super tensor product with R serving as the unit object. An associative, unital superalgebra over R can then be defined as a monoid in the category of R -supermodules. That is, a superalgebra is an R -supermodule A with two (even) morphisms

$$\mu : A \otimes A \rightarrow A$$

$$\eta : R \rightarrow A$$

for which the usual diagrams commute.

Notes

- [1] Kac, Martinez & Zelmanov (2001), p. 3 (<http://books.google.com/books?id=jTCNZz2Tk4cC&pg=PA3&dq=superalgebra>).
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Supergravity

In theoretical physics, **supergravity** (**supergravity theory**) is a field theory that combines the principles of supersymmetry and general relativity. Together, these imply that, in supergravity, the supersymmetry is a local symmetry (in contrast to non-gravitational supersymmetric theories, such as the Minimal Supersymmetric Standard Model (MSSM)). Since the generators of supersymmetry (SUSY) are convoluted with the Poincaré group to form a Super-Poincaré algebra it is very natural to see that SuperGravity follows naturally from supersymmetry.^[1]

Gravitons

Like any field theory of gravity, a supergravity theory contains a spin-2 field whose quantum is the graviton. Supersymmetry requires the graviton field to have a superpartner. This field has spin 3/2 and its quantum is the gravitino. The number of gravitino fields is equal to the number of supersymmetries. Supergravity theories are often said to be the only consistent theories of interacting massless spin 3/2 fields.

History

Four-dimensional SUGRA

SUGRA, or Super GRAvity, was initially proposed as a four-dimensional theory in 1976 by Daniel Z. Freedman, Peter van Nieuwenhuizen and Sergio Ferrara at Stony Brook University, but was quickly generalized to many different theories in various numbers of dimensions and greater number (N) of supersymmetry charges. Supergravity theories with $N > 1$ are usually referred to as extended supergravity (SUEGRA). Some supergravity theories were shown to be equivalent to certain higher-dimensional supergravity theories via dimensional reduction (e.g. $N = 1$ 11 dimensional supergravity is dimensionally reduced on S^7 to $N = 8$ $d = 4$ SUGRA). The resulting theories were sometimes referred to as Kaluza-Klein theories, as Kaluza and Klein constructed, nearly a century ago, a five-dimensional gravitational theory, that when dimensionally reduced on circle, its 4-dimensional non-massive modes describe electromagnetism coupled to gravity.

mSUGRA

mSUGRA means minimal Super GRAvity. The construction of a realistic model of particle interactions within the $N = 1$ supergravity framework where supersymmetry (SUSY) is broken by a super Higgs mechanism was carried out by Ali Chamseddine, Richard Arnowitt and Pran Nath in 1982. In these classes of models collectively now known as minimal supergravity Grand Unification Theories (mSUGRA GUT), gravity mediates the breaking of SUSY through the existence of a hidden sector. mSUGRA naturally generates the Soft SUSY breaking terms which are a consequence of the Super Higgs effect. Radiative breaking of electroweak symmetry through Renormalization Group Equations (RGEs) follows as an immediate consequence. mSUGRA is one of the most widely investigated models of particle physics due to its predictive power requiring only four input parameters and a sign, to determine the low energy Phenomenology from the scale of Grand Unification.

11d: the maximal SUGRA

One of these supergravities, the 11-dimensional theory, generated considerable excitement as the first potential candidate for the theory of everything. This excitement was built on four pillars, two of which have now been largely discredited:

- Werner Nahm showed that 11 dimensions was the largest number of dimensions consistent with a single graviton, and that a theory with more dimensions would also have particles with spins greater than 2. These problems are avoided in 12 dimensions if two of these dimensions are timelike, as has been often emphasized by Itzhak Bars.
- In 1981, Ed Witten showed that 11 was the smallest number of dimensions that was big enough to contain the gauge groups of the Standard Model, namely $SU(3)$ for the strong interactions and $SU(2)$ times $U(1)$ for the electroweak interactions. Today many techniques exist to embed the standard model gauge group in supergravity in any number of dimensions. For example, in the mid and late 1980s one often used the obligatory gauge symmetry in type I and heterotic string theories. In type II string theory they could also be obtained by compactifying on certain Calabi-Yau's. Today one may also use D-branes to engineer gauge symmetries.
- In 1978, Eugene Cremmer, Bernard Julia and Joel Scherk (CJS) of the École Normale Supérieure found the classical action for an 11-dimensional supergravity theory. This remains today the only known classical 11-dimensional theory with local supersymmetry and no fields of spin higher than two. Other 11-dimensional theories are known that are quantum-mechanically inequivalent to the CJS theory, but classically equivalent (that is, they reduce to the CJS theory when one imposes the classical equations of motion). For example, in the mid 1980s Bernard de Wit and Hermann Nicolai found an alternate theory in $D=11$ Supergravity with Local $SU(8)$ Invariance ^[2]. This theory, while not manifestly Lorentz-invariant, is in many ways superior to the CJS theory in that, for example, it dimensionally-reduces to the 4-dimensional theory without recourse to the classical equations of motion.
- In 1980, Peter G. O. Freund and M. A. Rubin showed that compactification from 11 dimensions preserving all the SUSY generators could occur in two ways, leaving only 4 or 7 macroscopic dimensions (the other 7 or 4 being compact). Unfortunately, the noncompact dimensions have to form an anti de Sitter space. Today it is understood that there are many possible compactifications, but that the Freund-Rubin compactifications are invariant under all of the supersymmetry transformations that preserve the action.

Thus, the first two results appeared to establish 11 dimensions uniquely, the third result appeared to specify the theory, and the last result explained why the observed universe appears to be four-dimensional.

Many of the details of the theory were fleshed out by Peter van Nieuwenhuizen, Sergio Ferrara and Daniel Z. Freedman.

The end of the SUGRA era

The initial excitement over 11-dimensional supergravity soon waned, as various failings were discovered, and attempts to repair the model failed as well. Problems included:

- The compact manifolds which were known at the time and which contained the standard model were not compatible with super-symmetry, and could not hold quarks or leptons. One suggestion was to replace the compact dimensions with the 7-sphere, with the symmetry group $SO(8)$, or the squashed 7-sphere, with symmetry group $SO(5)$ times $SU(2)$.
- Until recently, the physical neutrinos seen in the real world were believed to be massless, and appeared to be left-handed, a phenomenon referred to as the chirality of the Standard Model. It was very difficult to construct a chiral fermion from a compactification — the compactified manifold needed to have singularities, but physics near singularities did not begin to be understood until the advent of orbifold conformal field theories in the late 1980s.
- Supergravity models generically result in an unrealistically large cosmological constant in four dimensions, and that constant is difficult to remove, and so require fine-tuning. This is still a problem today.
- Quantization of the theory led to quantum field theory gauge anomalies rendering the theory inconsistent. In the intervening years physicists have learned how to cancel these anomalies.

Some of these difficulties could be avoided by moving to a 10-dimensional theory involving superstrings. However, by moving to 10 dimensions one loses the sense of uniqueness of the 11-dimensional theory.

The core breakthrough for the 10-dimensional theory, known as the first superstring revolution, was a demonstration by Michael B. Green, John H. Schwarz and David Gross that there are only three supergravity models in 10 dimensions which have gauge symmetries and in which all of the gauge and gravitational anomalies cancel. These were theories built on the groups $SO(32)$ and $E_8 \times E_8$, the direct product of two copies of E_8 . Today we know that, using D-branes for example, gauge symmetries can be introduced in other 10-dimensional theories as well.

The second superstring revolution

Initial excitement about the 10d theories, and the string theories that provide their quantum completion, died by the end of the 1980s. There were too many Calabi-Yaus to compactify on, many more than Yau had estimated, as he admitted in December 2005 at the 23rd International Solvay Conference in Physics. None quite gave the standard model, but it seemed as though one could get close with enough effort in many distinct ways. Plus no one understood the theory beyond the regime of applicability of string perturbation theory.

There was a comparatively quiet period at the beginning of the 1990s, during which, however, several important tools were developed. For example, it became apparent that the various superstring theories were related by "string dualities", some of which relate weak string-coupling (i.e. perturbative) physics in one model with strong string-coupling (i.e. non-perturbative) in another.

Then it all changed, in what is known as the second superstring revolution. Joseph Polchinski realized that obscure string theory objects, called D-branes, which he had discovered six years earlier, are stringy versions of the p-branes that were known in supergravity theories. The treatment of these p-branes was not restricted by string perturbation theory; in fact, thanks to supersymmetry, p-branes in supergravity were understood well beyond the limits in which string theory was understood.

Armed with this new nonperturbative tool, Edward Witten and many others were able to show that all of the perturbative string theories were descriptions of different states in a single theory which he named M-theory. Furthermore he argued that the long wavelength limit^{*} of M-theory should be described by the 11-dimensional supergravity that had fallen out of favor with the first superstring revolution 10 years earlier, accompanied by the 2- and 5-branes. [* = i.e. when the quantum wavelength associated to objects in the theory are much larger than the size of the 11th dimension].

Historically, then, supergravity has come "full circle". It is a commonly used framework in understanding features of string theories, M-theory and their compactifications to lower spacetime dimensions.

Relation to superstrings

Particular 10-dimensional supergravity theories are considered "low energy limits" of the 10-dimensional superstring theories; more precisely, these arise as the massless, tree-level approximation of string theories. True effective field theories of string theories, rather than truncations, are rarely available. Due to string dualities, the conjectured 11-dimensional M-theory is required to have 11-dimensional supergravity as a "low energy limit". However, this doesn't necessarily mean that string theory/M-theory is the only possible UV completion of supergravity; supergravity research is useful independent of those relations.

4D $N = 1$ SUGRA

Before we move on to SUGRA proper, let's recapitulate some important details about general relativity. We have a 4D differentiable manifold M with a $\text{Spin}(3,1)$ principal bundle over it. This principal bundle represents the local Lorentz symmetry. In addition, we have a vector bundle T over the manifold with the fiber having four real dimensions and transforming as a vector under $\text{Spin}(3,1)$. We have an invertible linear map from the tangent bundle TM to T . This map is the vierbein. The local Lorentz symmetry has a gauge connection associated with it, the spin connection.

The following discussion will be in superspace notation, as opposed to the component notation, which isn't manifestly covariant under SUSY. There are actually *many* different versions of SUGRA out there which are inequivalent in the sense that their actions and constraints upon the torsion tensor are different, but ultimately equivalent in that we can always perform a field redefinition of the supervierbeins and spin connection to get from one version to another.

In 4D $N=1$ SUGRA, we have a 4|4 real differentiable supermanifold M , i.e. we have 4 real bosonic dimensions and 4 real fermionic dimensions. As in the nonsupersymmetric case, we have a $\text{Spin}(3,1)$ principal bundle over M . We have an $\mathbf{R}^{4|4}$ vector bundle T over M . The fiber of T transforms under the local Lorentz group as follows; the four real bosonic dimensions transform as a vector and the four real fermionic dimensions transform as a Majorana spinor. This Majorana spinor can be reexpressed as a complex left-handed Weyl spinor and its complex conjugate right-handed Weyl spinor (they're not independent of each other). We also have a spin connection as before.

We will use the following conventions; the spatial (both bosonic and fermionic) indices will be indicated by $M, N, \dots$. The bosonic spatial indices will be indicated by $\mu, \nu, \dots$, the left-handed Weyl spatial indices by $\alpha, \beta, \dots$, and the right-handed Weyl spatial indices by $\dot{\alpha}, \dot{\beta}, \dots$. The indices for the fiber of T will follow a similar notation, except that they will be hatted like this: $\hat{M}, \hat{\alpha}$. See van der Waerden notation for more details. $\hat{M} = (\mu, \alpha, \dot{\alpha})$. The supervierbein is denoted by $e_{\hat{N}}^{\hat{M}}$, and the spin connection by $\omega_{\hat{M}\hat{N}P}$. The *inverse* supervierbein is denoted by $E_{\hat{M}}^{\hat{N}}$.

The supervierbein and spin connection are real in the sense that they satisfy the reality conditions

$$e_{\hat{N}}^{\hat{M}}(x, \bar{\theta}, \theta)^* = e_{\hat{N}^*}^{\hat{M}^*}(x, \theta, \bar{\theta}) \text{ where } \mu^* = \mu, \alpha^* = \dot{\alpha}, \text{ and } \dot{\alpha}^* = \alpha \text{ and } \omega(x, \bar{\theta}, \theta)^* = \omega(x, \theta, \bar{\theta})$$

The covariant derivative is defined as

$$D_{\hat{M}} f = E_{\hat{M}}^{\hat{N}} (\partial_N f + \omega_N[f]).$$

The covariant exterior derivative as defined over supermanifolds needs to be super graded. This means that every time we interchange two fermionic indices, we pick up a +1 sign factor, instead of -1.

The presence or absence of R symmetries is optional, but if R-symmetry exists, the integrand over the full superspace has to have an R-charge of 0 and the integrand over chiral superspace has to have an R-charge of 2.

A chiral superfield X is a superfield which satisfies $\bar{D}_{\hat{\alpha}} X = 0$. In order for this constraint to be consistent, we require the integrability conditions that $\left\{ \bar{D}_{\hat{\alpha}}, \bar{D}_{\hat{\beta}} \right\} = c_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} \bar{D}_{\hat{\gamma}}$ for some coefficients c .

Unlike nonSUSY GR, the torsion has to be nonzero, at least with respect to the fermionic directions. Already, even in flat superspace, $D_{\hat{\alpha}} e_{\hat{\alpha}} + \bar{D}_{\hat{\alpha}} e_{\hat{\alpha}} \neq 0$. In one version of SUGRA (but certainly not the only one), we have the following constraints upon the torsion tensor:

$$\begin{aligned} T_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} &= 0 \\ T_{\hat{\alpha}\hat{\beta}}^{\hat{\mu}} &= 0 \\ T_{\hat{\alpha}\hat{\beta}}^{\hat{\nu}} &= 0 \\ T_{\hat{\alpha}\hat{\beta}}^{\hat{\rho}} &= 2i\sigma_{\hat{\alpha}\hat{\beta}}^{\hat{\mu}} \\ T_{\hat{\mu}\hat{\alpha}}^{\hat{\nu}} &= 0 \\ T_{\hat{\mu}\hat{\nu}}^{\hat{\rho}} &= 0 \end{aligned}$$

Here, $\underline{\alpha}$ is a shorthand notation to mean the index runs over either the left or right Weyl spinors.

The superdeterminant of the supervierbein, $|e|$, gives us the volume factor for M. Equivalently, we have the volume 4|4-superform $e^{\hat{\mu}=0} \wedge \dots \wedge e^{\hat{\mu}=3} \wedge e^{\hat{\alpha}=1} \wedge e^{\hat{\alpha}=2} \wedge e^{\hat{\dot{\alpha}}=1} \wedge e^{\hat{\dot{\alpha}}=2}$.

If we complexify the superdiffeomorphisms, there is a gauge where $E_{\hat{\alpha}}^{\mu} = 0$, $E_{\hat{\alpha}}^{\beta} = 0$ and $E_{\hat{\alpha}}^{\dot{\beta}} = \delta_{\hat{\alpha}}^{\dot{\beta}}$. The resulting chiral superspace has the coordinates x and Θ .

R is a scalar valued chiral superfield derivable from the supervierbeins and spin connection. If f is any superfield, $(\bar{D}^2 - 8R)f$ is always a chiral superfield.

The action for a SUGRA theory with chiral superfields X , is given by

$$S = \int d^4x d^2\Theta d^2\mathcal{E} \left[\frac{3}{8} (\bar{D}^2 - 8R) e^{-K(\bar{X}, X)/3} + W(X) \right] + c.c.$$

where K is the Kähler potential and W is the superpotential, and $\mathcal{E}$ is the chiral volume factor. Unlike the case for flat superspace, adding a constant to either the Kähler or superpotential is now physical. A constant shift to the Kähler potential changes the effective Planck constant, while a constant shift to the superpotential changes the effective cosmological constant. As the effective Planck constant now depends upon the value of the chiral superfield X , we need to rescale the supervierbeins (a field redefinition) to get a constant Planck constant. This is called the **Einstein frame**.

Higher-dimensional SUGRA

See the article higher-dimensional supergravity for more details.

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Quantum statistical mechanics

Quantum statistical mechanics is the study of statistical ensembles of quantum mechanical systems. A statistical ensemble is described by a density operator S , which is a non-negative, self-adjoint, trace-class operator of trace 1 on the Hilbert space H describing the quantum system. This can be shown under various mathematical formalisms for quantum mechanics. One such formalism is provided by quantum logic.

Expectation

From classical probability theory, we know that the expectation of a random variable X is completely determined by its distribution D_X by

$$\text{Exp}(X) = \int_{\mathbb{R}} \lambda dD_X(\lambda)$$

assuming, of course, that the random variable is integrable or that the random variable is non-negative. Similarly, let A be an observable of a quantum mechanical system. A is given by a densely defined self-adjoint operator on H . The spectral measure of A defined by

$$E_A(U) = \int_U \lambda dE(\lambda),$$

uniquely determines A and conversely, is uniquely determined by A . E_A is a boolean homomorphism from the Borel subsets of $\mathbf{R}$ into the lattice $\mathcal{Q}$ of self-adjoint projections of H . In analogy with probability theory, given a state S , we introduce the *distribution* of A under S which is the probability measure defined on the Borel subsets of $\mathbf{R}$ by

$$D_A(U) = \text{Tr}(E_A(U)S).$$

Similarly, the expected value of A is defined in terms of the probability distribution D_A by

$$\text{Exp}(A) = \int_{\mathbb{R}} \lambda dD_A(\lambda).$$

Note that this expectation is relative to the mixed state S which is used in the definition of D_A .

Remark. For technical reasons, one needs to consider separately the positive and negative parts of A defined by the Borel functional calculus for unbounded operators.

One can easily show:

$$\text{Exp}(A) = \text{Tr}(AS) = \text{Tr}(SA).$$

Note that if S is a pure state corresponding to the vector ψ ,

$$\text{Exp}(A) = \langle \psi | A | \psi \rangle.$$

Von Neumann entropy

Of particular significance for describing randomness of a state is the von Neumann entropy of S *formally* defined by

$$H(S) = -\text{Tr}(S \log_2 S).$$

Actually, the operator $S \log_2 S$ is not necessarily trace-class. However, if S is a non-negative self-adjoint operator not of trace class we define $\text{Tr}(S) = +\infty$. Also note that any density operator S can be diagonalized, that it can be represented in some orthonormal basis by a (possibly infinite) matrix of the form

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & \cdots \\ 0 & \lambda_2 & \cdots & 0 & \cdots \\ & & \cdots & & \\ 0 & 0 & \cdots & \lambda_n & \cdots \\ & & \cdots & \cdots & \end{bmatrix}$$

and we define

$$H(S) = - \sum_i \lambda_i \log_2 \lambda_i.$$

The convention is that $0 \log_2 0 = 0$, since an event with probability zero should not contribute to the entropy. This value is an extended real number (that is in $[0, \infty]$) and this is clearly a unitary invariant of S .

Remark. It is indeed possible that $H(S) = +\infty$ for some density operator S . In fact T be the diagonal matrix

$$T = \begin{bmatrix} \frac{1}{2(\log_2 2)^2} & 0 & \cdots & 0 & \cdots \\ 0 & \frac{1}{3(\log_2 3)^2} & \cdots & 0 & \cdots \\ & & \cdots & & \\ 0 & 0 & \cdots & \frac{1}{n(\log_2 n)^2} & \cdots \\ & & \cdots & \cdots & \end{bmatrix}$$

T is non-negative trace class and one can show $T \log_2 T$ is not trace-class.

Theorem. Entropy is a unitary invariant.

In analogy with classical entropy (notice the similarity in the definitions), $H(S)$ measures the amount of randomness in the state S . The more dispersed the eigenvalues are, the larger the system entropy. For a system in which the space H is finite-dimensional, entropy is maximized for the states S which in diagonal form have the representation

$$\begin{bmatrix} \frac{1}{n} & 0 & \cdots & 0 \\ 0 & \frac{1}{n} & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & \frac{1}{n} \end{bmatrix}$$

For such an S , $H(S) = \log_2 n$. The state S is called the maximally mixed state.

Recall that a pure state is one of the form

$$S = |\psi\rangle\langle\psi|,$$

for ψ a vector of norm 1.

Theorem. $H(S) = 0$ if and only if S is a pure state.

For S is a pure state if and only if its diagonal form has exactly one non-zero entry which is a 1.

Entropy can be used as a measure of quantum entanglement.

Gibbs canonical ensemble

Consider an ensemble of systems described by a Hamiltonian H with average energy E . If H has pure-point spectrum and the eigenvalues E_n of H go to $+\infty$ sufficiently fast, e^{-rH} will be a non-negative trace-class operator for every positive r .

The *Gibbs canonical ensemble* is described by the state

$$S = \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})}$$

where β is such that the ensemble average of energy satisfies

$$\text{Tr}(SH) = E$$

,and

$$\text{Tr}(e^{-\beta H}) = \sum_n e^{-\beta E_n}$$

is the quantum mechanical version of the canonical partition function. The probability that a system chosen at random from the ensemble will be in a state corresponding to energy eigenvalue E_m is

$$\frac{e^{-\beta E_m}}{\sum_n e^{-\beta E_n}}.$$

Under certain conditions, the Gibbs canonical ensemble maximizes the von Neumann entropy of the state subject to the energy conservation requirement.

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Quantum thermodynamics

In the physical sciences, **quantum thermodynamics** is the study of heat and work dynamics in quantum systems. Approximately, quantum thermodynamics attempts to combine thermodynamics and quantum mechanics into a coherent whole. The essential point at which "quantum mechanics" began was when, in 1900, Max Planck outlined the "quantum hypothesis", i.e. that the energy of atomic systems can be quantized, as based on the first two laws of thermodynamics as described by Rudolf Clausius (1865) and Ludwig Boltzmann (1877).^[1] ^[2] See the history of quantum mechanics for a more detailed outline.

Overview

A central objective in quantum thermodynamics is the quantitative and qualitative determination of the laws of thermodynamics at the quantum level in which uncertainty and probability begin to take effect. A fundamental question is: what remains of thermodynamics if one goes to the extreme limit of small quantum systems having a few degrees of freedom? If thermodynamics applies at this level, does the second law of thermodynamics remain unchanged, or is there a more universal formulation than the many existing formulations, such as: the entropy of a closed system cannot decrease; heat flows from high to low temperature; systems evolve towards minimum potential energy wells; energy tends to dissipate; and so on.

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 2. Rudakov, E.S. (1998). *Molecular, Quantum and Evolution Thermodynamics: Development and Specialization of the Gibbs Method..* Donetsk State University Press. ISBN 966-02-0708-5.
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External links

- Quantum Thermodynamics and the Gibbs Paradox (<http://staff.science.uva.nl/~nieuwenh/QL2L.html>)
- Quantum Thermodynamics (<http://www.chaos.org.uk/~eddy/physics/heat.html>)
- On the Classical Limit of Quantum Thermodynamics in Finite Time (<http://www.fh.huji.ac.il/~ronnie/Papers/geva92.pdf>) [PDF-format]
- Quantum Thermodynamics (<http://www.quantumthermodynamics.org>) - list of good related articles

Supertheory

The **theory of everything** (TOE) is a putative theory of theoretical physics that fully explains and links together all known physical phenomena, and, ideally, has predictive power for the outcome of *any* experiment that could be carried out *in principle*.

Initially, the term was used with an ironic connotation to refer to various overgeneralized theories. For example, a great-grandfather of Ijon Tichy—a character from a cycle of Stanisław Lem's science fiction stories of the 1960s—was known to work on the "General Theory of Everything". Physicist John Ellis^[1] claims to have introduced the term into the technical literature in an article in *Nature* in 1986.^[2] Over time, the term stuck in popularizations of quantum physics to describe a theory that would unify or explain through a single model the theories of all fundamental interactions and of all particles of nature: general relativity for gravitation, and the standard model of elementary particle physics - which includes quantum mechanics - for electromagnetism, the two nuclear interactions, and the known elementary particles.

There have been many theories of everything proposed by theoretical physicists over the twentieth century, but none have been confirmed experimentally. The primary problem in producing a TOE is that the accepted theories of quantum mechanics and general relativity are hard to combine. Their mutual incompatibility makes their unification a difficult task. The combination is one of the unsolved problems in physics.

Based on theoretical holographic principle arguments from the 1990s, many physicists believe that 11-dimensional M-theory, which is described in many sectors by matrix string theory, in many other sectors by perturbative string theory, is the complete theory of everything. However, there is no widespread consensus on this issue, because M-theory is not a completed theory but rather an approach for producing one.

Historical antecedents

Ancient Greece to Einstein

Archimedes was possibly the first scientists to describe nature with axioms (or principles) and then to deduce new results from them. The putative theory of everything is expected to achieve the deduction of all phenomena from basic axioms.

Since ancient Greek times, philosophers have speculated that the apparent diversity of appearances conceals an underlying unity, and thus that the list of forces might be short, indeed might contain only a single entry. For example, the mechanical philosophy of the 17th century posited that all forces could be ultimately reduced to contact forces between tiny solid particles.^[3] This was abandoned after the acceptance of Isaac Newton's long-distance force of gravity; but at the same time, Newton's work in his *Principia* provided the first dramatic empirical evidence for the unification of apparently distinct forces: Galileo's work on terrestrial gravity, Kepler's laws of planetary motion, and the phenomenon of tides were all quantitatively explained by a single law of universal gravitation.

Building on these results, Laplace famously suggested that a sufficiently powerful intellect could, if it knew the position and velocity of every particle at a given time, along with the laws of nature, calculate the position of any

particle at any other time:

An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.

— *Essai philosophique sur les probabilités*, Introduction. 1814

Modern quantum mechanics implies that uncertainty is inescapable, and thus that Laplace's vision cannot be correct. A theory of everything this must include quantum mechanics and gravitation.

In 1820, Hans Christian Ørsted discovered a connection between electricity and magnetism, triggering decades of work that culminated in James Clerk Maxwell's theory of electromagnetism. Also during the 19th and early 20th centuries, it gradually became apparent that many common examples of forces—contact forces, elasticity, viscosity, friction, pressure—resulted from electrical interactions between the smallest particles of matter.

In the late 1920s, the new quantum mechanics showed that the chemical bonds between atoms were examples of (quantum) electrical forces, justifying Dirac's boast that "the underlying physical laws necessary for the mathematical theory of a large part of physics and the whole of chemistry are thus completely known".^[4]

Attempts to unify gravity with electromagnetism date back at least to Michael Faraday's experiments of 1849–50.^[5] After Albert Einstein's theory of gravity (general relativity) was published in 1915, the search for a unified field theory combining gravity with electromagnetism began in earnest. At the time, it seemed plausible that no other fundamental forces exist. Prominent contributors were Gunnar Nordström, Hermann Weyl, Arthur Eddington, Theodor Kaluza, Oskar Klein, and most notably, many attempts by Einstein and his collaborators. In his last years, Albert Einstein was intensely occupied in finding such a unifying theory. None of these attempts was successful.^[6]

The nuclear interactions

In the twentieth century, the search for a unifying theory was interrupted by the discovery of the strong and weak nuclear forces (or interactions), which differ both from gravity and from electromagnetism. A further hurdle was the acceptance that quantum mechanics had to be incorporated from the start, rather than emerging as a consequence of a deterministic unified theory, as Einstein had hoped.

Gravity and electromagnetism could always peacefully coexist as entries in a list of classical forces, but for many years it seemed that gravity could not even be incorporated into the quantum framework, let alone unified with the other fundamental forces. For this reason, work on unification, for much of the twentieth century, focused on understanding the three "quantum" forces: electromagnetism and the weak and strong forces. The first two were combined in 1967–68 by Sheldon Glashow, Steven Weinberg, and Abdus Salam into the "electroweak" force.^[7] However, while the strong and electroweak forces peacefully coexist in the Standard Model of particle physics, they remain distinct.

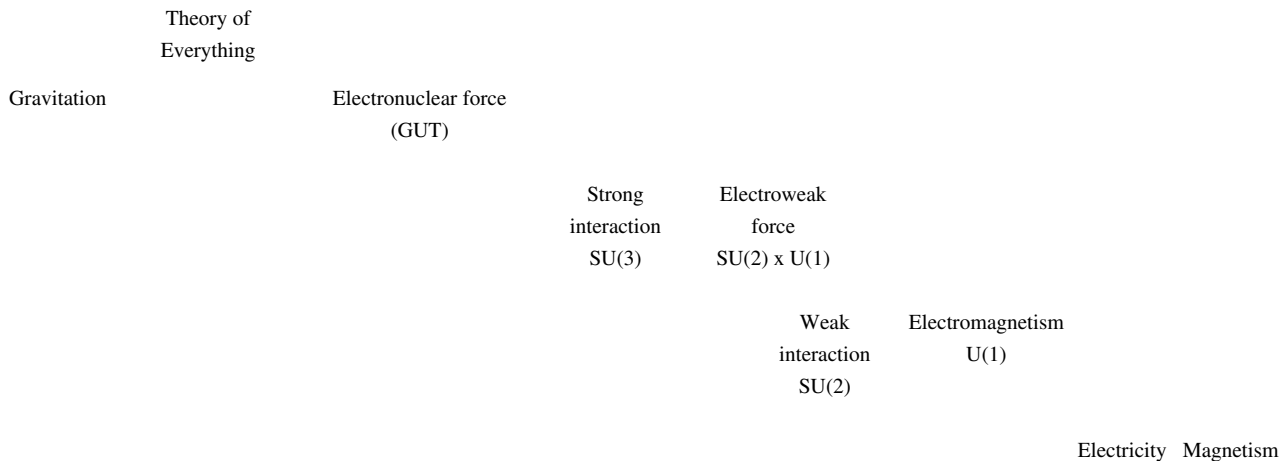
Electroweak unification is a broken symmetry: the electromagnetic and weak forces appear distinct at low energies because the particles carrying the weak force, the W and Z bosons, with masses of $80.4 \text{ GeV}/c^2$ and $91.2 \text{ GeV}/c^2$, whereas the photon, which carries the electromagnetic force, is massless. At higher energies Ws and Zs can be created easily and the unified nature of the force becomes apparent.

Several Grand Unified Theories (GUTs) have been proposed to unify electromagnetism and the weak and strong forces. Grand unification is expected to set in at energies of the order of 10^{16} GeV , far greater than could be reached by any possible Earth-based particle accelerator. Although the simplest GUTs have been experimentally ruled out, the general idea, especially when linked with supersymmetry, remains a favorite candidate in the theoretical physics community.

Modern physics

The conventional pattern of theories

A Theory of Everything would unify all the fundamental interactions of nature: gravitation, strong interaction, weak interaction, and electromagnetism. Because the weak interaction can transform elementary particles from one kind into another, the TOE should also yield a deep understanding of the various different kinds of possible particles. The usual assumed path of theories is given in the following graph, where each unification step leads one level higher:



In this graph, electroweak unification occurs at around 100 GeV, grand unification is predicted to occur at 10^{16} GeV, and unification of the GUT force with gravity is expected at the Planck energy, roughly 10^{19} GeV.

In addition to the forces listed in the graph, a TOE must also explain the status of at least to candidate forces suggested by modern cosmology: an inflationary force and dark energy. Furthermore, cosmological experiments also suggest the existence of dark matter, supposedly composed of fundamental particles outside the scheme of the standard model. However, the existence of these forces and particles has not been proven yet.

It may seem premature to be searching for a TOE when there is as yet no direct evidence for an electronuclear force, and while in any case there are many different proposed GUTs for this force. Nevertheless, most physicists believe that a GUT is possible, mainly due to the past history of convergence towards a single theory. Supersymmetric GUTs seem plausible not only for their theoretical "beauty", but because they naturally produce large quantities of dark matter, and the inflationary force may be related to GUT physics (although it does not seem to form an inevitable part of the theory). Yet GUTs are clearly not the final answer. Both the current standard model and all proposed GUTs are quantum field theories which require the problematic technique of renormalization to yield sensible answers. This is usually regarded as a sign that these are only effective field theories, omitting crucial phenomena relevant only at very high energies. Furthermore, the inconsistency between quantum mechanics and general relativity implies that one or both of these must be replaced by a theory incorporating quantum gravity.

String theory and M-theory

The mainstream theory of everything at the moment is superstring theory / M-theory. These theories attempt to deal with the renormalization problem by setting up some lower bound on the length scales possible.

String theories and supergravity (both believed to be limiting cases of the yet-to-be-defined M-theory) suppose that the universe actually has more dimensions than the easily observed three of space and one of time. The motivation behind this approach began with the Kaluza-Klein theory in which it was noted that applying general relativity to a five dimensional universe (with the usual four dimensions plus one small curled-up dimension) yields the equivalent of the usual general relativity in four dimensions together with Maxwell's equations (electromagnetism, also in four dimensions). This has led to efforts to work with theories with large number of dimensions in the hopes that this

would produce equations that are similar to known laws of physics. The notion of extra dimensions also helps to resolve the hierarchy problem, which is the question of why gravity is so much weaker than any other force. The common answer involves gravity leaking into the extra dimensions in ways that the other forces do not.

In the late 1990s, it was noted that one problem with several of the candidates for theories of everything (but particularly string theory) was that they did not constrain the characteristics of the predicted universe. For example, many theories of quantum gravity can create universes with arbitrary numbers of dimensions or with arbitrary cosmological constants. Even the "standard" ten-dimensional string theory allows the "curled up" dimensions to be compactified in an enormous number of different ways (one estimate is 10^{500}) each of which corresponds to a different collection of fundamental particles and low-energy forces. This array of theories is known as the string theory landscape.

A speculative solution is that many or all of these possibilities are realised in one or another of a huge number of universes, but that only a small number of them are habitable, and hence the fundamental constants of the universe are ultimately the result of the anthropic principle rather than a consequence of the theory of everything. This anthropic approach is often criticised in that, because the theory is flexible enough to encompass almost any observation, it cannot make useful (as in original, falsifiable, and verifiable) predictions. In this view, string theory would be considered a pseudoscience, where an unfalsifiable theory is constantly adapted to fit the experimental results.

Loop quantum gravity

Current research on loop quantum gravity may eventually play a fundamental role in a TOE, but that is not its primary aim.^[8] Loop quantum gravity is facing difficulties in incorporating electromagnetism and the nuclear interactions.

Other attempts

Any TOE must include general relativity and the standard model of particle physics. Outside the previously mentioned attempts, the best-known one is Garrett Lisi's E8 proposal.

Present status

At present, no convincing candidate for a TOE is available. Most particle physicists tend to state that the outcome of the ongoing experiments at the large particle accelerators, the LHC and the Tevatron, are needed in order to provide theoreticians with precise input for such a theory.

Arguments against a theory of everything

Gödel's incompleteness theorem

A small number of scientists claim that Gödel's incompleteness theorem proves that any attempt to construct a TOE is bound to fail. Gödel's theorem, informally stated, asserts that any formal theory expressive enough for elementary arithmetical facts to be expressed and strong enough for them to be proved is either inconsistent (both a statement and its denial can be derived from its axioms) or incomplete, in the sense that there is a true statement about natural numbers that can't be derived in the formal theory. In his 1966 book *The Relevance of Physics*, Stanley Jaki pointed out that, because any "theory of everything" will certainly be a consistent non-trivial mathematical theory, it must be incomplete. He claims that this dooms searches for a deterministic theory of everything.^[9] In a later reflection, Jaki states that it is wrong to say that a final theory is impossible, but rather that "when it is on hand one cannot know rigorously that it is a final theory."^[10]

Freeman Dyson has stated that

“Gödel's theorem implies that pure mathematics is inexhaustible. No matter how many problems we solve, there will always be other problems that cannot be solved within the existing rules. [...] Because of Gödel's theorem, physics is inexhaustible too. The laws of physics are a finite set of rules, and include the rules for doing mathematics, so that Gödel's theorem applies to them.”

—NYRB, May 13, 2004

Stephen Hawking was originally a believer in the Theory of Everything but, after considering Gödel's Theorem, concluded that one was not obtainable.

“Some people will be very disappointed if there is not an ultimate theory, that can be formulated as a finite number of principles. I used to belong to that camp, but I have changed my mind.”

—Gödel and the end of physics^[11], July 20, 2002

Jürgen Schmidhuber (1997) has argued against this view; he points out that Gödel's theorems are irrelevant for computable physics.^[12] In 2000, Schmidhuber explicitly constructed limit-computable, deterministic universes whose pseudo-randomness based on undecidable, Gödel-like halting problems is extremely hard to detect but does not at all prevent formal TOEs describable by very few bits of information.^{[13] [14]}

Related critique was offered by Solomon Feferman,^[15] among others. Douglas S. Robertson offers Conway's game of life as an example:^[16] The underlying rules are simple and complete, but there are formally undecidable questions about the game's behaviors. Analogously, it may (or may not) be possible to completely state the underlying rules of physics with a finite number of well-defined laws, but there is little doubt that there are questions about the behavior of physical systems which are formally undecidable on the basis of those underlying laws.

Since most physicists would consider the statement of the underlying rules to suffice as the definition of a "theory of everything", these researchers argue that Gödel's Theorem does *not* mean that a TOE cannot exist. On the other hand, the physicists invoking Gödel's Theorem appear, at least in some cases, to be referring not to the underlying rules, but to the understandability of the behavior of all physical systems, as when Hawking mentions arranging blocks into rectangles, turning the computation of prime numbers into a physical question.^[17] This definitional discrepancy may explain some of the disagreement among researchers.

Another approach to working with the limits of logic implied by Gödel's incompleteness theorems is to abandon the attempt to model reality using a formal system altogether. Process Physics^[18] is a notable example of a candidate TOE that takes this approach, where reality is modeled using self-organizing (purely semantic) information.

Fundamental limits in accuracy

No physical theory to date is believed to be precisely accurate. Instead, physics has proceeded by a series of "successive approximations" allowing more and more accurate predictions over a wider and wider range of phenomena. Some physicists believe that it is therefore a mistake to confuse theoretical models with the true nature of reality, and hold that the series of approximations will never terminate in the "truth". Einstein himself expressed this view on occasions.^[19] On this view, we may reasonably hope for a theory of everything which self-consistently incorporates all currently known forces, but should not expect it to be the final answer.

On the other hand it is often claimed that, despite the apparently ever-increasing complexity of the mathematics of each new theory, in a deep sense associated with their underlying gauge symmetry and the number of fundamental physical constants, the theories are becoming simpler. If so, the process of simplification cannot continue indefinitely.

Lack of fundamental laws

There is a philosophical debate within the physics community as to whether a theory of everything deserves to be called *the* fundamental law of the universe.^[20] One view is the hard reductionist position that the TOE is the fundamental law and that all other theories that apply within the universe are a consequence of the TOE. Another view is that emergent laws (called "free floating laws" by Steven Weinberg), which govern the behavior of complex systems, should be seen as equally fundamental. Examples are the second law of thermodynamics and the theory of natural selection. The point being that, although in our universe these laws describe systems whose behaviour could ("in principle") be predicted from a TOE, they would also hold in universes with different low-level laws, subject only to some very general conditions. Therefore it is of no help, even in principle, to invoke low-level laws when discussing the behavior of complex systems. Some argue that this attitude would violate Occam's Razor if a completely valid TOE were formulated. It is not clear that there is any point at issue in these debates (e.g., between Steven Weinberg and Philip Anderson) other than the right to apply the high-status word "fundamental" to their respective subjects of interest.

Impossibility of being "of everything"

Although the name "theory of everything" suggests the determinism of Laplace's quotation, this gives a very misleading impression. Determinism is frustrated by the probabilistic nature of quantum mechanical predictions, by the extreme sensitivity to initial conditions that leads to mathematical chaos, and by the extreme mathematical difficulty of applying the theory. Thus, although the current standard model of particle physics "in principle" predicts all known non-gravitational phenomena, in practice only a few quantitative results have been derived from the full theory (e.g., the masses of some of the simplest hadrons), and these results (especially the particle masses which are most relevant for low-energy physics) are less accurate than existing experimental measurements. The true TOE would almost certainly be even harder to apply. The main motive for seeking a TOE, apart from the pure intellectual satisfaction of completing a centuries-long quest, is that all prior successful unifications have predicted new phenomena, some of which (e.g., electrical generators) have proved of great practical importance. As in other cases of theory reduction, the TOE would also allow us to confidently define the domain of validity and residual error of low-energy approximations to the full theory which could be used for practical calculations.

Theory of everything and philosophy

The philosophical implication of a physical TOE are frequently debated. For example, if physicalism is true, a physical TOE will coincide with a philosophical theory of everything. Some philosophers (Aristotle, Plato, Hegel, Whitehead, et al.) have attempted to construct all-encompassing systems. Others are highly dubious about the very possibility of such an exercise.

Stephen Hawking wrote in *A Brief History of Time* that even if we had a TOE, it would necessarily be a set of equations. He wrote, "What is it that breathes fire into the equations and makes a universe for them to describe?"^[21]

While on his deathbed, Einstein still explored equations that he imagined to be candidates of a unified theory. Of course, the question would then be "why those equations?" One possible solution might be to adopt the point of view of ultimate ensemble, or modal realism, and say that those equations are not unique. Others doubt that the theory of everything will be in the form of equations at all.

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External links

- The Elegant Universe-Nova online (<http://www.pbs.org/wgbh/nova/elegant/program.html>) — a 3 hour PBS show about the search for the Theory of everything and string theory.
- 'Theory of Everything' (<http://www.vega.org.uk/video/programme/7>) Freeview video by the Vega Science Trust and the BBC/OU.

Quantum Algebra

Quantum algebra

Quantum algebra is one of the top-level mathematics categories used by the arXiv.

Subjects include:

- Quantum groups
- Skein theories
- Operadic algebra
- Diagrammatic algebra
- Quantum field theory

External links

- Quantum algebra at arxiv.org ^[1]

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Lie algebra

In mathematics, a **Lie algebra** (pronounced /'li:/ ("lee"), not /'laɪ/ ("lye")) is an algebraic structure whose main use is in studying geometric objects such as Lie groups and differentiable manifolds. Lie algebras were introduced to study the concept of infinitesimal transformations. The term "Lie algebra" (after Sophus Lie) was introduced by Hermann Weyl in the 1930s. In older texts, the name "**infinitesimal group**" is used.

Definition and first properties

A Lie algebra is a vector space $\mathfrak{g}$ over some field F together with a binary operation $[\cdot, \cdot]$

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

called the **Lie bracket**, which satisfies the following axioms:

- Bilinearity:

$$[ax + by, z] = a[x, z] + b[y, z], \quad [z, ax + by] = a[z, x] + b[z, y]$$

for all scalars a, b in F and all elements x, y, z in $\mathfrak{g}$.

- Alternating on $\mathfrak{g}$:

$$[x, x] = 0$$

for all x in $\mathfrak{g}$. This implies anticommutativity, or skew-symmetry (in fact the conditions are equivalent for any Lie algebra over any field whose characteristic is not 2):

$$[x, y] = -[y, x]$$

for all elements x, y in $\mathfrak{g}$.

- The Jacobi identity:
-

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all x, y, z in $\mathfrak{g}$.

For any associative algebra A with multiplication $*$, one can construct a Lie algebra $L(A)$. As a vector space, $L(A)$ is the same as A . The Lie bracket of two elements of $L(A)$ is defined to be their commutator in A :

$$[a, b] = a * b - b * a.$$

The associativity of the multiplication $*$ in A implies the Jacobi identity of the commutator in $L(A)$. In particular, the associative algebra of $n \times n$ matrices over a field F gives rise to the general linear Lie algebra $\mathfrak{gl}_n(F)$. The associative algebra A is called an **enveloping algebra** of the Lie algebra $L(A)$. It is known that every Lie algebra can be embedded into one that arises from an associative algebra in this fashion. See universal enveloping algebra.

Homomorphisms, subalgebras, and ideals

The Lie bracket is not an associative operation in general, meaning that $[[x, y], z]$ need not equal $[x, [y, z]]$. Nonetheless, much of the terminology that was developed in the theory of associative rings or associative algebras is commonly applied to Lie algebras. A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ that is closed under the Lie bracket is called a **Lie subalgebra**. If a subspace $I \subseteq \mathfrak{g}$ satisfies a stronger condition that

$$[\mathfrak{g}, I] \subseteq I,$$

then I is called an **ideal** in the Lie algebra $\mathfrak{g}$.^[1] A Lie algebra in which the commutator is not identically zero and which has no proper ideals is called **simple**. A **homomorphism** between two Lie algebras (over the same ground field) is a linear map that is compatible with the commutators:

$$f : \mathfrak{g} \rightarrow \mathfrak{g}', \quad f([x, y]) = [f(x), f(y)],$$

for all elements x and y in $\mathfrak{g}$. As in the theory of associative rings, ideals are precisely the kernels of homomorphisms, given a Lie algebra $\mathfrak{g}$ and an ideal I in it, one constructs the **factor algebra** $\mathfrak{g}/I$, and the first isomorphism theorem holds for Lie algebras. Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, their direct sum is the vector space $\mathfrak{g} \oplus \mathfrak{g}'$ consisting of the pairs (x, x') , $x \in \mathfrak{g}$, $x' \in \mathfrak{g}'$, with the operation

$$[(x, x'), (y, y')] = ([x, y], [x', y']), \quad x, y \in \mathfrak{g}, x', y' \in \mathfrak{g}'.$$

Examples

- Any vector space V endowed with the identically zero Lie bracket becomes a Lie algebra. Such Lie algebras are called abelian, cf. below. Any one-dimensional Lie algebra over a field is abelian, by the antisymmetry of the Lie bracket.
- The three-dimensional Euclidean space $\mathbf{R}^3$ with the Lie bracket given by the cross product of vectors becomes a three-dimensional Lie algebra.
- The Heisenberg algebra is a three-dimensional Lie algebra with generators (see also the definition at Generating set):

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

whose commutation relations are

$$[x, y] = z, \quad [x, z] = 0, \quad [y, z] = 0.$$

It is explicitly exhibited as the space of 3×3 strictly upper-triangular matrices.

- The subspace of the general linear Lie algebra $\mathfrak{gl}_n(F)$ consisting of matrices of trace zero is a subalgebra,^[2] the *special linear Lie algebra*, denoted $\mathfrak{sl}_n(F)$.

- Any Lie group G defines an associated real Lie algebra $\mathfrak{g} = \text{Lie}(G)$. The definition in general is somewhat technical, but in the case of real matrix groups, it can be formulated via the exponential map, or the matrix exponent. The Lie algebra $\mathfrak{g}$ consists of those matrices X for which

$$\exp(tX) \in G$$

for all real numbers t . The Lie bracket of $\mathfrak{g}$ is given by the commutator of matrices. As a concrete example, consider the special linear group $\text{SL}(n, \mathbf{R})$, consisting of all $n \times n$ matrices with real entries and determinant 1.

This is a matrix Lie group, and its Lie algebra consists of all $n \times n$ matrices with real entries and trace 0.

- The real vector space of all $n \times n$ skew-hermitian matrices is closed under the commutator and forms a real Lie algebra denoted $\mathfrak{u}(n)$. This is the Lie algebra of the unitary group $U(n)$.
- An important class of infinite-dimensional real Lie algebras arises in differential topology. The space of smooth vector fields on a differentiable manifold M forms a Lie algebra, where the Lie bracket is defined to be the commutator of vector fields. One way of expressing the Lie bracket is through the formalism of Lie derivatives, which identifies a vector field X with a first order partial differential operator L_X acting on smooth functions by letting $L_X(f)$ be the directional derivative of the function f in the direction of X . The Lie bracket $[X, Y]$ of two vector fields is the vector field defined through its action on functions by the formula:

$$L_{[X, Y]}f = L_X(L_Y f) - L_Y(L_X f).$$

This Lie algebra is related to the pseudogroup of diffeomorphisms of M .

- The commutation relations between the x , y , and z components of the angular momentum operator in quantum mechanics form a representation of a complex three-dimensional Lie algebra, which is the complexification of the Lie algebra $\mathfrak{so}(3)$ of the three-dimensional rotation group:

$$[L_x, L_y] = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y$$

- Kac–Moody algebra is an example of an infinite-dimensional Lie algebra.

Structure theory and classification

Every finite-dimensional real or complex Lie algebra has a faithful representation by matrices (Ado's theorem). Lie's fundamental theorems describe a relation between Lie groups and Lie algebras. In particular, any Lie group gives rise to a canonically determined Lie algebra (concretely, the tangent space at the identity), and conversely, for any Lie algebra there is a corresponding connected Lie group (Lie's third theorem). This Lie group is not determined uniquely, however, any two connected Lie groups with the same Lie algebra are *locally isomorphic*, and in particular, have the same universal cover. For instance, the special orthogonal group $\text{SO}(3)$ and the special unitary group $\text{SU}(2)$ give rise to the same Lie algebra, which is isomorphic to $\mathbf{R}^3$ with the cross-product, and $\text{SU}(2)$ is a simply-connected twofold cover of $\text{SO}(3)$. Real and complex Lie algebras can be classified to some extent, and this is often an important step toward the classification of Lie groups.

Abelian, nilpotent, and solvable

Analogously to abelian, nilpotent, and solvable groups, defined in terms of the derived subgroups, one can define abelian, nilpotent, and solvable Lie algebras.

A Lie algebra $\mathfrak{g}$ is **abelian** if the Lie bracket vanishes, i.e. $[x, y] = 0$, for all x and y in $\mathfrak{g}$. Abelian Lie algebras correspond to commutative (or abelian) connected Lie groups such as vector spaces K^n or tori T^n , and are all of the form $\mathfrak{k}^n$, meaning an n -dimensional vector space with the trivial Lie bracket.

A more general class of Lie algebras is defined by the vanishing of all commutators of given length. A Lie algebra $\mathfrak{g}$ is **nilpotent** if the lower central series

$$\mathfrak{g} > [\mathfrak{g}, \mathfrak{g}] > [[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] > [[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}], \mathfrak{g}] > \dots$$

becomes zero eventually. By Engel's theorem, a Lie algebra is nilpotent if and only if for every u in $\mathfrak{g}$ the adjoint endomorphism

$$\text{ad}(u) : \mathfrak{g} \rightarrow \mathfrak{g}, \quad \text{ad}(u)v = [u, v]$$

is nilpotent.

More generally still, a Lie algebra $\mathfrak{g}$ is said to be **solvable** if the derived series:

$$\mathfrak{g} > [\mathfrak{g}, \mathfrak{g}] > [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] > [[[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]], [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]]]] > \dots$$

becomes zero eventually.

Every finite-dimensional Lie algebra has a unique maximal solvable ideal, called its radical. Under the Lie correspondence, nilpotent (respectively, solvable) connected Lie groups correspond to nilpotent (respectively, solvable) Lie algebras.

Simple and semisimple

A Lie algebra is "simple" if it has no non-trivial ideals and is not abelian. A Lie algebra $\mathfrak{g}$ is called **semisimple** if its radical is zero. Equivalently, $\mathfrak{g}$ is semisimple if it does not contain any non-zero abelian ideals. In particular, a simple Lie algebra is semisimple. Conversely, it can be proven that any semisimple Lie algebra is the direct sum of its minimal ideals, which are canonically determined simple Lie algebras.

The concept of semisimplicity for Lie algebras is closely related with the complete reducibility of their representations. When the ground field F has characteristic zero, semisimplicity of a Lie algebra $\mathfrak{g}$ over F is equivalent to the complete reducibility of all finite-dimensional representations of $\mathfrak{g}$. An early proof of this statement proceeded via connection with compact groups (Weyl's unitary trick), but later entirely algebraic proofs were found.

Classification

In many ways, the classes of semisimple and solvable Lie algebras are at the opposite ends of the full spectrum of the Lie algebras. The Levi decomposition expresses an arbitrary Lie algebra as a semidirect sum of its solvable radical and a semisimple Lie algebra, almost in a canonical way. Semisimple Lie algebras over an algebraically closed field have been completely classified through their root systems. The classification of solvable Lie algebras is a 'wild' problem, and cannot be accomplished in general.

Cartan's criterion gives conditions for a Lie algebra to be nilpotent, solvable, or semisimple. It is based on the notion of the Killing form, a symmetric bilinear form on $\mathfrak{g}$ defined by the formula

$$K(u, v) = \text{tr}(\text{ad}(u)\text{ad}(v)),$$

where tr denotes the trace of a linear operator. A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form is nondegenerate. A Lie algebra $\mathfrak{g}$ is solvable if and only if $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

Relation to Lie groups

Although Lie algebras are often studied in their own right, historically they arose as a means to study Lie groups. Given a Lie group, a Lie algebra can be associated to it either by endowing the tangent space to the identity with the differential of the adjoint map, or by considering the left-invariant vector fields as mentioned in the examples. This association is functorial, meaning that homomorphisms of Lie groups lift to homomorphisms of Lie algebras, and various properties are satisfied by this lifting: it commutes with composition, it maps Lie subgroups, kernels, quotients and cokernels of Lie groups to subalgebras, kernels, quotients and cokernels of Lie algebras, respectively.

The functor which takes each Lie group to its Lie algebra and each homomorphism to its differential is a faithful and exact functor. This functor is not invertible; different Lie groups may have the same Lie algebra, for example $\text{SO}(3)$

and $SU(2)$ have isomorphic Lie algebras. Even worse, some Lie algebras need not have *any* associated Lie group. Nevertheless, when the Lie algebra is finite-dimensional, there is always at least one Lie group whose Lie algebra is the one under discussion, and a preferred Lie group can be chosen. Any finite-dimensional connected Lie group has a universal cover. This group can be constructed as the image of the Lie algebra under the exponential map. More generally, we have that the Lie algebra is homeomorphic to a neighborhood of the identity. But globally, if the Lie group is compact, the exponential will not be injective, and if the Lie group is not connected, simply connected or compact, the exponential map need not be surjective.

If the Lie algebra is infinite-dimensional, the issue is more subtle. In many instances, the exponential map is not even locally a homeomorphism (for example, in $\text{Diff}(S^1)$, one may find diffeomorphisms arbitrarily close to the identity which are not in the image of $\exp$). Furthermore, some infinite-dimensional Lie algebras are not the Lie algebra of any group.

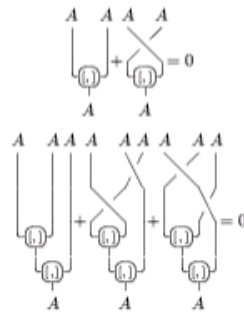
The correspondence between Lie algebras and Lie groups is used in several ways, including in the classification of Lie groups and the related matter of the representation theory of Lie groups. Every representation of a Lie algebra lifts uniquely to a representation of the corresponding connected, simply connected Lie group, and conversely every representation of any Lie group induces a representation of the group's Lie algebra; the representations are in one to one correspondence. Therefore, knowing the representations of a Lie algebra settles the question of representations of the group. As for classification, it can be shown that any connected Lie group with a given Lie algebra is isomorphic to the universal cover mod a discrete central subgroup. So classifying Lie groups becomes simply a matter of counting the discrete subgroups of the center, once the classification of Lie algebras is known (solved by Cartan et al. in the semisimple case).

Category theoretic definition

Using the language of category theory, a **Lie algebra** can be defined as an object A in **Vec**, the category of vector spaces together with a morphism $[\cdot, \cdot]: A \otimes A \rightarrow A$, where $\otimes$ refers to the monoidal product of **Vec**, such that

- $[\cdot, \cdot] \circ (\text{id} + \tau_{A,A}) = 0$
- $[\cdot, \cdot] \circ ([\cdot, \cdot] \otimes \text{id}) \circ (\text{id} + \sigma + \sigma^2) = 0$

where $\tau(a \otimes b) := b \otimes a$ and σ is the cyclic permutation braiding $(\text{id} \otimes \tau_{A,A}) \circ (\tau_{A,A} \otimes \text{id})$. In diagrammatic form:



Notes

- [1] Due to the anticommutativity of the commutator, the notions of a left and right ideal in a Lie algebra coincide.
 [2] Humphreys p.2

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Lie group

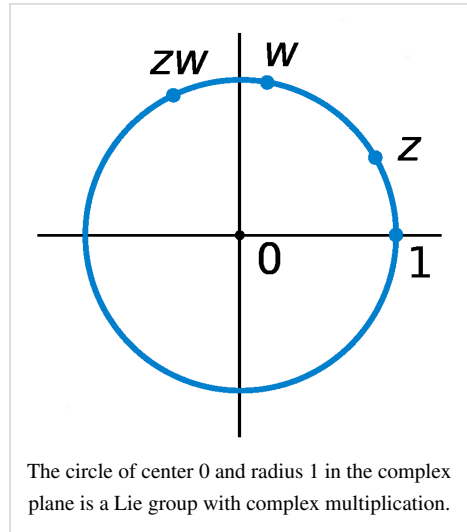
In mathematics, a **Lie group** (pronounced /'liː/: similar to "Lee") is a group which is also a differentiable manifold, with the property that the group operations are compatible with the smooth structure. Lie groups are named after Sophus Lie, who laid the foundations of the theory of continuous transformation groups.

Lie groups represent the best-developed theory of continuous symmetry of mathematical objects and structures, which makes them indispensable tools for many parts of contemporary mathematics, as well as for modern theoretical physics. They provide a natural framework for analysing the continuous symmetries of differential equations (Differential Galois theory), in much the same way as permutation groups are used in Galois theory for analysing the discrete symmetries of algebraic equations. An extension of Galois theory to the case of continuous symmetry groups was one of Lie's principal motivations.

Overview

Lie groups are smooth manifolds and, therefore, can be studied using differential calculus, in contrast with the case of more general topological groups. One of the key ideas in the theory of Lie groups, from Sophus Lie, is to replace the *global* object, the group, with its *local* or linearized version, which Lie himself called its "infinitesimal group" and which has since become known as its Lie algebra.

Lie groups play an enormous role in modern geometry, on several different levels. Felix Klein argued in his Erlangen program that one can consider various "geometries" by specifying an appropriate transformation group that leaves certain geometric properties invariant. Thus Euclidean geometry corresponds to the choice of the group $E(3)$ of distance-preserving transformations of the Euclidean space $\mathbf{R}^3$, conformal geometry corresponds to enlarging the group to the conformal group, whereas in projective geometry one is interested in the properties invariant under the projective group. This idea later led to the notion of a G -structure, where G is a Lie group of "local" symmetries of a manifold. On a "global" level, whenever a Lie group acts on a geometric object, such as a Riemannian or a symplectic manifold, this action provides a measure of rigidity and yields a rich algebraic structure. The presence of continuous symmetries expressed via a Lie group action on a manifold places strong constraints on its geometry and facilitates analysis on the manifold. Linear actions of Lie groups are especially important, and are studied in representation theory.



In the 1940s–1950s, Ellis Kolchin, Armand Borel and Claude Chevalley realised that many foundational results concerning Lie groups can be developed completely algebraically, giving rise to the theory of algebraic groups defined over an arbitrary field. This insight opened new possibilities in pure algebra, by providing a uniform construction for most finite simple groups, as well as in algebraic geometry. The theory of automorphic forms, an important branch of modern number theory, deals extensively with analogues of Lie groups over adèle rings; p -adic Lie groups play an important role, via their connections with Galois representations in number theory.

Definitions and examples

A **real Lie group** is a group which is also a finite-dimensional real smooth manifold, and in which the group operations of multiplication and inversion are smooth maps. Smoothness of the group multiplication

$$\mu : G \times G \rightarrow G \quad \mu(x, y) = xy$$

means that μ is a smooth mapping of the product manifold $G \times G$ into G . These two requirements can be combined to the single requirement that the mapping

$$(x, y) \mapsto x^{-1}y$$

be a smooth mapping of the product manifold into G .

First examples

- The 2×2 real invertible matrices form a group under multiplication, denoted by $GL_2(\mathbf{R})$:

$$GL_2(\mathbf{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det A = ad - bc \neq 0 \right\}.$$

This is a four-dimensional noncompact real Lie group. This group is disconnected; it has two connected components corresponding to the positive and negative values of the determinant.

- The rotation matrices form a subgroup of $GL_2(\mathbf{R})$, denoted by $SO_2(\mathbf{R})$. It is a Lie group in its own right: specifically, a one-dimensional compact connected Lie group which is diffeomorphic to the circle. Using the rotation angle φ as a parameter, this group can be parametrized as follows:

$$SO_2(\mathbf{R}) = \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} : \varphi \in \mathbf{R}/2\pi\mathbf{Z} \right\}.$$

Addition of the angles corresponds to multiplication of the elements of $SO_2(\mathbf{R})$, and taking the opposite angle corresponds to inversion. Thus both multiplication and inversion are differentiable maps.

- The orthogonal group also forms an interesting example of a Lie group.

All of the previous examples of Lie groups fall within the class of classical groups

Related concepts

A **complex Lie group** is defined in the same way using complex manifolds rather than real ones (example: $SL_2(\mathbf{C})$), and similarly one can define a **p -adic Lie group** over the p -adic numbers. Hilbert's fifth problem asked whether replacing differentiable manifolds with topological or analytic ones can yield new examples. The answer to this question turned out to be negative: in 1952, Gleason, Montgomery and Zippin showed that if G is a topological manifold with continuous group operations, then there exists exactly one analytic structure on G which turns it into a Lie group (see also Hilbert–Smith conjecture). If the underlying manifold is allowed to be infinite dimensional (for example, a Hilbert manifold) then one arrives at the notion of an infinite-dimensional Lie group. It is possible to define analogues of many Lie groups over finite fields, and these give most of the examples of finite simple groups.

The language of category theory provides a concise definition for Lie groups: a Lie group is a group object in the category of smooth manifolds. This is important, because it allows generalization of the notion of a Lie group to Lie supergroups.

More examples of Lie groups

Lie groups occur in abundance throughout mathematics and physics. Matrix groups or algebraic groups are (roughly) groups of matrices (for example, orthogonal and symplectic groups), and these give most of the more common examples of Lie groups.

Examples

- Euclidean space $\mathbf{R}^n$ with ordinary vector addition as the group operation becomes an n -dimensional noncompact abelian Lie group.
- The circle group $\mathbf{S}^1$ consisting of angles mod 2π under addition or, alternately, the complex numbers with absolute value 1 under multiplication. This is a one-dimensional compact connected abelian Lie group.
- The group $GL_n(\mathbf{R})$ of invertible matrices (under matrix multiplication) is a Lie group of dimension n^2 , called the general linear group. It has a closed connected subgroup $SL_n(\mathbf{R})$, the special linear group, consisting of matrices of determinant 1 which is also a Lie group.
- The orthogonal group $O_n(\mathbf{R})$, consisting of all $n \times n$ orthogonal matrices with real entries is an $n(n-1)/2$ -dimensional Lie group. This group is disconnected, but it has a connected subgroup $SO_n(\mathbf{R})$ of the same

dimension consisting of orthogonal matrices of determinant 1, called the special orthogonal group (for $n = 3$, the rotation group).

- The Euclidean group $E_n(\mathbf{R})$ is the Lie group of all Euclidean motions, i.e., isometric affine maps, of n -dimensional Euclidean space $\mathbf{R}^n$.
- The unitary group $U(n)$ consisting of $n \times n$ unitary matrices (with complex entries) is a compact connected Lie group of dimension n^2 . Unitary matrices of determinant 1 form a closed connected subgroup of dimension $n^2 - 1$ denoted $SU(n)$, the special unitary group.
- Spin groups are double covers of the special orthogonal groups, used for studying fermions in quantum field theory (among other things).
- The symplectic group $Sp_{2n}(\mathbf{R})$ consists of all $2n \times 2n$ matrices preserving a nondegenerate skew-symmetric bilinear form on $\mathbf{R}^{2n}$ (the *symplectic form*). It is a connected Lie group of dimension $2n^2 + n$. The fundamental group of the symplectic group is $\mathbf{Z}$ and this fact is related to the theory of Maslov index.
- The 3-sphere S^3 forms a Lie group by identification with the set of quaternions of unit norm, called versors. The only other spheres that admit the structure of a Lie group are the 0-sphere S^0 (real numbers with absolute value 1) and the circle S^1 (complex numbers with absolute value 1). For example, for even $n > 1$, S^n is not a Lie group because it does not admit a nonvanishing vector field and so *a fortiori* cannot be parallelizable as a differentiable manifold. Of the spheres only S^0 , S^1 , S^3 , and S^7 are parallelizable. The latter carries the structure of a Lie quasigroup (a nonassociative group), which can be identified with the set of unit octonions.
- The group of upper triangular n by n matrices is a solvable Lie group of dimension $n(n+1)/2$.
- The Lorentz group and the Poincare group are the groups of linear and affine isometries of the Minkowski space (interpreted as the spacetime of the special relativity). They are Lie groups of dimensions 6 and 10.
- The Heisenberg group is a connected nilpotent Lie group of dimension 3, playing a key role in quantum mechanics.
- The group $U(1) \times SU(2) \times SU(3)$ is a Lie group of dimension $1+3+8=12$ that is the gauge group of the Standard Model in particle physics. The dimensions of the factors correspond to the 1 photon + 3 vector bosons + 8 gluons of the standard model.
- The (3-dimensional) metaplectic group is a double cover of $SL_2(\mathbf{R})$ playing an important role in the theory of modular forms. It is a connected Lie group that cannot be faithfully represented by matrices of finite size, i.e., a nonlinear group.
- The exceptional Lie groups of types G_2, F_4, E_6, E_7, E_8 have dimensions 14, 52, 78, 133, and 248. There is also a group $E_{7\frac{1}{2}}$ of dimension 190.

Constructions

There are several standard ways to form new Lie groups from old ones:

- The product of two Lie groups is a Lie group.
- Any topologically closed subgroup of a Lie group is a Lie group. This is known as Cartan's theorem.
- The quotient of a Lie group by a closed normal subgroup is a Lie group.
- The universal cover of a connected Lie group is a Lie group. For example, the group $\mathbf{R}$ is the universal cover of the circle group S^1 . In fact any covering of a differentiable manifold is also a differentiable manifold, but by specifying *universal* cover, one guarantees a group structure (compatible with its other structures).

Related notions

Some examples of groups that are *not* Lie groups (except in the trivial sense that any group can be viewed as a 0-dimensional Lie group, with the discrete topology), are:

- Infinite dimensional groups, such as the additive group of an infinite dimensional real vector space. These are not Lie groups as they are not *finite dimensional* manifolds
- Some totally disconnected groups, such as the Galois group of an infinite extension of fields, or the additive group of the p -adic numbers. These are not Lie groups because their underlying spaces are not real manifolds. (Some of these groups are " p -adic Lie groups"). In general, only topological groups having similar local properties to $\mathbf{R}^n$ for some positive integer n can be Lie groups (of course they must also have a differentiable structure)

Early history

According to the most authoritative source on the early history of Lie groups (Hawkins, p. 1), Sophus Lie himself considered the winter of 1873–1874 as the birth date of his theory of continuous groups. Hawkins, however, suggests that it was "Lie's prodigious research activity during the four-year period from the fall of 1869 to the fall of 1873" that led to the theory's creation (*ibid*). Some of Lie's early ideas were developed in close collaboration with Felix Klein. Lie met with Klein every day from October 1869 through 1872: in Berlin from the end of October 1869 to the end of February 1870, and in Paris, Göttingen and Erlangen in the subsequent two years (*ibid*, p. 2). Lie stated that all of the principal results were obtained by 1884. But during the 1870s all his papers (except the very first note) were published in Norwegian journals, which impeded recognition of the work throughout the rest of Europe (*ibid*, p. 76). In 1884 a young German mathematician, Friedrich Engel, came to work with Lie on a systematic treatise to expose his theory of continuous groups. From this effort resulted the three-volume *Theorie der Transformationsgruppen*, published in 1888, 1890, and 1893.

Lie's ideas did not stand in isolation from the rest of mathematics. In fact, his interest in the geometry of differential equations was first motivated by the work of Carl Gustav Jacobi, on the theory of partial differential equations of first order and on the equations of classical mechanics. Much of Jacobi's work was published posthumously in the 1860s, generating enormous interest in France and Germany (Hawkins, p. 43). Lie's *idée fixe* was to develop a theory of symmetries of differential equations that would accomplish for them what Évariste Galois had done for algebraic equations: namely, to classify them in terms of group theory. Lie and other mathematicians showed that the most important equations for special functions and orthogonal polynomials tend to arise from group theoretical symmetries. Additional impetus to consider continuous groups came from ideas of Bernhard Riemann, on the foundations of geometry, and their further development in the hands of Klein. Thus three major themes in 19th century mathematics were combined by Lie in creating his new theory: the idea of symmetry, as exemplified by Galois through the algebraic notion of a group; geometric theory and the explicit solutions of differential equations of mechanics, worked out by Poisson and Jacobi; and the new understanding of geometry that emerged in the works of Plücker, Möbius, Grassmann and others, and culminated in Riemann's revolutionary vision of the subject.

Although today Sophus Lie is rightfully recognized as the creator of the theory of continuous groups, a major stride in the development of their structure theory, which was to have a profound influence on subsequent development of mathematics, was made by Wilhelm Killing, who in 1888 published the first paper in a series entitled *Die Zusammensetzung der stetigen endlichen Transformationsgruppen* (*The composition of continuous finite transformation groups*) (Hawkins, p. 100). The work of Killing, later refined and generalized by Élie Cartan, led to classification of semisimple Lie algebras, Cartan's theory of symmetric spaces, and Hermann Weyl's description of representations of compact and semisimple Lie groups using highest weights.

Weyl brought the early period of the development of the theory of Lie groups to fruition, for not only did he classify irreducible representations of semisimple Lie groups and connect the theory of groups with quantum mechanics, but he also put Lie's theory itself on firmer footing by clearly enunciating the distinction between Lie's *infinitesimal*

groups (i.e., Lie algebras) and the Lie groups proper, and began investigations of topology of Lie groups (Borel (2001),). The theory of Lie groups was systematically reworked in modern mathematical language in a monograph by Claude Chevalley.

The concept of a Lie group, and possibilities of classification

Lie groups may be thought of as smoothly varying families of symmetries. Examples of symmetries include rotation about an axis. What must be understood is the nature of 'small' transformations, e.g., rotations through tiny angles, that link nearby transformations. The mathematical object capturing this structure is called a Lie algebra (Lie himself called them "infinitesimal groups"). It can be defined because Lie groups are manifolds, so have tangent spaces at each point.

The Lie algebra of any compact Lie group (very roughly: one for which the symmetries form a bounded set) can be decomposed as a direct sum of an abelian Lie algebra and some number of simple ones. The structure of an abelian Lie algebra is mathematically uninteresting (since the Lie bracket is identically zero); the interest is in the simple summands. Hence the question arises: what are the simple Lie algebras of compact groups? It turns out that they mostly fall into four infinite families, the "classical Lie algebras" A_n , B_n , C_n and D_n , which have simple descriptions in terms of symmetries of Euclidean space. But there are also just five "exceptional Lie algebras" that do not fall into any of these families. E_8 is the largest of these.

Properties

- The diffeomorphism group of a Lie group acts transitively on the Lie group
- Every Lie group is parallelizable, and hence an orientable manifold (there is a bundle isomorphism between its tangent bundle and the product of itself with the tangent space at the identity)

Types of Lie groups and structure theory

Lie groups are classified according to their algebraic properties (simple, semisimple, solvable, nilpotent, abelian), their connectedness (connected or simply connected) and their compactness.

- Compact Lie groups are all known: they are finite central quotients of a product of copies of the circle group S^1 and simple compact Lie groups (which correspond to connected Dynkin diagrams).
- Any simply connected solvable Lie group is isomorphic to a closed subgroup of the group of invertible upper triangular matrices of some rank, and any finite dimensional irreducible representation of such a group is 1 dimensional. Solvable groups are too messy to classify except in a few small dimensions.
- Any simply connected nilpotent Lie group is isomorphic to a closed subgroup of the group of invertible upper triangular matrices with 1's on the diagonal of some rank, and any finite dimensional irreducible representation of such a group is 1 dimensional. Like solvable groups, nilpotent groups are too messy to classify except in a few small dimensions.
- Simple Lie groups are sometimes defined to be those that are simple as abstract groups, and sometimes defined to be connected Lie groups with a simple Lie algebra. For example, $SL_2(\mathbf{R})$ is simple according to the second definition but not according to the first. They have all been classified (for either definition).
- Semisimple Lie groups are Lie groups whose Lie algebra is a product of simple Lie algebras.^[1] They are central extensions of products of simple Lie groups.

The identity component of any Lie group is an open normal subgroup, and the quotient group is a discrete group. The universal cover of any connected Lie group is a simply connected Lie group, and conversely any connected Lie group is a quotient of a simply connected Lie group by a discrete normal subgroup of the center. Any Lie group G can be decomposed into discrete, simple, and abelian groups in a canonical way as follows. Write

G_{con} for the connected component of the identity

G_{sol} for the largest connected normal solvable subgroup

G_{nil} for the largest connected normal nilpotent subgroup

so that we have a sequence of normal subgroups

$$1 \subseteq G_{\text{nil}} \subseteq G_{\text{sol}} \subseteq G_{\text{con}} \subseteq G.$$

Then

G/G_{con} is discrete

$G_{\text{con}}/G_{\text{sol}}$ is a central extension of a product of simple connected Lie groups.

$G_{\text{sol}}/G_{\text{nil}}$ is abelian. A connected abelian Lie group is isomorphic to a product of copies of $\mathbf{R}$ and the circle group S^1 .

$G_{\text{nil}}/1$ is nilpotent, and therefore its ascending central series has all quotients abelian.

This can be used to reduce some problems about Lie groups (such as finding their unitary representations) to the same problems for connected simple groups and nilpotent and solvable subgroups of smaller dimension.

The Lie algebra associated with a Lie group

To every Lie group, we can associate a Lie algebra, whose underlying vector space is the tangent space of G at the identity element, which completely captures the local structure of the group. Informally we can think of elements of the Lie algebra as elements of the group that are "infinitesimally close" to the identity, and the Lie bracket is something to do with the commutator of two such infinitesimal elements. Before giving the abstract definition we give a few examples:

- The Lie algebra of the vector space $\mathbf{R}^n$ is just $\mathbf{R}^n$ with the Lie bracket given by

$$[A, B] = 0.$$

(In general the Lie bracket of a connected Lie group is always 0 if and only if the Lie group is abelian.)

- The Lie algebra of the general linear group $GL_n(\mathbf{R})$ of invertible matrices is the vector space $M_n(\mathbf{R})$ of square matrices with the Lie bracket given by

$$[A, B] = AB - BA.$$

If G is a closed subgroup of $GL_n(\mathbf{R})$ then the Lie algebra of G can be thought of informally as the matrices m of $M_n(\mathbf{R})$ such that $1 + \varepsilon m$ is in G , where ε is an infinitesimal positive number with $\varepsilon^2 = 0$ (of course, no such real number ε exists). For example, the orthogonal group $O_n(\mathbf{R})$ consists of matrices A with $AA^T = 1$, so the Lie algebra consists of the matrices m with $(1 + \varepsilon m)(1 + \varepsilon m)^T = 1$, which is equivalent to $m + m^T = 0$ because $\varepsilon^2 = 0$.

- Formally, when working over the reals, as here, this is accomplished by considering the limit as $\varepsilon \rightarrow 0$; but the "infinitesimal" language generalizes directly to Lie groups over general rings.

The concrete definition given above is easy to work with, but has some minor problems: to use it we first need to represent a Lie group as a group of matrices, but not all Lie groups can be represented in this way, and it is not obvious that the Lie algebra is independent of the representation we use. To get round these problems we give the general definition of the Lie algebra of any Lie group (in 4 steps):

1. Vector fields on any smooth manifold M can be thought of as derivations X of the ring of smooth functions on the manifold, and therefore form a Lie algebra under the Lie bracket $[X, Y] = XY - YX$, because the Lie bracket of any two derivations is a derivation.
2. If G is any group acting smoothly on the manifold M , then it acts on the vector fields, and the vector space of vector fields fixed by the group is closed under the Lie bracket and therefore also forms a Lie algebra.
3. We apply this construction to the case when the manifold M is the underlying space of a Lie group G , with G acting on $G = M$ by left translations $L_g(h) = gh$. This shows that the space of left invariant vector fields (vector fields satisfying $L_{g*}X_h = X_{gh}$ for every h in G , where L_{g*} denotes the differential of L_g) on a Lie group is a Lie

algebra under the Lie bracket of vector fields.

4. Any tangent vector at the identity of a Lie group can be extended to a left invariant vector field by left translating the tangent vector to other points of the manifold. Specifically, the left invariant extension of an element v of the tangent space at the identity is the vector field defined by $v^\wedge_g = L_{g*}v$. This identifies the tangent space T_e at the identity with the space of left invariant vector fields, and therefore makes the tangent space at the identity into a Lie algebra, called the Lie algebra of G , usually denoted by a Fraktur $\mathfrak{g}$. Thus the Lie bracket on $\mathfrak{g}$ is given explicitly by $[v, w] = [v^\wedge, w^\wedge]_e$.

This Lie algebra $\mathfrak{g}$ is finite-dimensional and it has the same dimension as the manifold G . The Lie algebra of G determines G up to "local isomorphism", where two Lie groups are called **locally isomorphic** if they look the same near the identity element. Problems about Lie groups are often solved by first solving the corresponding problem for the Lie algebras, and the result for groups then usually follows easily. For example, simple Lie groups are usually classified by first classifying the corresponding Lie algebras.

We could also define a Lie algebra structure on T_e using right invariant vector fields instead of left invariant vector fields. This leads to the same Lie algebra, because the inverse map on G can be used to identify left invariant vector fields with right invariant vector fields, and acts as -1 on the tangent space T_e .

The Lie algebra structure on T_e can also be described as follows: the commutator operation

$$(x, y) \rightarrow xyx^{-1}y^{-1}$$

on $G \times G$ sends (e, e) to e , so its derivative yields a bilinear operation on $T_e G$. This bilinear operation is actually the zero map, but the second derivative, under the proper identification of tangent spaces, yields an operation that satisfies the axioms of a Lie bracket, and it is equal to twice the one defined through left-invariant vector fields.

Homomorphisms and isomorphisms

If G and H are Lie groups, then a Lie-group homomorphism $f: G \rightarrow H$ is a smooth group homomorphism. (It is equivalent to require only that f be continuous rather than smooth.) The composition of two such homomorphisms is again a homomorphism, and the class of all Lie groups, together with these morphisms, forms a category. Two Lie groups are called *isomorphic* if there exists a bijective homomorphism between them whose inverse is also a homomorphism. Isomorphic Lie groups are essentially the same; they only differ in the notation for their elements.

Every homomorphism $f: G \rightarrow H$ of Lie groups induces a homomorphism between the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. The association $G \mapsto \mathfrak{g}$ is a functor (mapping between categories satisfying certain axioms).

One version of Ado's theorem is that every finite dimensional Lie algebra is isomorphic to a matrix Lie algebra. For every finite dimensional matrix Lie algebra, there is a linear group (matrix Lie group) with this algebra as its Lie algebra. So every abstract Lie algebra is the Lie algebra of some (linear) Lie group.

The *global structure* of a Lie group is not determined by its Lie algebra; for example, if Z is any discrete subgroup of the center of G then G and G/Z have the same Lie algebra (see the table of Lie groups for examples). A *connected* Lie group is simple, semisimple, solvable, nilpotent, or abelian if and only if its Lie algebra has the corresponding property.

If we require that the Lie group be simply connected, then the global structure is determined by its Lie algebra: for every finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbf{F}$ there is a simply connected Lie group G with $\mathfrak{g}$ as Lie algebra, unique up to isomorphism. Moreover every homomorphism between Lie algebras lifts to a unique homomorphism between the corresponding simply connected Lie groups.

The exponential map

The exponential map from the Lie algebra $M_n(\mathbf{R})$ of the general linear group $GL_n(\mathbf{R})$ to $GL_n(\mathbf{R})$ is defined by the usual power series:

$$\exp(A) = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

for matrices A . If G is any subgroup of $GL_n(\mathbf{R})$, then the exponential map takes the Lie algebra of G into G , so we have an exponential map for all matrix groups.

The definition above is easy to use, but it is not defined for Lie groups that are not matrix groups, and it is not clear that the exponential map of a Lie group does not depend on its representation as a matrix group. We can solve both problems using a more abstract definition of the exponential map that works for all Lie groups, as follows.

Every vector v in $\mathfrak{g}$ determines a linear map from $\mathbf{R}$ to $\mathfrak{g}$ taking 1 to v , which can be thought of as a Lie algebra homomorphism. Because $\mathbf{R}$ is the Lie algebra of the simply connected Lie group $\mathbf{R}$, this induces a Lie group homomorphism $c : \mathbf{R} \rightarrow G$ so that

$$c(s+t) = c(s)c(t)$$

for all s and t . The operation on the right hand side is the group multiplication in G . The formal similarity of this formula with the one valid for the exponential function justifies the definition

$$\exp(v) = c(1).$$

This is called the *exponential map*, and it maps the Lie algebra $\mathfrak{g}$ into the Lie group G . It provides a diffeomorphism between a neighborhood of 0 in $\mathfrak{g}$ and a neighborhood of e in G . This exponential map is a generalization of the exponential function for real numbers (because $\mathbf{R}$ is the Lie algebra of the Lie group of positive real numbers with multiplication), for complex numbers (because $\mathbf{C}$ is the Lie algebra of the Lie group of non-zero complex numbers with multiplication) and for matrices (because $M_n(\mathbf{R})$ with the regular commutator is the Lie algebra of the Lie group $GL_n(\mathbf{R})$ of all invertible matrices).

Because the exponential map is surjective on some neighbourhood N of e , it is common to call elements of the Lie algebra **infinitesimal generators** of the group G . The subgroup of G generated by N is the identity component of G .

The exponential map and the Lie algebra determine the *local group structure* of every connected Lie group, because of the Baker–Campbell–Hausdorff formula: there exists a neighborhood U of the zero element of $\mathfrak{g}$, such that for u, v in U we have

$$\exp(u)\exp(v) = \exp(u + v + 1/2 [u, v] + 1/12 [[u, v], v] - 1/12 [[u, v], u] - \dots)$$

where the omitted terms are known and involve Lie brackets of four or more elements. In case u and v commute, this formula reduces to the familiar exponential law $\exp(u)\exp(v) = \exp(u + v)$.

The exponential map from the Lie algebra to the Lie group is not always onto, even if the group is connected (though it does map onto the Lie group for connected groups that are either compact or nilpotent). For example, the exponential map of $SL_2(\mathbf{R})$ is not surjective.

Infinite dimensional Lie groups

Lie groups are finite dimensional by definition, but there are many groups that resemble Lie groups, except for being infinite dimensional. There is very little "general theory" of such groups, but some of the examples that have been studied include:

- The group of diffeomorphisms of a manifold. Quite a lot is known about the group of diffeomorphisms of the circle. Its Lie algebra is (more or less) the Witt algebra, which has a central extension called the Virasoro algebra, used in string theory and conformal field theory. Very little is known about the diffeomorphism groups of manifolds of larger dimension. The diffeomorphism group of spacetime sometimes appears in attempts to quantize gravity.

- The group of smooth maps from a manifold to a finite dimensional Lie group is called a gauge group (with operation of pointwise multiplication), and is used in quantum field theory and Donaldson theory. If the manifold is a circle these are called loop groups, and have central extensions whose Lie algebras are (more or less) Kac–Moody algebras.
- There are infinite dimensional analogues of general linear groups, orthogonal groups, and so on. One important aspect is that these may have *simpler* topological properties: see for example Kuiper's theorem.
- Just as calculus in finite-dimensional real vector spaces can be extended to calculus in Banach spaces, the definition of finite-dimensional smooth manifolds can be extended to give a definition of Banach analytic manifolds. Similarly, the standard finite-dimensional definition of Lie groups can be extended to give a definition of Banach analytic Lie groups. In this case, we have a Banach analytic manifold which simultaneously has a group structure such that multiplication and inversion are analytic maps. Some of the theorems of finite-dimensional Lie groups do not carry over to the Banach analytic case, and in particular the relation between Lie groups and Lie algebras is much more subtle in the infinite dimensional case. However, it is true that "for infinite dimensional Lie groups modeled on Banach spaces there is a well-developed theory ... which is closely parallel to the theory of finite dimensional Lie groups."^[2]

Notes

[1] Sigurdur Helgason, "Differential Geometry, Lie Groups, and Symmetric Spaces", Academic Press, 1978, page 131.

[2] Andrew Pressley and Graeme Segal, *Loop Groups*, Oxford Science Publications, 1986, page 26.

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Hopf algebra

In mathematics, a **Hopf algebra**, named after Heinz Hopf, is a structure that is simultaneously a (unital associative) algebra, a (counital coassociative) coalgebra, with these structures compatible making it a bialgebra, and moreover is equipped with an antiautomorphism satisfying a certain property.

Hopf algebras occur naturally in algebraic topology, where they originated and are related to the H-space concept, in group scheme theory, in group theory (via the concept of a group ring), and in numerous other places, making them probably the most familiar type of bialgebra. Hopf algebras are also studied in their own right, with much work on specific classes of examples on the one hand and classification problems on the other.

Formal definition

Formally, a Hopf algebra is a bialgebra H over a field K together with a K -linear map $S: H \rightarrow H$ (called the **antipode**) such that the following diagram commutes:

$$\begin{array}{ccccc}
 & H \otimes H & \xrightarrow{S \otimes \text{id}} & H \otimes H & \\
 \Delta \nearrow & & & & \searrow \nabla \\
 H & \xrightarrow{\epsilon} & K & \xrightarrow{\eta} & H \\
 \Delta \searrow & & & & \nearrow \nabla \\
 & H \otimes H & \xrightarrow{\text{id} \otimes S} & H \otimes H &
 \end{array}$$

Here Δ is the comultiplication of the bialgebra, ∇ its multiplication, η its unit and ϵ its counit. In the sumless Sweedler notation, this property can also be expressed as

$$S(c_{(1)})c_{(2)} = c_{(1)}S(c_{(2)}) = \epsilon(c)1 \quad \text{for all } c \in H.$$

As for algebras, one can replace the underlying field K with a commutative ring R in the above definition.

The definition of Hopf algebra is self-dual (as reflected in the symmetry of the above diagram), so if one can define a dual of H (which is always possible if H is finite-dimensional), then it is automatically a Hopf algebra.

Properties of the antipode

The antipode S is sometimes required to have a K -linear inverse, which is automatic in the finite-dimensional case, or if H is commutative or cocommutative (or more generally quasitriangular).

In general, S is an antihomomorphism,^[1] so S^2 is a homomorphism, which is therefore an automorphism if S was invertible (as may be required).

If $S^2 = \text{id}$, then the Hopf algebra is said to be **involutive** (and the underlying algebra with involution is a $*$ -algebra). If H is finite-dimensional semisimple over a field of characteristic zero, commutative, or cocommutative, then it is involutive.

If a bialgebra B admits an antipode S , then S is unique ("a bialgebra admits at most 1 Hopf algebra structure").^[2]

The antipode is an analog to the inversion map on a group that sends g to g^{-1} .^[3]

Hopf subalgebras

A subalgebra K (not to be confused with the Field K in the notation above) of a Hopf algebra H is a Hopf subalgebra if it is a subcoalgebra of H and the antipode S maps K into K . In other words, a Hopf subalgebra K is a Hopf algebra in its own right when the multiplication, comultiplication, counit and antipode of H is restricted to K (and additionally the identity 1 is required to be in K). The Nichols-Zoeller Freeness theorem established (in 1989) that either natural K -module H is free of finite rank if H is finite dimensional: a generalization of Lagrange's theorem for subgroups. As a corollary of this and integral theory, a Hopf subalgebra of a semisimple finite dimensional Hopf algebra is automatically semisimple.

A Hopf subalgebra K is said to be right normal in a Hopf algebra H if it satisfies the condition of stability, $ad_r(h)(K) \subseteq K$ for all h in H , where the right adjoint mapping ad_r is defined by $ad_r(h)(k) = S(h_{(1)})kh_{(2)}$ for all k in K , h in H . Similarly, a Hopf subalgebra K is left normal in H if it is stable under the left adjoint mapping defined by $ad_\ell(h)(k) = h_{(1)}kS(h_{(2)})$. The two conditions of normality are equivalent if the antipode S is bijective, in which case K is said to be a normal Hopf subalgebra.

A normal Hopf subalgebra K in H satisfies the condition (of equality of subsets of H): $HK^+ = K^+H$ where K^+ denotes the kernel of the counit on K . This normality condition implies that HK^+ is a Hopf ideal of H (i.e. an algebra ideal in the kernel of the counit, a coalgebra coideal and stable under the antipode). As a consequence one has a quotient Hopf algebra H/HK^+ and epimorphism $H \rightarrow H/K^+H$, a theory analogous to that of normal subgroups and quotient groups in group theory.^[4]

Examples

Group algebra. Suppose G is a group. The group algebra KG is a unital associative algebra over K . It turns into a Hopf algebra if we define

- $\Delta : KG \rightarrow KG \otimes KG$ by $\Delta(g) = g \otimes g$ for all g in G
- $\varepsilon : KG \rightarrow K$ by $\varepsilon(g) = 1$ for all g in G
- $S : KG \rightarrow KG$ by $S(g) = g^{-1}$ for all g in G .

Functions on a finite group. Suppose now that G is a *finite* group. Then the set K^G of all functions from G to K with pointwise addition and multiplication is a unital associative algebra over K , and $K^G \otimes K^G$ is naturally isomorphic to $K^{G \times G}$ (for G infinite, $K^G \otimes K^G$ is a proper subset of $K^{G \times G}$). The set K^G becomes a Hopf algebra if we define

- $\Delta : K^G \rightarrow K^{G \times G}$ by $\Delta(f)(x,y) = f(xy)$ for all f in K^G and all x,y in G
- $\varepsilon : K^G \rightarrow K$ by $\varepsilon(f) = f(e)$ for every f in K^G [here e is the identity element of G]
- $S : K^G \rightarrow K^G$ by $S(f)(x) = f(x^{-1})$ for all f in K^G and all x in G .

Note that functions on a finite group can be identified with the group ring, though these are more naturally thought of as dual – the group ring consists of *finite* sums of elements, and thus pairs with functions on the group by evaluating the function on the summed elements.

Regular functions on an algebraic group. Generalizing the previous example, we can use the same formulas to show that for a given algebraic group G over K , the set of all regular functions on G forms a Hopf algebra.

Universal enveloping algebra. Suppose g is a Lie algebra over the field K and U is its universal enveloping algebra. U becomes a Hopf algebra if we define

- $\Delta : U \rightarrow U \otimes U$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for every x in g (this rule is compatible with commutators and can therefore be uniquely extended to all of U).
- $\varepsilon : U \rightarrow K$ by $\varepsilon(x) = 0$ for all x in g (again, extended to U)
- $S : U \rightarrow U$ by $S(x) = -x$ for all x in g .

Cohomology of Lie groups

The cohomology algebra of a Lie group is a Hopf algebra: the multiplication is provided by the cup-product, and the comultiplication

$$H^*(G) \rightarrow H^*(G \times G) \cong H^*(G) \otimes H^*(G)$$

by the group multiplication $G \times G \rightarrow G$. This observation was actually a source of the notion of Hopf algebra. Using this structure, Hopf proved a structure theorem for the cohomology algebra of Lie groups.

Theorem (Hopf)^[5] Let A be a finite-dimensional, graded commutative, graded cocommutative Hopf algebra over a field of characteristic 0. Then A (as an algebra) is a free exterior algebra with generators of odd degree.

Quantum groups and non-commutative geometry

All examples above are either commutative (i.e. the multiplication is commutative) or co-commutative (i.e. $\Delta = T \circ \Delta$ where $T: H \otimes H \rightarrow H \otimes H$ is defined by $T(x \otimes y) = y \otimes x$). Other interesting Hopf algebras are certain "deformations" or "quantizations" of those from example 3 which are neither commutative nor co-commutative. These Hopf algebras are often called *quantum groups*, a term that is so far only loosely defined. They are important in noncommutative geometry, the idea being the following: a standard algebraic group is well described by its standard Hopf algebra of regular functions; we can then think of the deformed version of this Hopf algebra as describing a certain "non-standard" or "quantized" algebraic group (which is not an algebraic group at all). While there does not seem to be a direct way to define or manipulate these non-standard objects, one can still work with their Hopf algebras, and indeed one *identifies* them with their Hopf algebras. Hence the name "quantum group".

Related concepts

Graded Hopf algebras are often used in algebraic topology: they are the natural algebraic structure on the direct sum of all homology or cohomology groups of an H-space.

Locally compact quantum groups generalize Hopf algebras and carry a topology. The algebra of all continuous functions on a Lie group is a locally compact quantum group.

Quasi-Hopf algebras are generalizations of Hopf algebras, where coassociativity only holds up to a twist.

Weak Hopf algebras, or quantum groupoids, are generalizations of Hopf algebras. Like Hopf algebras, weak Hopf algebras form a self-dual class of algebras; i.e., if H is a (weak) Hopf algebra, so is H^* , the dual space of linear forms on H (with respect to the algebra-coalgebra structure obtained from the natural pairing with H and its coalgebra-algebra structure).

A weak Hopf algebra H is usually taken to be a 1) finite dimensional algebra and coalgebra with coproduct $\Delta: H \rightarrow H \otimes H$ and counit $\epsilon: H \rightarrow k$ satisfying all the axioms of Hopf algebra except possibly $\Delta(1) \neq 1 \otimes 1$ or $\epsilon(ab) \neq \epsilon(a)\epsilon(b)$ for some a, b in H . Instead one requires that $(\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (\Delta \otimes \text{Id})\Delta(1)$ and $\epsilon(abc) = \sum \epsilon(ab_{(1)})\epsilon(b_{(2)}c) = \sum \epsilon(ab_{(2)})\epsilon(b_{(1)}c)$ for all a, b , and c in H .

2) H has a weakened antipode $S: H \rightarrow H$ satisfying the axioms (a) - (c): (a) $S(a_{(1)})a_{(2)} = 1_{(1)}\epsilon(a1_{(2)})$ for all a in H (the right-hand side is the interesting projection usually denoted by $\Pi^R(a)$ or $\epsilon_s(a)$ with image a separable subalgebra denoted by H^R or H_s); (b) $a_{(1)}S(a_{(2)}) = \epsilon(1_{(1)}a)1_{(2)}$ for all a in H (another interesting projection usually denoted by $\Pi^L(a)$ or $\epsilon_t(a)$ with image a separable algebra H^L or H_t , anti-isomorphic to H^L via S); and (c) $S(a_{(1)})a_{(2)}S(a_{(3)}) = S(a)$ for all a in H . Note that if $\Delta(1) = 1 \otimes 1$, these conditions reduce to the two usual conditions on the antipode of a Hopf algebra. The axioms are partly chosen so that the category of H -modules is a rigid tensor category. The unit H -module is the separable algebra H^L mentioned above.

For example, a finite groupoid algebra is a weak Hopf algebra. In particular, the groupoid algebra on $[n]$ with one pair of invertible arrows e_{ij} and e_{ji} between i and j in $[n]$ is isomorphic to the algebra H of $n \times n$ matrices. The weak Hopf algebra structure on this particular H is given by coproduct $\Delta(e_{ij}) = e_{ij} \otimes e_{ij}$, counit $\epsilon(e_{ij}) = 1$ and antipode $S(e_{ij}) = e_{ji}$. The separable subalgebras H^L and H^R coincide and are non-central commutative algebras in this particular case (the subalgebra of diagonal matrices).

Early theoretical contributions to weak Hopf algebras are to be found in [6] as well as [7]

Hopf algebroids introduced by J.-H. Lu in 1996 as a result on work on groupoids in Poisson geometry (later shown equivalent in nontrivial way to a construction of Takeuchi from the 1970s and another by Xu around the year 2000): Hopf algebroids generalize weak Hopf algebras and certain skew Hopf algebras. They may be loosely thought of as Hopf algebras over a noncommutative base ring, where weak Hopf algebras become Hopf algebras over a separable algebra. It is a theorem that a Hopf algebroid satisfying a finite projectivity condition over a separable algebra is a weak Hopf algebra, and conversely a weak Hopf algebra H is a Hopf algebroid over its separable subalgebra H^L . The antipode axioms have been changed by G. Böhm and K. Szlachanyi (J. Algebra) in 2004 for tensor categorical reasons and to accommodate examples associated to depth two Frobenius algebra extensions.

A left Hopf algebroid (H, R) is a left bialgebroid together with an antipode: the bialgebroid (H, R) consists of a total algebra H and a base algebra R and two mappings, an algebra homomorphism $s : R \rightarrow H$ called a source map, an algebra anti-homomorphism $t : R \rightarrow H$ called a target map, such that the commutativity condition $s(r_1)t(r_2) = t(r_2)s(r_1)$ is satisfied for all $r_1, r_2 \in R$. The axioms resemble those of a Hopf algebra but are complicated by the possibility that R is a noncommutative algebra or its images under s and t are not in the center of H . In particular a left bialgebroid (H, R) has an R - R -bimodule structure on H which prefers the left side as follows: $r_1 \cdot h \cdot r_2 = s(r_1)t(r_2)h$ for all h in H , $r_1, r_2 \in R$. There is a coproduct $\Delta : H \rightarrow H \otimes_R H$ and counit $\epsilon : H \rightarrow R$ that make (H, R, Δ, ϵ) an R -coring (with axioms like that of a coalgebra such that all mappings are R - R -bimodule homomorphisms and all tensors over R). Additionally the bialgebroid (H, R) must satisfy $\Delta(ab) = \Delta(a)\Delta(b)$ for all a, b in H , and a condition to make sure this last condition makes sense: every image point $\Delta(a)$ satisfies $a_{(1)}t(r) \otimes a_{(2)} = a_{(1)} \otimes a_{(2)}s(r)$ for all r in R . Also $\Delta(1) = 1 \otimes 1$. The counit is required to satisfy $\epsilon(1) = 1$ and the condition $\epsilon(ab) = \epsilon(as(\epsilon(b))) = \epsilon(at(\epsilon(b)))$. The antipode $S : H \rightarrow H$ is usually taken to be an algebra anti-automorphism satisfying conditions of exchanging the source and target maps and satisfying two axioms like Hopf algebra antipode axioms; see the references in Lu or in Böhm-Szlachanyi for a more example-category friendly, though somewhat more complicated, set of axioms for the antipode S . The latter set of axioms depend on the axioms of a right bialgebroid as well, which are a straightforward switching of left to right, s with t , of the axioms for a left bialgebroid given above.

As an example of left bialgebroid, take R to be any algebra over a field k . Let H be its algebra of linear self-mappings. Let $s(r)$ be left multiplication by r on R ; let $t(r)$ be right multiplication by r on R . H is a left bialgebroid over R , which may be seen as follows. From the fact that $H \otimes_R H \cong \text{Hom}_k(R \otimes R, R)$ one may define a coproduct by $\Delta(f)(r \otimes u) = f(ru)$ for each linear transformation f from R to itself and all r, u in R . Coassociativity of the coproduct follows from associativity of the product on R . A counit is given by $\epsilon(f) = f(1)$. The counit axioms of a coring follow from the identity element condition on multiplication in R . The reader will be amused, or at least edified, to check that (H, R) is a left bialgebroid. In case R is an Azumaya algebra, in which case H is isomorphic to $R \otimes R$, an antipode comes from transposing tensors, which makes H a Hopf algebroid over R . Multiplier Hopf algebras introduced by Alfons Van Daele in 1994^[8] are generalizations of Hopf algebras where comultiplication from an algebra (with or without unit) to the multiplier algebra of tensor product algebra of the algebra with itself.

Hopf group-(co)algebras introduced by V.G.Turaev in 2000 are also generalizations of Hopf algebras.

Analogy with groups

Groups can be axiomatized by the same diagrams (equivalently, operations) as a Hopf algebra, where G is taken to be a set instead of a module. In this case:

- the field K is replaced by the 1-point set
- there is a natural counit (map to 1 point)
- there is a natural comultiplication (the diagonal map)
- the unit is the identity element of the group
- the multiplication is the multiplication in the group
- the antipode is the inverse

In this philosophy, a group can be thought of as a Hopf algebra over the "field with one element".^[9]

See also

- Quasitriangular Hopf algebra
- Algebra/set analogy
- Representation theory of Hopf algebras
- Ribbon Hopf algebra
- Superalgebra
- Supergroup
- Anyonic Lie algebra

Notes

- [1] Dăscălescu, Năstăsescu & Raianu (2001), Prop. 4.2.6, p. 153 (<http://books.google.com/books?id=pBJ6sbPHA0IC&pg=PA153&dq=is+an+antimorphism+of+algebras>)
- [2] Dăscălescu, Năstăsescu & Raianu (2001), Remarks 4.2.3, p. 151 (<http://books.google.com/books?id=pBJ6sbPHA0IC&pg=PA151&dq=the+antipode+is+unique>)
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- [9] Group = Hopf algebra « Secret Blogging Seminar (<http://sbseminar.wordpress.com/2007/10/07/group-hopf-algebra/>), Group objects and Hopf algebras (<http://www.youtube.com/watch?v=p3kkm5dYH-w>), video of Simon Willerton.

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Quantum group

In mathematics and theoretical physics, the term **quantum group** denotes various kinds of noncommutative algebra with additional structure. In general, a quantum group is some kind of Hopf algebra. There is no single, all-encompassing definition, but instead a family of broadly similar objects.

The term "quantum group" often denotes a kind of noncommutative algebra with additional structure that first appeared in the theory of quantum integrable systems, and which was then formalized by Vladimir Drinfel'd and Michio Jimbo as a particular class of Hopf algebra. The same term is also used for other Hopf algebras that deform or are close to classical Lie groups or Lie algebras, such as a 'bicrossproduct' class of quantum groups introduced by Shahn Majid a little after the work of Drinfeld and Jimbo.

In Drinfeld's approach, quantum groups arise as Hopf algebras depending on an auxiliary parameter q or h , which become universal enveloping algebras of a certain Lie algebra, frequently semisimple or affine, when $q = 1$ or $h = 0$. Closely related are certain dual objects, also Hopf algebras and also called quantum groups, deforming the algebra of functions on the corresponding semisimple algebraic group or a compact Lie group.

Just as groups often appear as symmetries, quantum groups act on many other mathematical objects and it has become fashionable to introduce the adjective *quantum* in such cases; for example there are quantum planes and quantum Grassmannians.

Intuitive meaning

The discovery of quantum groups was quite unexpected, since it was known for a long time that compact groups and semisimple Lie algebras are "rigid" objects, in other words, they cannot be "deformed". One of the ideas behind quantum groups is that if we consider a structure that is in a sense equivalent but larger, namely a group algebra or a universal enveloping algebra, then a group or enveloping algebra can be "deformed", although the deformation will no longer remain a group or enveloping algebra. More precisely, deformation can be accomplished within the category of Hopf algebras that are not required to be either commutative or cocommutative. One can think of the deformed object as an algebra of functions on a "noncommutative space", in the spirit of the noncommutative geometry of Alain Connes. This intuition, however, came after particular classes of quantum groups had already proved their usefulness in the study of the quantum Yang-Baxter equation and quantum inverse scattering method developed by the Leningrad School (Ludwig Faddeev, Leon Takhtajan, Evgenii Sklyanin, Nicolai Reshetikhin and Korepin) and related work by the Japanese School.^[1] The intuition behind the second, bicrossproduct, class of quantum groups was different and came from the search for self-dual objects as an approach to quantum gravity^[2].

Drinfel'd-Jimbo type quantum groups

One type of objects commonly called a "quantum group" appeared in the work of Vladimir Drinfel'd and Michio Jimbo as a deformation of the universal enveloping algebra of a semisimple Lie algebra or, more generally, a Kac-Moody algebra, in the category of Hopf algebras. The resulting algebra has additional structure, making it into a quasitriangular Hopf algebra.

Let $A = (a_{ij})$ be the Cartan matrix of the Kac-Moody algebra, and let q be a nonzero complex number distinct from 1, then the quantum group, $U_q(G)$, where G is the Lie algebra whose Cartan matrix is A , is defined as the unital associative algebra with generators k_λ (where λ is an element of the weight lattice, *i.e.* $2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \in \mathbb{Z}$ for all i), and e_i and f_i (for simple roots, α_i), subject to the following relations:

- $k_0 = 1$,
- $k_\lambda k_\mu = k_{\lambda+\mu}$,
- $k_\lambda e_i k_\lambda^{-1} = q^{(\lambda, \alpha_i)} e_i$,

- $k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda, \alpha_i)} f_i,$
- $[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$
- $\sum_{n=0}^{1-a_{ij}} (-1)^n \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-n]_{q_i}! [n]_{q_i}!} e_i^n e_j e_i^{1-a_{ij}-n} = 0, \text{ for } i \neq j,$
- $\sum_{n=0}^{1-a_{ij}} (-1)^n \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-n]_{q_i}! [n]_{q_i}!} f_i^n f_j f_i^{1-a_{ij}-n} = 0, \text{ for } i \neq j,$

where $k_i = k_{\alpha_i}$, $q_i = q^{\frac{1}{2}(\alpha_i, \alpha_i)}$, $[0]_{q_i}! = 1$, $[n]_{q_i}! = \prod_{m=1}^n [m]_{q_i}$ for all positive integers n , and $[m]_{q_i} = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$. These are the q -factorial and q -number, respectively, the q -analogs of the ordinary factorial.

The last two relations above are the q -Serre relations, the deformations of the Serre relations.

In the limit as $q \rightarrow 1$, these relations approach the relations for the universal enveloping algebra $U(G)$, where $k_\lambda \rightarrow 1$ and $\frac{k_\lambda - k_{-\lambda}}{q - q^{-1}} \rightarrow t_\lambda$ as $q \rightarrow 1$, where the element, t_λ , of the Cartan subalgebra satisfies

$(t_\lambda, h) = \lambda(h)$ for all h in the Cartan subalgebra.

There are various coassociative coproducts under which these algebras are Hopf algebras, for example,

- $\Delta_1(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta_1(e_i) = 1 \otimes e_i + e_i \otimes k_i$, $\Delta_1(f_i) = k_i^{-1} \otimes f_i + f_i \otimes 1$,
- $\Delta_2(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta_2(e_i) = k_i^{-1} \otimes e_i + e_i \otimes 1$, $\Delta_2(f_i) = 1 \otimes f_i + f_i \otimes k_i$,
- $\Delta_3(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta_3(e_i) = k_i^{-\frac{1}{2}} \otimes e_i + e_i \otimes k_i^{\frac{1}{2}}$, $\Delta_3(f_i) = k_i^{-\frac{1}{2}} \otimes f_i + f_i \otimes k_i^{\frac{1}{2}}$, where the set of generators has been extended, if required, to include k_λ for λ which is expressible as the sum of an element of the weight lattice and half an element of the root lattice.

In addition, any Hopf algebra leads to another with reversed coproduct $T \circ \Delta$, where T is given by $T(x \otimes y) = y \otimes x$, giving three more possible versions.

The counit on $U_q(A)$ is the same for all these coproducts: $\epsilon(k_\lambda) = 1$, $\epsilon(e_i) = 0$, $\epsilon(f_i) = 0$, and the respective antipodes for the above coproducts are given by

- $S_1(k_\lambda) = k_{-\lambda}$, $S_1(e_i) = -e_i k_i^{-1}$, $S_1(f_i) = -k_i f_i$,
- $S_2(k_\lambda) = k_{-\lambda}$, $S_2(e_i) = -k_i e_i$, $S_2(f_i) = -f_i k_i^{-1}$,
- $S_3(k_\lambda) = k_{-\lambda}$, $S_3(e_i) = -q_i e_i$, $S_3(f_i) = -q_i^{-1} f_i$.

Alternatively, the quantum group $U_q(G)$ can be regarded as an algebra over the field $\mathbb{C}(q)$, the field of all rational functions of an indeterminate q over $\mathbb{C}$.

Similarly, the quantum group $U_q(G)$ can be regarded as an algebra over the field $\mathbb{Q}(q)$, the field of all rational functions of an indeterminate q over $\mathbb{Q}$ (see below in the section on quantum groups at $q = 0$).

Representation theory

Just as there are many different types of representations for Kac-Moody algebras and their universal enveloping algebras, so there are many different types of representation for quantum groups.

As is the case for all Hopf algebras, $U_q(G)$ has an adjoint representation on itself as a module, with the action being given by $\text{Ad}_x \cdot y = \sum_{(x)} x_{(1)} y S(x_{(2)})$, where $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$.

Case 1: q is not a root of unity

One important type of representation is a weight representation, and the corresponding module is called a weight module. A weight module is a module with a basis of weight vectors. A weight vector is a nonzero vector v such that $k_\lambda.v = d_\lambda v$ for all λ , where d_λ are complex numbers for all weights λ such that

- $d_0 = 1$,
- $d_\lambda d_\mu = d_{\lambda+\mu}$, for all weights λ and μ .

A weight module is called integrable if the actions of e_i and f_i are locally nilpotent (*i.e.* for any vector v in the module, there exists a positive integer k , possibly dependent on v , such that $e_i^k.v = f_i^k.v = 0$ for all i). In the case of integrable modules, the complex numbers d_λ associated with a weight vector satisfy $d_\lambda = c_\lambda q^{(\lambda, \nu)}$, where ν is an element of the weight lattice, and c_λ are complex numbers such that

- $c_0 = 1$,
- $c_\lambda c_\mu = c_{\lambda+\mu}$, for all weights λ and μ ,
- $c_{2\alpha_i} = 1$ for all i .

Of special interest are highest weight representations, and the corresponding highest weight modules. A highest weight module is a module generated by a weight vector v , subject to $k_\lambda.v = d_\lambda v$ for all weights λ , and $e_i.v = 0$ for all i . Similarly, a quantum group can have a lowest weight representation and lowest weight module, *i.e.* a module generated by a weight vector v , subject to $k_\lambda.v = d_\lambda v$ for all weights λ , and $f_i.v = 0$ for all i .

Define a vector v to have weight ν if $k_\lambda.v = q^{(\lambda, \nu)}v$ for all λ in the weight lattice.

If G is a Kac-Moody algebra, then in any irreducible highest weight representation of $U_q(G)$, with highest weight ν , the multiplicities of the weights are equal to their multiplicities in an irreducible representation of $U(G)$ with equal highest weight. If the highest weight is dominant and integral (a weight μ is dominant and integral if μ satisfies the condition that $2(\mu, \alpha_i)/(\alpha_i, \alpha_i)$ is a non-negative integer for all i), then the weight spectrum of the irreducible representation is invariant under the Weyl group for G , and the representation is integrable.

Conversely, if a highest weight module is integrable, then its highest weight vector v satisfies $k_\lambda.v = c_\lambda q^{(\lambda, \nu)}v$, where c_λ are complex numbers such that

- $c_0 = 1$,
- $c_\lambda c_\mu = c_{\lambda+\mu}$, for all weights λ and μ ,
- $c_{2\alpha_i} = 1$ for all i ,

and ν is dominant and integral.

As is the case for all Hopf algebras, the tensor product of two modules is another module. For an element x of $U_q(G)$, and for vectors v and w in the respective modules, $x.(v \otimes w) = \Delta(x).(v \otimes w)$, so that $k_\lambda.(v \otimes w) = k_\lambda.v \otimes k_\lambda.w$, and in the case of coproduct Δ_1 , $e_i.(v \otimes w) = k_i.v \otimes e_i.w + e_i.v \otimes w$ and $f_i.(v \otimes w) = v \otimes f_i.w + f_i.v \otimes k_i^{-1}.w$.

The integrable highest weight module described above is a tensor product of a one-dimensional module (on which $k_\lambda = c_\lambda$ for all λ , and $e_i = f_i = 0$ for all i) and a highest weight module generated by a nonzero vector v_0 , subject to $k_\lambda.v_0 = q^{(\lambda, \nu)}v_0$ for all weights λ , and $e_i.v_0 = 0$ for all i .

In the specific case where G is a finite-dimensional Lie algebra (as a special case of a Kac-Moody algebra), then the irreducible representations with dominant integral highest weights are also finite-dimensional.

In the case of a tensor product of highest weight modules, its decomposition into submodules is the same as for the tensor product of the corresponding modules of the Kac-Moody algebra (the highest weights are the same, as are their multiplicities).

Quasitriangularity

Case 1: q is not a root of unity

Strictly, the quantum group $U_q(G)$ is not quasitriangular, but it can be thought of as being "nearly quasitriangular" in that there exists an infinite formal sum which plays the role of an R -matrix. This infinite formal sum is expressible in terms of generators e_i and f_i , and Cartan generators t_λ , where k_λ is formally identified with q^{t_λ} . The infinite formal sum is the product of two factors, $q^{\eta \sum_j t_{\lambda_j} \otimes t_{\mu_j}}$, and an infinite formal sum, where $\{\lambda_j\}$ is a basis for the dual space to the Cartan subalgebra, and $\{\mu_j\}$ is the dual basis, and η is a sign (+1 or -1). The formal infinite sum which plays the part of the R -matrix has a well-defined action on the tensor product of two irreducible highest weight modules, and also on the tensor product of two lowest weight modules. Specifically, if v has weight α and w has weight β , then $q^{\eta \sum_j t_{\lambda_j} \otimes t_{\mu_j}}.(v \otimes w) = q^{\eta(\alpha, \beta)} v \otimes w$, and the fact that the modules are both highest weight modules or both lowest weight modules reduces the action of the other factor on $v \otimes w$ to a finite sum.

Specifically, if V is a highest weight module, then the formal infinite sum, R , has a well-defined, and invertible, action on $V \otimes V$, and this value of R (as an element of $\text{Hom}(V) \otimes \text{Hom}(V)$) satisfies the Yang-Baxter equation, and therefore allows us to determine a representation of the braid group, and to define quasi-invariants for knots, links and braids.

Quantum groups at $q = 0$

Masaki Kashiwara has researched the limiting behaviour of quantum groups as $q \rightarrow 0$.

As a consequence of the defining relations for the quantum group $U_q(G)$, $U_q(G)$ can be regarded as a Hopf algebra over $\mathbb{Q}(q)$, the field of all rational functions of an indeterminate q over $\mathbb{Q}$.

For simple root α_i and non-negative integer n , define $e_i^{(n)} = e_i^n / [n]_{q_i}!$ and $f_i^{(n)} = f_i^n / [n]_{q_i}!$ (specifically, $e_i^{(0)} = f_i^{(0)} = 1$). In an integrable module M , and for weight λ , a vector $u \in M_\lambda$ (i.e. a vector u in M with weight λ) can be uniquely decomposed into the sums

$$\bullet \quad u = \sum_{n=0}^{\infty} f_i^{(n)} u_n = \sum_{n=0}^{\infty} e_i^{(n)} v_n,$$

where $u_n \in \ker(e_i) \cap M_{\lambda+n\alpha_i}$, $v_n \in \ker(f_i) \cap M_{\lambda-n\alpha_i}$, $u_n \neq 0$ only if $n + \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \geq 0$, and $v_n \neq 0$ only if $n - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \geq 0$. Linear mappings $\tilde{e}_i : M \rightarrow M$ and $\tilde{f}_i : M \rightarrow M$ can be defined on M_λ by

$$\bullet \quad \tilde{e}_i u = \sum_{n=1}^{\infty} f_i^{(n-1)} u_n = \sum_{n=0}^{\infty} e_i^{(n+1)} v_n,$$

$$\bullet \quad \tilde{f}_i u = \sum_{n=0}^{\infty} f_i^{(n+1)} u_n = \sum_{n=1}^{\infty} e_i^{(n-1)} v_n.$$

Let $\mathcal{A}$ be the integral domain of all rational functions in $\mathbb{Q}(q)$ which are regular at $q = 0$ (i.e. a rational function $f(q)$ is an element of $\mathcal{A}$ if and only if there exist polynomials $g(q)$ and $h(q)$ in the polynomial ring $\mathbb{Q}[q]$ such that $h(0) \neq 0$, and $f(q) = g(q)/h(q)$). A **crystal base** for M is an ordered pair (L, B) , such that

- L is a free $\mathcal{A}$ -submodule of M such that $M = \mathbb{Q}(q) \otimes_{\mathcal{A}} L$;
- B is a $\mathbb{Q}$ -basis of the vector space L/qL over $\mathbb{Q}$,
- $L = \bigoplus_{\lambda} L_{\lambda}$ and $B = \sqcup_{\lambda} B_{\lambda}$, where $L_{\lambda} = L \cap M_{\lambda}$ and $B_{\lambda} = B \cap (L_{\lambda}/qL_{\lambda})$,
- $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for all i ,

- $\tilde{e}_i B \subset B \cup \{0\}$ and $\tilde{f}_i B \subset B \cup \{0\}$ for all i ,
- for all $b \in B$ and $b' \in B$, and for all i , $\tilde{e}_i b = b'$ if and only if $\tilde{f}_i b' = b$.

To put this into a more informal setting, the actions of $e_i f_i$ and $f_i e_i$ are generally singular at $q = 0$ on an integrable module M . The linear mappings $\tilde{e}_i$ and $\tilde{f}_i$ on the module are introduced so that the actions of $\tilde{e}_i \tilde{f}_i$ and $\tilde{f}_i \tilde{e}_i$ are regular at $q = 0$ on the module. There exists a $\mathbb{Q}(q)$ -basis of weight vectors $\tilde{B}$ for M , with respect to which the actions of $\tilde{e}_i$ and $\tilde{f}_i$ are regular at $q = 0$ for all i . The module is then restricted to the free A -module generated by the basis, and the basis vectors, the A -submodule and the actions of $\tilde{e}_i$ and $\tilde{f}_i$ are evaluated at $q = 0$. Furthermore, the basis can be chosen such that at $q = 0$, for all i , $\tilde{e}_i$ and $\tilde{f}_i$ are represented by

mutual transposes, and map basis vectors to basis vectors or 0. A crystal base can be represented by a directed graph with labelled edges. Each vertex of the graph represents an element of the $\mathbb{Q}$ -basis B of L/qL , and a directed edge, labelled by i , and directed from vertex v_1 to vertex v_2 , represents that $b_2 = \tilde{f}_i b_1$ (and, equivalently, that $b_1 = \tilde{e}_i b_2$), where b_1 is the basis element represented by v_1 , and b_2 is the basis element represented by v_2 . The graph completely determines the actions of $\tilde{e}_i$ and $\tilde{f}_i$ at $q = 0$. If an integrable module has a crystal base, then the module is irreducible if and only if the graph representing the crystal base is connected (a graph is called "connected" if the set of vertices cannot be partitioned into the union of nontrivial disjoint subsets V_1 and V_2 such that there are no edges joining any vertex in V_1 to any vertex in V_2).

For any integrable module with a crystal base, the weight spectrum for the crystal base is the same as the weight spectrum for the module, and therefore the weight spectrum for the crystal base is the same as the weight spectrum for the corresponding module of the appropriate Kac-Moody algebra. The multiplicities of the weights in the crystal base are also the same as their multiplicities in the corresponding module of the appropriate Kac-Moody algebra.

It is a theorem of Kashiwara that every integrable highest weight module has a crystal base. Similarly, every integrable lowest weight module has a crystal base.

Tensor products of crystal bases

Let M be an integrable module with crystal base (L, B) and M' be an integrable module with crystal base (L', B') . For crystal bases, the coproduct Δ , given by $\Delta(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta(e_i) = e_i \otimes k_i^{-1} + 1 \otimes e_i$, $\Delta(f_i) = f_i \otimes 1 + k_i \otimes f_i$, is adopted. The integrable module $M \otimes_{\mathbb{Q}(q)} M'$ has crystal base $(L \otimes_A L', B \otimes B')$, where

$B \otimes B' = \{b \otimes_{\mathbb{Q}} b' : b \in B, b' \in B'\}$. For a basis vector $b \in B$, define $\epsilon_i(b) = \max\{n \geq 0 : \tilde{e}_i^n b \neq 0\}$ and $\phi_i(b) = \max\{n \geq 0 : \tilde{f}_i^n b \neq 0\}$. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ on

$$\begin{aligned}
 \bullet \quad \tilde{e}_i(b \otimes b') &= \begin{cases} \tilde{e}_i b \otimes b', & \text{if } \phi_i(b) \geq \epsilon_i(b'), \\ b \otimes \tilde{e}_i b', & \text{if } \phi_i(b) < \epsilon_i(b'), \end{cases} \\
 \bullet \quad \tilde{f}_i(b \otimes b') &= \begin{cases} \tilde{f}_i b \otimes b', & \text{if } \phi_i(b) > \epsilon_i(b'), \\ b \otimes \tilde{f}_i b', & \text{if } \phi_i(b) \leq \epsilon_i(b'). \end{cases}
 \end{aligned}$$

The decomposition of the product two integrable highest weight modules into irreducible submodules is determined by the decomposition of the graph of the crystal base into its connected components (*i.e.* the highest weights of the submodules are determined, and the multiplicity of each highest weight is determined).

Compact matrix quantum groups

See also compact quantum group.

S.L. Woronowicz introduced compact matrix quantum groups. Compact matrix quantum groups are abstract structures on which the "continuous functions" on the structure are given by elements of a C^* -algebra. The geometry of a compact matrix quantum group is a special case of a noncommutative geometry.

The continuous complex-valued functions on a compact Hausdorff topological space form a commutative C^* -algebra. By the Gelfand theorem, a commutative C^* -algebra is isomorphic to the C^* -algebra of continuous complex-valued functions on a compact Hausdorff topological space, and the topological space is uniquely determined by the C^* -algebra up to homeomorphism.

For a compact topological group, G , there exists a C^* -algebra homomorphism $\Delta : C(G) \rightarrow C(G) \otimes C(G)$ (where $C(G) \otimes C(G)$ is the C^* -algebra tensor product - the completion of the algebraic tensor product of $C(G)$ and $C(G)$), such that $\Delta(f)(x, y) = f(xy)$ for all $f \in C(G)$, and for all $x, y \in G$ (where $(f \otimes g)(x, y) = f(x)g(y)$ for all $f, g \in C(G)$ and all $x, y \in G$). There also exists a linear multiplicative mapping $\kappa : C(G) \rightarrow C(G)$, such that $\kappa(f)(x) = f(x^{-1})$ for all $f \in C(G)$ and all $x \in G$. Strictly, this does not make $C(G)$ a Hopf algebra, unless G is finite. On the other hand, a finite-dimensional representation of G can be used to generate a $*$ -subalgebra of $C(G)$ which is also a Hopf $*$ -algebra. Specifically, if $g \mapsto (u_{ij}(g))_{i,j}$ is an n -dimensional representation of G , then $u_{ij} \in C(G)$ for all i, j , and $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all i, j . It follows that the $*$ -algebra generated by u_{ij} for all i, j and $\kappa(u_{ij})$ for all i, j is a Hopf $*$ -algebra: the counit is determined by $\epsilon(u_{ij}) = \delta_{ij}$ for all i, j (where δ_{ij} is the Kronecker delta), the antipode is κ , and the unit is given by $1 = \sum_k u_{1k} \kappa(u_{k1}) = \sum_k \kappa(u_{1k}) u_{k1}$. As a generalization, a compact matrix quantum group is defined as a pair (C, u) , where C is a C^* -algebra and $u = (u_{ij})_{i,j=1,\dots,n}$ is a matrix with entries in C such that

- The $*$ -subalgebra, C_0 , of C , which is generated by the matrix elements of u , is dense in C ;
- There exists a C^* -algebra homomorphism $\Delta : C \rightarrow C \otimes C$ (where $C \otimes C$ is the C^* -algebra tensor product - the completion of the algebraic tensor product of C and C) such that $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all i, j (Δ is called the comultiplication);
- There exists a linear antimultiplicative map $\kappa : C_0 \rightarrow C_0$ (the coinverse) such that $\kappa(\kappa(v^*)^*) = v$ for all $v \in C_0$ and $\sum_k \kappa(u_{ik}) u_{kj} = \sum_k u_{ik} \kappa(u_{kj}) = \delta_{ij} I$, where I is the identity element of C . Since κ is antimultiplicative, then $\kappa(vw) = \kappa(w) \kappa(v)$ for all $v, w \in C_0$.

As a consequence of continuity, the comultiplication on C is coassociative.

In general, C is not a bialgebra, and C_0 is a Hopf $*$ -algebra.

Informally, C can be regarded as the $*$ -algebra of continuous complex-valued functions over the compact matrix quantum group, and u can be regarded as a finite-dimensional representation of the compact matrix quantum group.

A representation of the compact matrix quantum group is given by a corepresentation of the Hopf $*$ -algebra (a corepresentation of a counital coassociative coalgebra A is a square matrix $v = (v_{ij})_{i,j=1,\dots,n}$ with entries in A

(so $v \in M_n(A)$) such that $\Delta(v_{ij}) = \sum_{k=1}^n v_{ik} \otimes v_{kj}$ for all i, j and $\epsilon(v_{ij}) = \delta_{ij}$ for all i, j). Furthermore,

a representation v , is called unitary if the matrix for v is unitary (or equivalently, if $\kappa(v_{ij}) = v_{ji}^*$ for all i, j).

An example of a compact matrix quantum group is $SU_\mu(2)$, where the parameter μ is a positive real number. So

$SU_\mu(2) = (C(SU_\mu(2)), u)$, where $C(SU_\mu(2))$ is the C^* -algebra generated by α and γ , subject to

$$\gamma\gamma^* = \gamma^*\gamma, \quad \alpha\gamma = \mu\gamma\alpha, \quad \alpha\gamma^* = \mu\gamma^*\alpha, \quad \alpha\alpha^* + \mu\gamma^*\gamma = \alpha^*\alpha + \mu^{-1}\gamma^*\gamma = I,$$

and $u = \begin{pmatrix} \alpha & \gamma \\ -\gamma^* & \alpha^* \end{pmatrix}$, so that the comultiplication is determined by $\Delta(\alpha) = \alpha \otimes \alpha - \gamma \otimes \gamma^*$, $\Delta(\gamma) = \alpha \otimes \gamma + \gamma \otimes \alpha^*$, and the coinverse is determined by $\kappa(\alpha) = \alpha^*$, $\kappa(\gamma) = -\mu^{-1}\gamma$, $\kappa(\gamma^*) = -\mu\gamma^*$, $\kappa(\alpha^*) = \alpha$. Note that u is a representation, but not a unitary representation. u is equivalent to the unitary representation $v = \begin{pmatrix} \alpha & \sqrt{\mu}\gamma \\ -\frac{1}{\sqrt{\mu}}\gamma^* & \alpha^* \end{pmatrix}$.

Equivalently, $SU_\mu(2) = (C(SU_\mu(2)), w)$, where $C(SU_\mu(2))$ is the C^* -algebra generated by α and β , subject to

$$\beta\beta^* = \beta^*\beta, \alpha\beta = \mu\beta\alpha, \alpha\beta^* = \mu\beta^*\alpha, \alpha\alpha^* + \mu^2\beta^*\beta = \alpha^*\alpha + \beta^*\beta = I,$$

and $w = \begin{pmatrix} \alpha & \mu\beta \\ -\beta^* & \alpha^* \end{pmatrix}$, so that the comultiplication is determined by $\Delta(\alpha) = \alpha \otimes \alpha - \mu\beta \otimes \beta^*$,

$\Delta(\beta) = \alpha \otimes \beta + \beta \otimes \alpha^*$, and the coinverse is determined by $\kappa(\alpha) = \alpha^*$, $\kappa(\beta) = -\mu^{-1}\beta$, $\kappa(\beta^*) = -\mu\beta^*$, $\kappa(\alpha^*) = \alpha$. Note that w is a unitary representation. The realizations can be identified by equating $\gamma = \sqrt{\mu}\beta$.

When $\mu = 1$, then $SU_\mu(2)$ is equal to the algebra $C(SU(2))$ of functions on the concrete compact group $SU(2)$.

Bicrossproduct quantum groups

Whereas compact matrix pseudogroups are typically versions of Drinfeld-Jimbo quantum groups in a dual function algebra formulation, with additional structure, the bicrossproduct ones are a distinct second family of quantum groups of increasing importance as deformations of solvable rather than semisimple Lie groups. They are associated to Lie splittings of Lie algebras or local factorisations of Lie groups and can be viewed as the cross product or Mackey quantisation of one of the factors acting on the other for the algebra and a similar story for the coproduct Δ with the second factor acting back on the first. The very simplest nontrivial example corresponds to two copies of $\mathbb{R}$ locally acting on each other and results in a quantum group (given here in an algebraic form) with generators p, K, K^{-1} , say, and coproduct

$$[p, K] = \hbar K(K - 1), \Delta p = p \otimes K + 1 \otimes p, \Delta K = K \otimes K$$

where $\hbar$ is the deformation parameter. This quantum group was linked to a toy model of Planck scale physics implementing Born reciprocity when viewed as a deformation of the Heisenberg algebra of quantum mechanics. Also, starting with any compact real form of a semisimple Lie algebra $\mathfrak{g}$ its complexification as a real Lie algebra of twice the dimension splits into $\mathfrak{g}$ and a certain solvable Lie algebra (the Iwasawa decomposition), and this provides a canonical bicrossproduct quantum group associated to $\mathfrak{g}$. For $\mathfrak{su}(2)$ one obtains a quantum group deformation of the Euclidean group $E(3)$ of motions in 3 dimensions.

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Affine quantum group

In mathematics, a **quantum affine algebra** (or **affine quantum group**) is a Hopf algebra that is a q -deformation of the universal enveloping algebra of an affine Lie algebra. They were introduced independently by Drinfeld (1985) and Jimbo (1985) as a special case of their general construction of a quantum group from a Cartan matrix. One of their principal applications has been to the theory of solvable lattice models in quantum statistical mechanics, where the Yang-Baxter equation occurs with a spectral parameter. Combinatorial aspects of the representation theory of quantum affine algebras can be described simply using crystal bases, which correspond to the degenerate case when the deformation parameter q vanishes and the Hamiltonian of the associated lattice model can be explicitly diagonalized.

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Affine Lie algebra

In mathematics, an **affine Lie algebra** is an infinite-dimensional Lie algebra that is constructed in a canonical fashion out of a finite-dimensional simple Lie algebra. It is a Kac–Moody algebra whose generalized Cartan matrix is positive semi-definite and has corank 1. From purely mathematical point of view, affine Lie algebras are interesting because their representation theory, like representation theory of finite dimensional, semisimple Lie algebras is much better understood than that of general Kac–Moody algebras. As observed by Victor Kac, the character formula for representations of affine Lie algebras implies certain combinatorial identities, the **Macdonald identities**.

Affine Lie algebras play an important role in string theory and conformal field theory due to the way they are constructed: starting from a simple Lie algebra $\mathfrak{g}$, one considers the **loop algebra**, $L\mathfrak{g}$, formed by the $\mathfrak{g}$ -valued functions on a circle (interpreted as the closed string) with pointwise commutator. The affine Lie algebra $\hat{\mathfrak{g}}$ is obtained by adding one extra dimension to the loop algebra and modifying a commutator in a non-trivial way, which physicists call a **quantum anomaly**. The point of view of string theory helps to understand many deep properties of affine Lie algebras, such as the fact that the characters of their representations are given by modular forms.

Affine Lie algebras from simple Lie algebras

Construction

If $\mathfrak{g}$ is a finite dimensional simple Lie algebra, the corresponding affine Lie algebra $\hat{\mathfrak{g}}$ is constructed as a central extension of the infinite-dimensional Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, with one-dimensional center $\mathbb{C}c$. As a vector space,

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $\mathbb{C}[t, t^{-1}]$ is the complex vector space of Laurent polynomials in the indeterminate t . The Lie bracket is defined by the formula

$$[a \otimes t^n + \alpha c, b \otimes t^m + \beta c] = [a, b] \otimes t^{n+m} + \langle a|b \rangle n \delta_{m+n,0} c$$

for all $a, b \in \mathfrak{g}$, $\alpha, \beta \in \mathbb{C}$ and $n, m \in \mathbb{Z}$, where $[a, b]$ is the Lie bracket in the Lie algebra $\mathfrak{g}$ and $\langle \cdot | \cdot \rangle$ is the Cartan-Killing form on $\mathfrak{g}$.

The affine Lie algebra corresponding to a finite-dimensional semisimple Lie algebra is the direct sum of the affine Lie algebras corresponding to its simple summands.

Constructing the Dynkin diagrams

The Dynkin diagram of each affine Lie algebra consists of that of the corresponding simple Lie algebra plus an additional node, which corresponds to the addition of an imaginary root. Of course, such a node cannot be attached to the Dynkin diagram in just any location, but for each simple Lie algebra there exists a number of possible attachments equal to the cardinality of the group of outer automorphisms of the Lie algebra. In particular, this group always contains the identity element, and the corresponding affine Lie algebra is called an **untwisted** affine Lie algebra. When the simple algebra admits automorphisms that are not inner automorphisms, one may obtain other Dynkin diagrams and these correspond to twisted affine Lie algebras.

Classifying the central extensions

The attachment of an extra node to the Dynkin diagram of the corresponding simple Lie algebra corresponds to the following construction. An affine Lie algebra can always be constructed as a central extension of the loop algebra of the corresponding simple Lie algebra. If one wishes to begin instead with a semisimple Lie algebra, then one needs to centrally extend by a number of elements equal to the number of simple components of the semisimple algebra. In physics, one often considers instead the direct sum of a semisimple algebra and an abelian algebra $\mathbb{C}^n$. In this case one also needs to add n further central elements for the n abelian generators.

The second integral cohomology of the loop group of the corresponding simple compact Lie group is isomorphic to the integers. Central extensions of the affine Lie group by a single generator are topologically circle bundles over this free loop group, which are classified by a two-class known as the first Chern class of the fibration. Therefore the central extensions of an affine Lie group are classified by a single parameter k which is called the central charge in the physics literature, where it first appeared. Unitary highest weight representations of the affine compact groups only exist when k is a natural number. More generally, if one considers a semi-simple algebra, there is a central charge for each simple component.

Applications

They appear naturally in theoretical physics (for example, in conformal field theories such as the WZW model and coset models and even on the worldsheet of the heterotic string), geometry, and elsewhere in mathematics.

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Quantum affine algebra

In mathematics, a **quantum affine algebra** (or **affine quantum group**) is a Hopf algebra that is a q -deformation of the universal enveloping algebra of an affine Lie algebra. They were introduced independently by Drinfeld (1985) and Jimbo (1985) as a special case of their general construction of a quantum group from a Cartan matrix. One of their principal applications has been to the theory of solvable lattice models in quantum statistical mechanics, where the Yang-Baxter equation occurs with a spectral parameter. Combinatorial aspects of the representation theory of quantum affine algebras can be described simply using crystal bases, which correspond to the degenerate case when the deformation parameter q vanishes and the Hamiltonian of the associated lattice model can be explicitly diagonalized.

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Operator algebra

In functional analysis, an **operator algebra** is an algebra of continuous linear operators on a topological vector space with the multiplication given by the composition of mappings. Although it is usually classified as a branch of functional analysis, it has direct applications to representation theory, differential geometry, quantum statistical mechanics and quantum field theory.

Such algebras can be used to study arbitrary sets of operators with little algebraic relation *simultaneously*. From this point of view, operator algebras can be regarded as a generalization of spectral theory of a single operator. In general operator algebras are non-commutative rings.

An operator algebra is typically required to be closed in a specified operator topology inside the algebra of the whole continuous linear operators. In particular, it is a set of operators with both algebraic and topological closure properties. In some disciplines such properties are axiomized and algebras with certain topological structure become the subject of the research.

Though algebras of operators are studied in various contexts (for example, algebras of pseudo-differential operators acting on spaces of distributions), the term *operator algebra* is usually used in reference to algebras of bounded operators on a Banach space or, even more specially in reference to algebras of operators on a separable Hilbert space, endowed with the operator norm topology.

In the case of operators on a Hilbert space, the adjoint map on operators gives a natural involution which provides an additional algebraic structure which can be imposed on the algebra. In this context, the best studied examples are self-adjoint operator algebras, meaning that they are closed under taking adjoints. These include C^* -algebras and von Neumann algebras. C^* -algebras can be easily characterized abstractly by a condition relating the norm, involution and multiplication. Such abstractly defined C^* -algebras can be identified to a certain closed subalgebra of the algebra of the continuous linear operators on a suitable Hilbert space. A similar result holds for von Neumann algebras.

Commutative self-adjoint operator algebras can be regarded as the algebra of complex valued continuous functions on a locally compact space, or that of measurable functions on a standard measurable space. Thus, general operator algebras are often regarded as a noncommutative generalizations of these algebras, or the structure of the *base space* on which the functions are defined. This point of view is elaborated as the philosophy of noncommutative geometry, which tries to study various non-classical and/or pathological objects by noncommutative operator algebras.

Examples of operator algebras which are not self-adjoint include:

- nest algebras
- many commutative subspace lattice algebras
- many limit algebras

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Clifford algebra

In mathematics, **Clifford algebras** are a type of associative algebra. They can be thought of as one of the possible generalizations of the complex numbers and quaternions. The theory of Clifford algebras is intimately connected with the theory of quadratic forms and orthogonal transformations. Clifford algebras have important applications in a variety of fields including geometry and theoretical physics. They are named after the English geometer William Kingdon Clifford.

Introduction and basic properties

Specifically, a Clifford algebra is a unital associative algebra which contains and is generated by a vector space V equipped with a quadratic form Q . The Clifford algebra $Cl(V, Q)$ is the "freest" algebra generated by V subject to the condition^[1]

$$v^2 = Q(v)1 \text{ for all } v \in V.$$

If the characteristic of the ground field K is not 2, then one can rewrite this fundamental identity in the form

$$uv + vu = 2\langle u, v \rangle \text{ for all } u, v \in V,$$

where $\langle u, v \rangle = \frac{1}{2}(Q(u+v) - Q(u) - Q(v))$ is the symmetric bilinear form associated to Q , via the polarization identity. The idea of being the "freest" or "most general" algebra subject to this identity can be formally expressed through the notion of a universal property, as done below.

Quadratic forms and Clifford algebras in characteristic 2 form an exceptional case. In particular, if $\text{char } K = 2$ it is not true that a quadratic form determines a symmetric bilinear form, or that every quadratic form admits an orthogonal basis. Many of the statements in this article include the condition that the characteristic is not 2, and are false if this condition is removed.

As quantization of exterior algebra

Clifford algebras are closely related to exterior algebras. In fact, if $Q = 0$ then the Clifford algebra $Cl(V, Q)$ is just the exterior algebra $\Lambda(V)$. For nonzero Q there exists a canonical *linear* isomorphism between $\Lambda(V)$ and $Cl(V, Q)$ whenever the ground field K does not have characteristic two. That is, they are naturally isomorphic as vector spaces, but with different multiplications (in the case of characteristic two, they are still isomorphic as vector spaces, just not naturally). Clifford multiplication is strictly richer than the exterior product since it makes use of the extra information provided by Q .

More precisely, Clifford algebras may be thought of as *quantizations* (cf. quantization (physics), Quantum group) of the exterior algebra, in the same way that the Weyl algebra is a quantization of the symmetric algebra.

Weyl algebras and Clifford algebras admit a further structure of a $*$ -algebra, and can be unified as even and odd terms of a superalgebra, as discussed in CCR and CAR algebras.

Universal property and construction

Let V be a vector space over a field K , and let $Q : V \rightarrow K$ be a quadratic form on V . In most cases of interest the field K is either $\mathbf{R}$, $\mathbf{C}$ or a finite field.

A Clifford algebra $Cl(V, Q)$ is a unital associative algebra over K together with a linear map $i : V \rightarrow Cl(V, Q)$ satisfying $i(v)^2 = Q(v)1$ for all $v \in V$, defined by the following universal property: Given any associative algebra A over K and any linear map $j : V \rightarrow A$ such that

$$j(v)^2 = Q(v)1 \text{ for all } v \in V$$

(where 1 denotes the multiplicative identity of A), there is a unique algebra homomorphism $f: Cl(V, Q) \rightarrow A$ such that the following diagram commutes (i.e. such that $f \circ i = j$):

$$\begin{array}{ccc} V & \xrightarrow{i} & Cl(V, Q) \\ & \searrow j & \downarrow f \\ & & A \end{array}$$

Working with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ instead of Q (in characteristic not 2), the requirement on j is

$$j(v)j(w) + j(w)j(v) = 2\langle v, w \rangle \text{ for all } v, w \in V.$$

A Clifford algebra as described above always exists and can be constructed as follows: start with the most general algebra that contains V , namely the tensor algebra $T(V)$, and then enforce the fundamental identity by taking a suitable quotient. In our case we want to take the two-sided ideal I_Q in $T(V)$ generated by all elements of the form

$$v \otimes v - Q(v)1 \text{ for all } v \in V$$

and define $Cl(V, Q)$ as the quotient

$$Cl(V, Q) = T(V)/I_Q.$$

It is then straightforward to show that $Cl(V, Q)$ contains V and satisfies the above universal property, so that Cl is unique up to a unique isomorphism; thus one speaks of "the" Clifford algebra $Cl(V, Q)$. It also follows from this construction that i is injective. One usually drops the i and considers V as a linear subspace of $Cl(V, Q)$.

The universal characterization of the Clifford algebra shows that the construction of $Cl(V, Q)$ is *functorial* in nature. Namely, Cl can be considered as a functor from the category of vector spaces with quadratic forms (whose morphisms are linear maps preserving the quadratic form) to the category of associative algebras. The universal property guarantees that linear maps between vector spaces (preserving the quadratic form) extend uniquely to algebra homomorphisms between the associated Clifford algebras.

Basis and dimension

If the dimension of V is n and $\{e_1, \dots, e_n\}$ is a basis of V , then the set

$$\{e_{i_1} e_{i_2} \cdots e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n \text{ and } 0 \leq k \leq n\}$$

is a basis for $Cl(V, Q)$. The empty product ($k = 0$) is defined as the multiplicative identity element. For each value of k there are $\binom{n}{k}$ basis elements, so the total dimension of the Clifford algebra is

$$\dim Cl(V, Q) = \sum_{k=0}^n \binom{n}{k} = 2^n.$$

Since V comes equipped with a quadratic form, there is a set of privileged bases for V : the orthogonal ones. An orthogonal basis is one such that

$$\langle e_i, e_j \rangle = 0 \quad i \neq j.$$

where $\langle \cdot, \cdot \rangle$ is the symmetric bilinear form associated to Q . The fundamental Clifford identity implies that for an orthogonal basis

$$e_i e_j = -e_j e_i \quad i \neq j.$$

This makes manipulation of orthogonal basis vectors quite simple. Given a product $e_{i_1} e_{i_2} \cdots e_{i_k}$ of *distinct* orthogonal basis vectors, one can put them into standard order by including an overall sign corresponding to the number of flips needed to correctly order them (i.e. the signature of the ordering permutation).

If the characteristic is not 2 then an orthogonal basis for V exists, and one can easily extend the quadratic form on V to a quadratic form on all of $Cl(V, Q)$ by requiring that distinct elements $e_{i_1} e_{i_2} \cdots e_{i_k}$ are orthogonal to one another whenever the $\{e_i\}$'s are orthogonal. Additionally, one sets

$$Q(e_{i_1}e_{i_2}\cdots e_{i_k}) = Q(e_{i_1})Q(e_{i_2})\cdots Q(e_{i_k}).$$

The quadratic form on a scalar is just $Q(\lambda) = \lambda^2$. Thus, orthogonal bases for V extend to orthogonal bases for $Cl(V, Q)$. The quadratic form defined in this way is actually independent of the orthogonal basis chosen (a basis-independent formulation will be given later).

Examples: real and complex Clifford algebras

The most important Clifford algebras are those over real and complex vector spaces equipped with nondegenerate quadratic forms. The geometric interpretation of nondegenerate Clifford algebras is known as geometric algebra.

Every nondegenerate quadratic form on a finite-dimensional real vector space is equivalent to the standard diagonal form:

$$Q(v) = v_1^2 + \cdots + v_p^2 - v_{p+1}^2 - \cdots - v_{p+q}^2$$

where $n = p + q$ is the dimension of the vector space. The pair of integers (p, q) is called the signature of the quadratic form. The real vector space with this quadratic form is often denoted $\mathbf{R}^{p,q}$. The Clifford algebra on $\mathbf{R}^{p,q}$ is denoted $Cl_{p,q}(\mathbf{R})$. The symbol $Cl_n(\mathbf{R})$ means either $Cl_{n,0}(\mathbf{R})$ or $Cl_{0,n}(\mathbf{R})$ depending on whether the author prefers positive definite or negative definite spaces.

A standard orthonormal basis $\{e_i\}$ for $\mathbf{R}^{p,q}$ consists of $n = p + q$ mutually orthogonal vectors, p of which have norm +1 and q of which have norm -1. The algebra $Cl_{p,q}(\mathbf{R})$ will therefore have p vectors which square to +1 and q vectors which square to -1.

Note that $Cl_{0,0}(\mathbf{R})$ is naturally isomorphic to $\mathbf{R}$ since there are no nonzero vectors. $Cl_{0,1}(\mathbf{R})$ is a two-dimensional algebra generated by a single vector e_1 which squares to -1, and therefore is isomorphic to $\mathbf{C}$, the field of complex numbers. The algebra $Cl_{0,2}(\mathbf{R})$ is a four-dimensional algebra spanned by $\{1, e_1, e_2, e_1e_2\}$. The latter three elements square to -1 and all anticommute, and so the algebra is isomorphic to the quaternions $\mathbf{H}$. The next algebra in the sequence is $Cl_{0,3}(\mathbf{R})$ is an 8-dimensional algebra isomorphic to the direct sum $\mathbf{H} \oplus \mathbf{H}$ called split-biquaternions.

One can also study Clifford algebras on complex vector spaces. Every nondegenerate quadratic form on a complex vector space is equivalent to the standard diagonal form

$$Q(z) = z_1^2 + z_2^2 + \cdots + z_n^2$$

where $n = \dim V$, so there is essentially only one Clifford algebra in each dimension. We will denote the Clifford algebra on $\mathbf{C}^n$ with the standard quadratic form by $Cl_n(\mathbf{C})$. One can show that the algebra $Cl_n(\mathbf{C})$ may be obtained as the complexification of the algebra $Cl_{p,q}(\mathbf{R})$ where $n = p + q$:

$$Cl_n(\mathbf{C}) \cong Cl_{p,q}(\mathbf{R}) \otimes \mathbf{C} \cong Cl(\mathbf{C}^{p+q}, Q \otimes \mathbf{C}).$$

Here Q is the real quadratic form of signature (p, q) . Note that the complexification does not depend on the signature. The first few cases are not hard to compute. One finds that

$$Cl_0(\mathbf{C}) = \mathbf{C}$$

$$Cl_1(\mathbf{C}) = \mathbf{C} \oplus \mathbf{C}$$

$$Cl_2(\mathbf{C}) = M_2(\mathbf{C})$$

where $M_2(\mathbf{C})$ denotes the algebra of 2x2 matrices over $\mathbf{C}$.

It turns out that every one of the algebras $Cl_{p,q}(\mathbf{R})$ and $Cl_n(\mathbf{C})$ is isomorphic to a matrix algebra over $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ or to a direct sum of two such algebras. For a complete classification of these algebras see classification of Clifford algebras.

Properties

Relation to the exterior algebra

Given a vector space V one can construct the exterior algebra $\Lambda(V)$, whose definition is independent of any quadratic form on V . It turns out that if F does not have characteristic 2 then there is a natural isomorphism between $\Lambda(V)$ and $Cl(V, Q)$ considered as vector spaces (and there exists an isomorphism in characteristic two, which may not be natural). This is an algebra isomorphism if and only if $Q = 0$. One can thus consider the Clifford algebra $Cl(V, Q)$ as an enrichment (or more precisely, a quantization, cf. the Introduction) of the exterior algebra on V with a multiplication that depends on Q (one can still define the exterior product independent of Q).

The easiest way to establish the isomorphism is to choose an *orthogonal* basis $\{e_i\}$ for V and extend it to an orthogonal basis for $Cl(V, Q)$ as described above. The map $Cl(V, Q) \rightarrow \Lambda(V)$ is determined by

$$e_{i_1} e_{i_2} \cdots e_{i_k} \mapsto e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}.$$

Note that this only works if the basis $\{e_i\}$ is orthogonal. One can show that this map is independent of the choice of orthogonal basis and so gives a natural isomorphism.

If the characteristic of K is 0, one can also establish the isomorphism by antisymmetrizing. Define functions $f_k : V \times \cdots \times V \rightarrow Cl(V, Q)$ by

$$f_k(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(k)}$$

where the sum is taken over the symmetric group on k elements. Since f_k is alternating it induces a unique linear map $\Lambda^k(V) \rightarrow Cl(V, Q)$. The direct sum of these maps gives a linear map between $\Lambda(V)$ and $Cl(V, Q)$. This map can be shown to be a linear isomorphism, and it is natural.

A more sophisticated way to view the relationship is to construct a filtration on $Cl(V, Q)$. Recall that the tensor algebra $T(V)$ has a natural filtration: $F^0 \subset F^1 \subset F^2 \subset \dots$ where F^k contains sums of tensors with rank $\leq k$. Projecting this down to the Clifford algebra gives a filtration on $Cl(V, Q)$. The associated graded algebra

$$Gr_F Cl(V, Q) = \bigoplus_k F^k / F^{k-1}$$

is naturally isomorphic to the exterior algebra $\Lambda(V)$. Since the associated graded algebra of a filtered algebra is always isomorphic to the filtered algebra as filtered vector spaces (by choosing complements of F^k in F^{k+1} for all k), this provides an isomorphism (although not a natural one) in any characteristic, even two.

Grading

In the following, assume that the characteristic is not 2.^[2]

Clifford algebras are $\mathbf{Z}_2$ -graded algebras (also known as superalgebras). Indeed, the linear map on V defined by $v \mapsto -v$ (reflection through the origin) preserves the quadratic form Q and so by the universal property of Clifford algebras extends to an algebra automorphism

$$\alpha : Cl(V, Q) \rightarrow Cl(V, Q).$$

Since α is an involution (i.e. it squares to the identity) one can decompose $Cl(V, Q)$ into positive and negative eigenspaces

$$Cl(V, Q) = Cl^0(V, Q) \oplus Cl^1(V, Q)$$

where $Cl^i(V, Q) = \{x \in Cl(V, Q) \mid \alpha(x) = (-1)^i x\}$. Since α is an automorphism it follows that

$$Cl^i(V, Q) Cl^j(V, Q) = Cl^{i+j}(V, Q)$$

where the superscripts are read modulo 2. This gives $Cl(V, Q)$ the structure of a $\mathbf{Z}_2$ -graded algebra. The subspace $Cl^0(V, Q)$ forms a subalgebra of $Cl(V, Q)$, called the *even subalgebra*. The subspace $Cl^1(V, Q)$ is called the *odd part* of $Cl(V, Q)$ (it is not a subalgebra). The $\mathbf{Z}_2$ -grading plays an important role in the analysis and application of Clifford

algebras. The automorphism α is called the *main involution* or *grade involution*.

Remark. In characteristic not 2 the underlying vector space of $Cl(V, Q)$ inherits a $\mathbf{Z}$ -grading from the canonical isomorphism with the underlying vector space of the exterior algebra $\Lambda(V)$. It is important to note, however, that this is a *vector space grading only*. That is, Clifford multiplication does not respect the $\mathbf{Z}$ -grading, only the $\mathbf{Z}_2$ -grading: for instance if $Q(v) \neq 0$, then $v \in Cl^1(V, Q)$, but $v^2 \in Cl^0(V, Q)$, not in $Cl^2(V, Q)$. Happily, the gradings are related in the natural way: $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$. Further, the Clifford algebra is $\mathbf{Z}$ -filtered: $Cl^{\leq i}(V, Q) \cdot Cl^{\leq j}(V, Q) \subset Cl^{\leq i+j}(V, Q)$. The *degree* of a Clifford number usually refers to the degree in the $\mathbf{Z}$ -grading. Elements which are pure in the $\mathbf{Z}_2$ -grading are simply said to be even or odd. The even subalgebra $Cl^0(V, Q)$ of a Clifford algebra is itself a Clifford algebra^[3]. If V is the orthogonal direct sum of a vector a of norm $Q(a)$ and a subspace U , then $Cl^0(V, Q)$ is isomorphic to $Cl(U, -Q(a)Q)$, where $-Q(a)Q$ is the form Q restricted to U and multiplied by $-Q(a)$. In particular over the reals this implies that

$$Cl_{p,q}^0(\mathbb{R}) \cong Cl_{p,q-1}(\mathbb{R}) \text{ for } q > 0, \text{ and}$$

$$Cl_{p,q}^0(\mathbb{R}) \cong Cl_{q,p-1}(\mathbb{R}) \text{ for } p > 0.$$

In the negative-definite case this gives an inclusion $Cl_{0,n-1}(\mathbb{R}) \subset Cl_{0,n}(\mathbb{R})$ which extends the sequence

$$\mathbf{R} \subset \mathbf{C} \subset \mathbf{H} \subset \mathbf{H} \oplus \mathbf{H} \subset \dots$$

Likewise, in the complex case, one can show that the even subalgebra of $Cl_n(\mathbf{C})$ is isomorphic to $Cl_{n-1}(\mathbf{C})$.

Antiautomorphisms

In addition to the automorphism α , there are two antiautomorphisms which play an important role in the analysis of Clifford algebras. Recall that the tensor algebra $T(V)$ comes with an antiautomorphism that reverses the order in all products:

$$v_1 \otimes v_2 \otimes \dots \otimes v_k \mapsto v_k \otimes \dots \otimes v_2 \otimes v_1.$$

Since the ideal I_Q is invariant under this reversal, this operation descends to an antiautomorphism of $Cl(V, Q)$ called the *transpose* or *reversal* operation, denoted by x^t . The transpose is an antiautomorphism: $(xy)^t = y^t x^t$. The transpose operation makes no use of the $\mathbf{Z}_2$ -grading so we define a second antiautomorphism by composing α and the transpose. We call this operation *Clifford conjugation* denoted $\bar{x}$

$$\bar{x} = \alpha(x^t) = \alpha(x)^t.$$

Of the two antiautomorphisms, the transpose is the more fundamental.^[4]

Note that all of these operations are involutions. One can show that they act as ± 1 on elements which are pure in the $\mathbf{Z}$ -grading. In fact, all three operations depend only on the degree modulo 4. That is, if x is pure with degree k then

$$\alpha(x) = \pm x \quad x^t = \pm x \quad \bar{x} = \pm x$$

where the signs are given by the following table:

$k \bmod 4$	0	1	2	3	
$\alpha(x)$	+	-	+	-	$(-1)^k$
x^t	+	+	-	-	$(-1)^{k(k-1)/2}$
$\bar{x}$	+	-	-	+	$(-1)^{k(k+1)/2}$

The Clifford scalar product

When the characteristic is not 2 the quadratic form Q on V can be extended to a quadratic form on all of $Cl(V, Q)$ as explained earlier (which we also denoted by Q). A basis independent definition is

$$Q(x) = \langle x^t x \rangle$$

where $\langle a \rangle$ denotes the scalar part of a (the grade 0 part in the $\mathbf{Z}$ -grading). One can show that

$$Q(v_1 v_2 \cdots v_k) = Q(v_1) Q(v_2) \cdots Q(v_k)$$

where the v_i are elements of V — this identity is *not* true for arbitrary elements of $Cl(V, Q)$.

The associated symmetric bilinear form on $Cl(V, Q)$ is given by

$$\langle x, y \rangle = \langle x^t y \rangle.$$

One can check that this reduces to the original bilinear form when restricted to V . The bilinear form on all of $Cl(V, Q)$ is nondegenerate if and only if it is nondegenerate on V .

It is not hard to verify that the transpose is the adjoint of left/right Clifford multiplication with respect to this inner product. That is,

$$\begin{aligned} \langle ax, y \rangle &= \langle x, a^t y \rangle, \text{ and} \\ \langle xa, y \rangle &= \langle x, ya^t \rangle. \end{aligned}$$

Structure of Clifford algebras

In this section we assume that the vector space V is finite dimensional and that the bilinear form of Q is non-singular. A central simple algebra over K is a matrix algebra over a (finite dimensional) division algebra with center K . For example, the central simple algebras over the reals are matrix algebras over either the reals or the quaternions.

- If V has even dimension then $Cl(V, Q)$ is a central simple algebra over K .
- If V has even dimension then $Cl^0(V, Q)$ is a central simple algebra over a quadratic extension of K or a sum of two isomorphic central simple algebras over K .
- If V has odd dimension then $Cl(V, Q)$ is a central simple algebra over a quadratic extension of K or a sum of two isomorphic central simple algebras over K .
- If V has odd dimension then $Cl^0(V, Q)$ is a central simple algebra over K .

The structure of Clifford algebras can be worked out explicitly using the following result. Suppose that U has even dimension and a non-singular bilinear form with discriminant d , and suppose that V is another vector space with a quadratic form. The Clifford algebra of $U+V$ is isomorphic to the tensor product of the Clifford algebras of U and $(-1)^{\dim(U)/2}dV$, which is the space V with its quadratic form multiplied by $(-1)^{\dim(U)/2}d$. Over the reals, this implies in particular that

$$\begin{aligned} Cl_{p+2, q}(\mathbb{R}) &= M_2(\mathbb{R}) \otimes Cl_{q, p}(\mathbb{R}) \\ Cl_{p+1, q+1}(\mathbb{R}) &= M_2(\mathbb{R}) \otimes Cl_{p, q}(\mathbb{R}) \\ Cl_{p, q+2}(\mathbb{R}) &= \mathbb{H} \otimes Cl_{q, p}(\mathbb{R}). \end{aligned}$$

These formulas can be used to find the structure of all real Clifford algebras; see the classification of Clifford algebras.

Notably, the Morita equivalence class of a Clifford algebra (its representation theory: the equivalence class of the category of modules over it) depends only on the signature $p - q \bmod 8$. This is an algebraic form of Bott periodicity.

The Clifford group Γ

In this section we assume that V is finite dimensional and the quadratic form Q is nondegenerate.

The invertible elements of the Clifford algebra act on it by twisted conjugation: conjugation by x maps $y \mapsto xy\alpha(x)^{-1}$.

The Clifford group Γ is defined to be the set of invertible elements x that *stabilize vectors*, meaning that

$$xv\alpha(x)^{-1} \in V$$

for all v in V .

This formula also defines an action of the Clifford group on the vector space V that preserves the norm Q , and so gives a homomorphism from the Clifford group to the orthogonal group. The Clifford group contains all elements r of V of nonzero norm, and these act on V by the corresponding reflections that take v to $v - \langle v, r \rangle Q(r)$ (In characteristic 2 these are called orthogonal transvections rather than reflections.)

The Clifford group Γ is the disjoint union of two subsets Γ^0 and Γ^1 , where Γ^i is the subset of elements of degree i . The subset Γ^0 is a subgroup of index 2 in Γ .

If V is a finite dimensional real vector space with positive definite (or negative definite) quadratic form then the Clifford group maps onto the orthogonal group of V with respect to the form (by the Cartan-Dieudonné theorem) and the kernel consists of the nonzero elements of the field K . This leads to exact sequences

$$\begin{aligned} 1 \rightarrow K^* \rightarrow \Gamma \rightarrow O_V(K) \rightarrow 1, \\ 1 \rightarrow K^* \rightarrow \Gamma^0 \rightarrow SO_V(K) \rightarrow 1. \end{aligned}$$

Over other fields or with indefinite forms, the map is not in general onto, and the failure is captured by the spinor norm.

Spinor norm

In arbitrary characteristic, the spinor norm Q is defined on the Clifford group by

$$Q(x) = x^t x.$$

It is a homomorphism from the Clifford group to the group K^* of non-zero elements of K . It coincides with the quadratic form Q of V when V is identified with a subspace of the Clifford algebra. Several authors define the spinor norm slightly differently, so that it differs from the one here by a factor of -1 , 2 , or -2 on Γ^1 . The difference is not very important in characteristic other than 2.

The nonzero elements of K have spinor norm in the group K^{*2} of squares of nonzero elements of the field K . So when V is finite dimensional and non-singular we get an induced map from the orthogonal group of V to the group K^*/K^{*2} , also called the spinor norm. The spinor norm of the reflection of a vector r has image $Q(r)$ in K^*/K^{*2} , and this property uniquely defines it on the orthogonal group. This gives exact sequences:

$$\begin{aligned} 1 \rightarrow \{\pm 1\} \rightarrow \text{Pin}_V(K) \rightarrow O_V(K) \rightarrow K^*/K^{*2}, \\ 1 \rightarrow \{\pm 1\} \rightarrow \text{Spin}_V(K) \rightarrow SO_V(K) \rightarrow K^*/K^{*2}. \end{aligned}$$

Note that in characteristic 2 the group $\{\pm 1\}$ has just one element.

From the point of view of Galois cohomology of algebraic groups, the spinor norm is a connecting homomorphism on cohomology. Writing μ_2 for the algebraic group of square roots of 1 (over a field of characteristic not 2 it is roughly the same as a two-element group with trivial Galois action), the short exact sequence

$$1 \rightarrow \mu_2 \rightarrow \text{Pin}_V \rightarrow O_V \rightarrow 1$$

yields a long exact sequence on cohomology, which begins

$$1 \rightarrow H^0(\mu_2; K) \rightarrow H^0(\text{Pin}_V; K) \rightarrow H^0(O_V; K) \rightarrow H^1(\mu_2; K).$$

The 0th Galois cohomology group of an algebraic group with coefficients in K is just the group of K -valued points: $H^0(G; K) = G(K)$, and $H^1(\mu_2; K) \cong K^*/K^{*2}$, which recovers the previous sequence

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Pin}_V(K) \rightarrow O_V(K) \rightarrow K^*/K^{*2},$$

where the spinor norm is the connecting homomorphism $H^0(O_V; K) \rightarrow H^1(\mu_2; K)$.

Spin and Pin groups

In this section we assume that V is finite dimensional and its bilinear form is non-singular. (If K has characteristic 2 this implies that the dimension of V is even.)

The Pin group $\text{Pin}_V(K)$ is the subgroup of the Clifford group Γ of elements of spinor norm 1, and similarly the Spin group $\text{Spin}_V(K)$ is the subgroup of elements of Dickson invariant 0 in $\text{Pin}_V(K)$. When the characteristic is not 2, these are the elements of determinant 1. The Spin group usually has index 2 in the Pin group.

Recall from the previous section that there is a homomorphism from the Clifford group onto the orthogonal group. We define the special orthogonal group to be the image of Γ^0 . If K does not have characteristic 2 this is just the group of elements of the orthogonal group of determinant 1. If K does have characteristic 2, then all elements of the orthogonal group have determinant 1, and the special orthogonal group is the set of elements of Dickson invariant 0.

There is a homomorphism from the Pin group to the orthogonal group. The image consists of the elements of spinor norm 1 $\in K^*/K^{*2}$. The kernel consists of the elements $+1$ and -1 , and has order 2 unless K has characteristic 2. Similarly there is a homomorphism from the Spin group to the special orthogonal group of V .

In the common case when V is a positive or negative definite space over the reals, the spin group maps onto the special orthogonal group, and is simply connected when V has dimension at least 3. Further the kernel of this homomorphism consists of 1 and -1. So in this case the spin group, $\text{Spin}(n)$, is a double cover of $\text{SO}(n)$. Please note, however, that the simple connectedness of the spin group is not true in general: if V is $\mathbb{R}^{p,q}$ for p and q both at least 2 then the spin group is not simply connected. In this case the algebraic group $\text{Spin}_{p,q}$ is simply connected as an algebraic group, even though its group of real valued points $\text{Spin}_{p,q}(\mathbb{R})$ is not simply connected. This is a rather subtle point, which completely confused the authors of at least one standard book about spin groups.

Spinors

Clifford algebras $\mathbb{C}\ell_{p,q}(\mathbb{C})$, with $p+q=2n$ even, are matrix algebras which have a complex representation of dimension 2^n . By restricting to the group $\text{Pin}_{p,q}(\mathbb{R})$ we get a complex representation of the Pin group of the same dimension, called the spin representation. If we restrict this to the spin group $\text{Spin}_{p,q}(\mathbb{R})$ then it splits as the sum of two *half spin representations* (or *Weyl representations*) of dimension 2^{n-1} .

If $p+q=2n+1$ is odd then the Clifford algebra $\mathbb{C}\ell_{p,q}(\mathbb{C})$ is a sum of two matrix algebras, each of which has a representation of dimension 2^n , and these are also both representations of the Pin group $\text{Pin}_{p,q}(\mathbb{R})$. On restriction to the spin group $\text{Spin}_{p,q}(\mathbb{R})$ these become isomorphic, so the spin group has a complex spinor representation of dimension 2^n .

More generally, spinor groups and pin groups over any field have similar representations whose exact structure depends on the structure of the corresponding Clifford algebras: whenever a Clifford algebra has a factor that is a matrix algebra over some division algebra, we get a corresponding representation of the pin and spin groups over that division algebra. For examples over the reals see the article on spinors.

Real spinors

To describe the real spin representations, one must know how the spin group sits inside its Clifford algebra. The Pin group, $Pin_{p,q}$ is the set of invertible elements in $Cl_{p,q}$ which can be written as a product of unit vectors:

$$Pin_{p,q} = \{v_1 v_2 \dots v_r \mid \forall i, \|v_i\| = \pm 1\}.$$

Comparing with the above concrete realizations of the Clifford algebras, the Pin group corresponds to the products of arbitrarily many reflections: it is a cover of the full orthogonal group $O(p,q)$. The Spin group consists of those elements of $Pin_{p,q}$ which are products of an even number of unit vectors. Thus by the Cartan-Dieudonné theorem $Spin$ is a cover of the group of proper rotations $SO(p,q)$.

Let $\alpha : Cl \rightarrow Cl$ be the automorphism which is given by $-Id$ acting on pure vectors. Then in particular, $Spin_{p,q}$ is the subgroup of $Pin_{p,q}$ whose elements are fixed by α . Let

$$Cl_{p,q}^0 = \{x \in Cl_{p,q} \mid \alpha(x) = x\}.$$

(These are precisely the elements of even degree in $Cl_{p,q}$.) Then the spin group lies within $Cl_{p,q}^0$.

The irreducible representations of $Cl_{p,q}$ restrict to give representations of the pin group. Conversely, since the pin group is generated by unit vectors, all of its irreducible representations are induced in this manner. Thus the two representations coincide. For the same reasons, the irreducible representations of the spin coincide with the irreducible representations of $Cl_{p,q}^0$.

To classify the pin representations, one need only appeal to the classification of Clifford algebras. To find the spin representations (which are representations of the even subalgebra), one can first make use of either of the isomorphisms (see above)

$$\begin{aligned} Cl_{p,q}^0 &\approx Cl_{p,q-1}, \text{ for } q > 0 \\ Cl_{p,q}^0 &\approx Cl_{q,p-1}, \text{ for } p > 0 \end{aligned}$$

and realize a spin representation in signature (p,q) as a pin representation in either signature $(p,q-1)$ or $(q,p-1)$.

Applications

Differential geometry

One of the principal applications of the exterior algebra is in differential geometry where it is used to define the bundle of differential forms on a smooth manifold. In the case of a (pseudo-)Riemannian manifold, the tangent spaces come equipped with a natural quadratic form induced by the metric. Thus, one can define a Clifford bundle in analogy with the exterior bundle. This has a number of important applications in Riemannian geometry. Perhaps more importantly is the link to a spin manifold, its associated spinor bundle and $spin^c$ manifolds.

Physics

Clifford algebras have numerous important applications in physics. Physicists usually consider a Clifford algebra to be an algebra spanned by matrices $\gamma_1, \dots, \gamma_n$ called Dirac matrices which have the property that

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\eta_{ij}$$

where η is the matrix of a quadratic form of signature (p,q) — typically (1,3) when working in Minkowski space. These are exactly the defining relations for the Clifford algebra $Cl_{1,3}(C)$ (up to an unimportant factor of 2), which by the classification of Clifford algebras is isomorphic to the algebra of 4 by 4 complex matrices.

The Dirac matrices were first written down by Paul Dirac when he was trying to write a relativistic first-order wave equation for the electron, and give an explicit isomorphism from the Clifford algebra to the algebra of complex matrices. The result was used to define the Dirac equation and introduce the Dirac operator. The entire Clifford algebra shows up in quantum field theory in the form of Dirac field bilinears.

Computer Vision

Recently, Clifford algebras have been applied in the problem of action recognition and classification in computer vision. Rodriguez et al. [5] propose a Clifford embedding to generalize traditional MACH filters to video (3D spatiotemporal volume), and vector-valued data such as optical flow. Vector-valued data is analyzed using the Clifford Fourier transform. Based on these vectors action filters are synthesized in the Clifford Fourier domain and recognition of actions is performed using Clifford Correlation. The authors demonstrate the effectiveness of the Clifford embedding by recognizing actions typically performed in classic feature films and sports broadcast television.

Notes

- [1] Mathematicians who work with real Clifford algebras and prefer positive definite quadratic forms (especially those working in index theory) sometimes use a different choice of sign in the fundamental Clifford identity. That is, they take $v^2 = -Q(v)$. One must replace Q with $-Q$ in going from one convention to the other.
- [2] Thus the group algebra $\mathbf{K}[\mathbf{Z}/2]$ is semisimple and the Clifford algebra splits into eigenspaces of the main involution.
- [3] We are still assuming that the characteristic is not 2.
- [4] The opposite is true when using the alternate $(-)$ sign convention for Clifford algebras: it is the conjugate which is more important. In general, the meanings of conjugation and transpose are interchanged when passing from one sign convention to the other. For example, in the convention used here the inverse of a vector is given by $v^{-1} = v^t / Q(v)$ while in the $(-)$ convention it is given by $v^{-1} = \bar{v} / Q(v)$.
- [5] Rodriguez, Mikel; Shah, M (2008). "Action MACH: A Spatio-Temporal Maximum Average Correlation Height Filter for Action Classification". *Computer Vision and Pattern Recognition (CVPR)*.

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External links

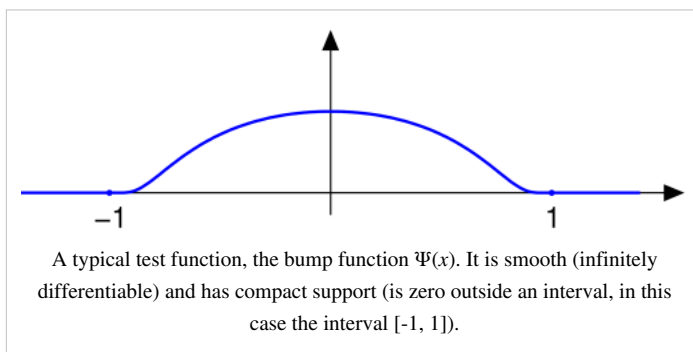
- Planetmath entry on Clifford algebras (<http://planetmath.org/encyclopedia/CliffordAlgebra2.html>)
- A history of Clifford algebras (<http://members.fortunecity.com/jonhays/clifhistory.htm>) (unverified)
- John Baez on Clifford algebras (<http://www.math.ucr.edu/home/baez/octonions/node6.html>)

Distributions

In mathematical analysis, **distributions** (or **generalized functions**) are objects that generalize functions. Distributions make it possible to differentiate functions whose derivatives do not exist in the classical sense. In particular, any locally integrable function has a distributional derivative. Distributions are widely used to formulate generalized solutions of partial differential equations. Where a classical solution may not exist or be very difficult to establish, a distribution solution to a differential equation is often much easier. Distributions are also important in physics and engineering where many problems naturally lead to differential equations whose solutions or initial conditions are distributions, such as the Dirac delta distribution.

Generalized functions were introduced by Sergei Sobolev in 1935. They were re-introduced in the late 1940s by Laurent Schwartz, who developed a comprehensive theory of distributions.

Basic idea



Distributions are a class of linear functionals that map a set of *test functions* (conventional and well-behaved functions) onto the set of real numbers. In the simplest case, the set of test functions considered is $D(\mathbf{R})$, which is the set of functions from $\mathbf{R}$ to $\mathbf{R}$ having two properties:

- The function is smooth (infinitely differentiable);
- The function has compact support (is identically zero outside some interval).

Then, a distribution d is a mapping from $D(\mathbf{R})$ to $\mathbf{R}$. Instead of writing $d(\varphi)$, where φ is a test function in $D(\mathbf{R})$, it is conventional to write $\langle d, \varphi \rangle$. A simple example of a distribution is the Dirac delta δ , defined by

$$\delta(\varphi) = \langle \delta, \varphi \rangle = \varphi(0).$$

There are straightforward mappings from both locally integrable functions and probability distributions to corresponding distributions, as discussed below. However, not all distributions can be formed in this manner.

Suppose that

$$f : \mathbf{R} \rightarrow \mathbf{R}$$

is a locally integrable function, and let

$$\varphi : \mathbf{R} \rightarrow \mathbf{R}$$

be a test function in $D(\mathbf{R})$. We can then define a corresponding distribution T_f by:

$$\langle T_f, \varphi \rangle = \int_{\mathbf{R}} f \varphi \, dx.$$

This integral is a real number which linearly and continuously depends on φ . This suggests the requirement that a distribution should be linear and continuous over the space of test functions $D(\mathbf{R})$, which completes the definition. In a conventional abuse of notation, f may be used to represent both the original function f and the distribution T_f derived from it.

Similarly, if P is a probability distribution on the reals and φ is a test function, then a corresponding distribution T_P may be defined by:

$$\langle T_P, \varphi \rangle = \int_{\mathbf{R}} \varphi dP$$

Again, this integral continuously and linearly depends on φ , so that T_P is in fact a distribution.

Such distributions may be multiplied with real numbers and can be added together, so they form a real vector space. In general it is not possible to define a multiplication for distributions, but distributions may be multiplied with infinitely differentiable functions.

It's desirable to choose a definition for the derivative of a distribution which, at least for distributions derived from locally integrable functions, has the property that $(T_f)' = T_{f'}$. If φ is a test function, we can show that

$$\langle T_{f'}, \varphi \rangle = \int_{\mathbf{R}} f' \varphi dx = - \int_{\mathbf{R}} f \varphi' dx = - \langle T_f, \varphi' \rangle$$

using integration by parts and noting that $[f'(x)\varphi(x)]_{-\infty}^{\infty} = 0$, since φ is zero outside of a bounded set. This suggests that if S is a *distribution*, we should define its derivative S' by

$$\langle S', \varphi \rangle = - \langle S, \varphi' \rangle.$$

It turns out that this is the proper definition; it extends the ordinary definition of derivative, every distribution becomes infinitely differentiable and the usual properties of derivatives hold.

Example: Recall that the Dirac delta (so-called Dirac delta function) is the distribution defined by

$$\langle \delta, \varphi \rangle = \varphi(0)$$

It is the derivative of the distribution corresponding to the Heaviside step function H : For any test function φ ,

$$\langle (T_H)', \varphi \rangle = - \langle T_H, \varphi' \rangle = - \int_{-\infty}^{\infty} H(x) \varphi'(x) dx = - \int_0^{\infty} \varphi'(x) dx = \varphi(0) - \varphi(\infty) = \varphi(0) = \langle \delta, \varphi \rangle,$$

so $(T_H)' = \delta$. Note, $\varphi(\infty) = 0$ because of compact support. Similarly, the derivative of the Dirac delta is the distribution

$$\langle \delta', \varphi \rangle = -\varphi'(0).$$

This latter distribution is our first example of a distribution which is derived from neither a function nor a probability distribution.

Test functions and distributions

In the sequel, real-valued distributions on an open subset U of $\mathbf{R}^n$ will be formally defined. With minor modifications, one can also define complex-valued distributions, and one can replace $\mathbf{R}^n$ by any (paracompact) smooth manifold.

The first object to define is the space $D(U)$ of test functions on U . Once this is defined, it is then necessary to equip it with a topology by defining the limit of a sequence of elements of $D(U)$. The space of distributions will then be given as the space of continuous linear functionals on $D(U)$.

Test function space

The space $D(U)$ of **test functions** on U is defined as follows. A function $\varphi: U \rightarrow \mathbf{R}$ is said to have compact support if there exists a compact subset K of U such that $\varphi(x) = 0$ for all x in $U \setminus K$. The elements of $D(U)$ are the infinitely differentiable functions $\varphi: U \rightarrow \mathbf{R}$ with compact support — also known as bump functions. This is a real vector space. It can be given a topology by defining the limit of a sequence of elements of $D(U)$. A sequence (φ_k) in $D(U)$ is said to converge to $\varphi \in D(U)$ if the following two conditions hold (Gelfand & Shilov 1966-1968, v. 1, §1.2):

- There is a compact set $K \subset U$ containing the supports of all φ_k :

$$\bigcup_k \text{supp}(\varphi_k) \subset K.$$

- For each multiindex α , the sequence of partial derivatives $D^\alpha \varphi_k$ tends uniformly to $D^\alpha \varphi$.

With this definition, $D(U)$ becomes a complete locally convex topological vector space satisfying the Heine–Borel property (Rudin 1991, §6.4-5). If U_i is a countable nested family of open subsets of U with compact closures $K_i = \bar{U}_i$, then

$$D(U) = \bigcup_i D_{K_i}$$

where D_{K_i} is the set of all smooth functions with support lying in K_i . The topology on $D(U)$ is the final topology of the family of nested metric spaces D_{K_i} and so $D(U)$ is an LF-space. The topology is not metrizable by the Baire category theorem, since $D(U)$ is the union of subspaces of the first category in $D(U)$ (Rudin 1991, §6.9).

Distributions

A **distribution** on U is a linear functional $S : D(U) \rightarrow \mathbf{R}$ (or $S : D(U) \rightarrow \mathbf{C}$), such that

$$\lim_{n \rightarrow \infty} S(\varphi_n) = S\left(\lim_{n \rightarrow \infty} \varphi_n\right)$$

for any convergent sequence φ_n in $D(U)$. The space of all distributions on U is denoted by $D'(U)$. Equivalently, the vector space $D'(U)$ is the continuous dual space of the topological vector space $D(U)$.

The dual pairing between a distribution S in $D'(U)$ and a test function φ in $D(U)$ is denoted using angle brackets thus:

$$D'(U) \times D(U) \ni (S, \varphi) \mapsto \langle S, \varphi \rangle \in \mathbf{R}.$$

Equipped with the weak-* topology, the space $D'(U)$ is a locally convex topological vector space. In particular, a sequence (S_k) in $D'(U)$ converges to a distribution S if and only if

$$\langle S_k, \varphi \rangle \rightarrow \langle S, \varphi \rangle$$

for all test functions φ . This is the case if and only if S_k converges uniformly to S on all bounded subsets of $D(U)$. (A subset E of $D(U)$ is bounded if there exists a compact subset K of U and numbers d_n such that every φ in E has its support in K and has its n -th derivatives bounded by d_n .)

Functions as distributions

The function $f : U \rightarrow \mathbf{R}$ is called **locally integrable** if it is Lebesgue integrable over every compact subset K of U . This is a large class of functions which includes all continuous functions and all L^p functions. The topology on $D(U)$ is defined in such a fashion that any locally integrable function f yields a continuous linear functional on $D(U)$ — that is, an element of $D'(U)$ — denoted here by T_f , whose value on the test function φ is given by the Lebesgue integral:

$$\langle T_f, \varphi \rangle = \int_U f \varphi dx.$$

Conventionally, one abuses notation by identifying T_f with f , provided no confusion can arise, and thus the pairing between f and φ is often written

$$\langle f, \varphi \rangle = \langle T_f, \varphi \rangle.$$

If f and g are two locally integrable functions, then the associated distributions T_f and T_g are equal to the same element of $D'(U)$ if and only if f and g are equal almost everywhere (see, for instance, Hörmander (1983, Theorem 1.2.5)). In a similar manner, every Radon measure μ on U defines an element of $D'(U)$ whose value on the test function φ is $\int \varphi d\mu$. As above, it is conventional to abuse notation and write the pairing between a Radon

measure μ and a test function φ as $\langle \mu, \varphi \rangle$. Conversely, essentially by the Riesz representation theorem, every distribution which is non-negative on non-negative functions is of this form for some (positive) Radon measure.

The test functions are themselves locally integrable, and so define distributions. As such they are dense in $D'(U)$ with respect to the topology on $D'(U)$ in the sense that for any distribution $S \in D'(U)$, there is a sequence $\varphi_n \in D(U)$ such that

$$\langle \varphi_n, \psi \rangle \rightarrow \langle S, \psi \rangle$$

for all $\psi \in D(U)$. This follows at once from the Hahn–Banach theorem, since by an elementary fact about weak topologies the dual of $D'(U)$ with its weak-* topology is the space $D(U)$ (Rudin 1991, Theorem 3.10). This can also be proven more constructively by a convolution argument.

Operations on distributions

Many operations which are defined on smooth functions with compact support can also be defined for distributions. In general,, if

$$T : D(U) \rightarrow D(U)$$

is a linear mapping of vector spaces which is continuous with respect to the weak-* topology, then it is possible to extend T to a mapping

$$T : D'(U) \rightarrow D'(U)$$

by passing to the limit. (This approach works for more general non-linear mappings as well, provided they are assumed to be uniformly continuous.)

In practice, however, it is more convenient to define operations on distributions by means of the transpose (or adjoint transformation) (Strichartz 1994, §2.3; Trèves 1967). If $T : D(U) \rightarrow D(U)$ is a continuous linear operator, then the transpose is an operator $T^* : D(U) \rightarrow D(U)$ such that

$$\langle T\varphi, \psi \rangle = \langle \varphi, T^*\psi \rangle$$

for all $\varphi, \psi \in D(U)$. If such an operator T^* exists, and is continuous, then the original operator T may be extended to distributions by defining

$$Tf(\psi) = f(T^*\psi).$$

Differentiation

If $T : D(U) \rightarrow D(U)$ is given by the partial derivative

$$T\varphi = \frac{\partial \varphi}{\partial x_k}.$$

By integration by parts, if φ and ψ are in $D(U)$, then

$$\langle T\varphi, \psi \rangle = \left\langle \frac{\partial \varphi}{\partial x_k}, \psi \right\rangle = - \left\langle \varphi, \frac{\partial \psi}{\partial x_k} \right\rangle$$

so that $T^* = -T$. This is a continuous linear transformation $D(U) \rightarrow D(U)$. So, if $S \in D'(U)$ is a distribution, then the partial derivative of S with respect to the coordinate x_k is defined by the formula

$$\left\langle \frac{\partial S}{\partial x_k}, \varphi \right\rangle = - \left\langle S, \frac{\partial \varphi}{\partial x_k} \right\rangle$$

for all test functions φ . In this way, every distribution is infinitely differentiable, and the derivative in the direction x_k is a linear operator on $D'(U)$. In general, if $\alpha = (\alpha_1, \dots, \alpha_n)$ is an arbitrary multi-index and ∂^α denotes the associated mixed partial derivative operator, the mixed partial derivative $\partial^\alpha S$ of the distribution $S \in D'(U)$ is defined by

$$\langle \partial^\alpha S, \varphi \rangle = (-1)^{|\alpha|} \langle S, \partial^\alpha \varphi \rangle \text{ for all } \varphi \in D(U).$$

Differentiation of distributions is a continuous operator on $D'(U)$; this is an important and desirable property that is not shared by most other notions of differentiation.

Multiplication by a smooth function

If $m : U \rightarrow \mathbf{R}$ is an infinitely differentiable function and S is a distribution on U , then the product mS is defined by $(mS)(\varphi) = S(m\varphi)$ for all test functions φ . This definition coincides with the transpose transformation of

$$T_m : \varphi \mapsto m\varphi$$

for $\varphi \in D(U)$. Then, for any test function ψ

$$\langle T_m \varphi, \psi \rangle = \int_U m(x) \varphi(x) \psi(x) dx = \langle \varphi, T_m^* \psi \rangle$$

so that $T_m^* = T_m^*$. Multiplication of a distribution S by the smooth function m is therefore defined by

$$mS(\psi) = \langle mS, \psi \rangle = \langle S, m\psi \rangle = S(m\psi).$$

Under multiplication by smooth functions, $D'(U)$ is a module over the ring $C^\infty(D)$. With this definition of multiplication by a smooth function, the ordinary product rule of calculus remains valid. However, a number of unusual identities also arise. For example, the Dirac delta distribution δ is defined on $\mathbf{R}$ by $\langle \delta, \varphi \rangle = \varphi(0)$, and its derivative is given by $\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = -\varphi'(0)$. However, the product $m\delta'$ is the distribution

$$m\delta' = m(0)\delta' - m'\delta.$$

This definition of multiplication also makes it possible to define the operation of a linear differential operator with smooth coefficients on a distribution. A linear differential operator takes a distribution $S \in D'(U)$ to another distribution given by a sum of the form

$$PS = \sum_{|\alpha| \leq k} p_\alpha \partial^\alpha S$$

where the coefficients p_α are smooth functions in U . If P is a given differential operator, then the minimum integer k for which such an expansion holds for every distribution S is called the **order** of P . The transpose of P is given by

$$\left\langle \varphi, \sum_\alpha p_\alpha \partial^\alpha S \right\rangle = \left\langle \sum_\alpha (-1)^{|\alpha|} \partial^\alpha (p_\alpha \varphi), S \right\rangle.$$

The space $D'(U)$ is a D -module with respect to the action of the ring of linear differential operators.

Composition with a smooth function

Let S be a distribution on an open set $U \subset \mathbf{R}^n$. Let V be an open set in $\mathbf{R}^n$, and $F : V \rightarrow U$. Then provided F is a submersion, it is possible to define

$$S \circ F \in D'(V).$$

This is the **composition** of the distribution S with F , and is also called the **pullback** of S along F , sometimes written

$$F^\# : S \mapsto F^\# S = S \circ F.$$

The pullback is often denoted F^* , but this notation risks confusion with the above use of '*' to denote the transpose of a linear mapping.

The condition that F be a submersion is equivalent to the requirement that the Jacobian derivative $dF(x)$ of F is a surjective linear map for every $x \in V$. A necessary (but not sufficient) condition for extending $F^\#$ to distributions is that F be an open mapping (Hörmander 1983, Theorem 6.1.1). The inverse function theorem ensures that a submersion satisfies this condition.

If F is a submersion, then $F^\#$ is defined on distributions by finding the transpose map. Uniqueness of this extension is guaranteed since $F^\#$ is a continuous linear operator on $D(U)$. Existence, however, requires using the change of variables formula, the inverse function theorem (locally) and a partition of unity argument; see Hörmander (1983,

Theorem 6.1.2).

In the special case when F is a diffeomorphism from an open subset V of $\mathbf{R}^n$ onto an open subset U of $\mathbf{R}^n$ change of variables under the integral gives

$$\int_V \varphi \circ F(x) \psi(x) dx = \int_U \varphi(x) \psi(F^{-1}(x)) |\det dF^{-1}(x)| dx.$$

In this particular case, then, $F^\#$ is defined by the transpose formula:

$$\langle F^\# S, \varphi \rangle = \langle S, |\det d(F^{-1})| \varphi \circ F^{-1} \rangle.$$

Localization of distributions

There is no way to define the value of a distribution in $D'(U)$ at a particular point of U . However, as is the case with functions, distributions on U restrict to give distributions on open subsets of U . Furthermore, distributions are *locally determined* in the sense that a distribution on all of U can be assembled from a distribution on an open cover of U satisfying some compatibility conditions on the overlap. Such a structure is known as a sheaf.

Restriction

Let U and V be open subsets of $\mathbf{R}^n$ with $V \subset U$. Let $E_{VU} : D(V) \rightarrow D(U)$ be the operator which *extends by zero* a given smooth function compactly supported in V to a smooth function compactly supported in the larger set U . Then the restriction mapping ρ_{VU} is defined to be the transpose of E_{VU} . Thus for any distribution $S \in D'(U)$, the restriction $\rho_{VU} S$ is a distribution in the dual space $D'(V)$ defined by

$$\langle \rho_{VU} S, \varphi \rangle = \langle S, E_{VU} \varphi \rangle$$

for all test functions $\varphi \in D(V)$.

Unless $U = V$, the restriction to V is neither injective nor surjective. Lack of surjectivity follows since distributions can blow up towards the boundary of V . For instance, if $U = \mathbf{R}$ and $V = (0, 2)$, then the distribution

$$S(x) = \sum_{n=1}^{\infty} n \delta \left(x - \frac{1}{n} \right)$$

is in $D'(V)$ but admits no extension to $D'(U)$.

Support of a distribution

Let $S \in D'(U)$ be a distribution on an open set U . Then S is said to vanish on an open set V of U if S lies in the kernel of the restriction map ρ_{VU} . Explicitly S vanishes on V if

$$\langle S, \varphi \rangle = 0$$

for all test functions $\varphi \in C^\infty(U)$ with support in V . Let V be a maximal open set on which the distribution S vanishes; i.e., V is the union of every open set on which S vanishes. The **support** of S is the complement of V in U . Thus

$$\text{supp } S = U - \bigcup \{V \mid \rho_{VU} S = 0\}.$$

The distribution S has **compact support** if its support is a compact set. Explicitly, S has compact support if there is a compact subset K of U such that for every test function φ whose support is completely outside of K , we have $S(\varphi) = 0$. Compactly supported distributions define continuous linear functions on the space $C^\infty(U)$; the topology on $C^\infty(U)$ is defined such that a sequence of test functions φ_k converges to 0 if and only if all derivatives of φ_k converge uniformly to 0 on every compact subset of U . Conversely, it can be shown that every continuous linear functional on this space defines a distribution of compact support.

Tempered distributions and Fourier transform

By using a larger space of test functions, one can define the **tempered distributions**, a subspace of $D'(\mathbf{R}^n)$. These distributions are useful if one studies the Fourier transform in generality: all tempered distributions have a Fourier transform, but not all distributions have one.

The space of test functions employed here, the so-called Schwartz space $S(\mathbf{R}^n)$, is the function space of all infinitely differentiable functions that are rapidly decreasing at infinity along with all partial derivatives. Thus $\varphi : \mathbf{R}^n \rightarrow \mathbf{R}$ is in the Schwartz space provided that any derivative of φ , multiplied with any power of $|x|$, converges towards 0 for $|x| \rightarrow \infty$. These functions form a complete topological vector space with a suitably defined family of seminorms. More precisely, let

$$p_{\alpha,\beta}(\varphi) = \sup_{x \in \mathbf{R}^n} |x^\alpha D^\beta \varphi(x)|$$

for α, β multi-indices of size n . Then φ is a Schwartz function if all the values

$$p_{\alpha,\beta}(\varphi) < \infty.$$

The family of seminorms $p_{\alpha,\beta}$ defines a locally convex topology on the Schwartz-space. The seminorms are, in fact, norms on the Schwartz space, since Schwartz functions are smooth. The Schwartz space is metrizable and complete.

The space of **tempered distributions** is defined as the (continuous) dual of the Schwartz space. In other words, a distribution F is a tempered distribution if and only if

$$\lim_{m \rightarrow \infty} F(\varphi_m) = 0.$$

is true whenever,

$$\lim_{m \rightarrow \infty} p_{\alpha,\beta}(\varphi_m) = 0$$

holds for all multi-indices α, β .

The derivative of a tempered distribution is again a tempered distribution. Tempered distributions generalize the bounded (or slow-growing) locally integrable functions; all distributions with compact support and all square-integrable functions are tempered distributions. All locally integrable functions f with at most polynomial growth, i.e. such that $f(x) = O(|x|^r)$ for some r , are tempered distributions. This includes all functions in $L^p(\mathbf{R}^n)$ for $p \geq 1$.

The *tempered distributions* can also be characterized as *slowly growing*. This characterization is *dual* to the *rapidly falling* behaviour, e.g. $\propto |x|^n \cdot \exp(-x^2)$, of the test functions.

To study the Fourier transform, it is best to consider *complex-valued* test functions and complex-linear distributions. The ordinary continuous Fourier transform F yields then an automorphism of Schwartz function space, and we can define the **Fourier transform** of the tempered distribution S by $(FS)(\psi) = S(F\psi)$ for every test function ψ . FS is thus again a tempered distribution. The Fourier transform is a continuous, linear, bijective operator from the space of tempered distributions to itself. This operation is compatible with differentiation in the sense that

$$F \frac{dS}{dx} = ixFS$$

and also with convolution: if S is a tempered distribution and ψ is a *slowly increasing* infinitely differentiable function on $\mathbf{R}^n$ (meaning that all derivatives of ψ grow at most as fast as polynomials), then $S\psi$ is again a tempered distribution and

$$F(S\psi) = FS * F\psi$$

is the convolution of FS and $F\psi$.

Convolution

Under some circumstances, it is possible to define the convolution of a function with a distribution, or even the convolution of two distributions.

Convolution of a test function with a distribution

If $f \in D(\mathbf{R}^n)$ is a compactly supported smooth test function, then convolution with f defines an operator

$$C_f : D(\mathbf{R}^n) \rightarrow D(\mathbf{R}^n)$$

defined by $C_f g = f * g$, which is linear (and continuous with respect to the LF space topology on $D(\mathbf{R}^n)$.)

Convolution of f with a distribution $S \in D'(\mathbf{R}^n)$ can be defined by taking the transpose of C_f relative to the duality pairing of $D(\mathbf{R}^n)$ with the space $D'(\mathbf{R}^n)$ of distributions (Trèves 1967, Chapter 27). If $f, g, \varphi \in D(\mathbf{R}^n)$, then by Fubini's theorem

$$\langle C_f g, \varphi \rangle = \int_{\mathbf{R}^n} \varphi(x) \int_{\mathbf{R}^n} f(x-y)g(y) dy dx = \langle g, C_{\tilde{f}} \varphi \rangle$$

where $\tilde{f}(x) = f(-x)$. Extending by continuity, the convolution of f with a distribution S is defined by

$$\langle f * S, \varphi \rangle = \langle S, \tilde{f} * \varphi \rangle$$

for all test functions $\varphi \in D(\mathbf{R}^n)$.

An alternative way to define the convolution of a function f and a distribution S is to use the translation operator τ_x defined on test functions by

$$\tau_x \varphi(y) = \varphi(y-x)$$

and extended by the transpose to distributions in the obvious way (Rudin 1991, §6.29). The convolution of the compactly supported function f and the distribution S is then the function defined for each $x \in \mathbf{R}^n$ by

$$(f * S)(x) = \langle S, \tau_x \tilde{f} \rangle.$$

It can be shown that the convolution of a compactly supported function and a distribution is a smooth function. If the distribution S has compact support as well, then $f * S$ is a compactly supported function, and the Titchmarsh convolution theorem (Hörmander 1983, Theorem 4.3.3) implies that

$$\text{ch}(f * S) = \text{ch} f + \text{ch} S$$

where ch denotes the convex hull.

Distribution of compact support

It is also possible to define the convolution of two distributions S and T on $\mathbf{R}^n$, provided one of them has compact support. Informally, in order to define $S * T$ where T has compact support, the idea is to extend the definition of the convolution $*$ to a linear operation on distributions so that the associativity formula

$$S * (T * \varphi) = (S * T) * \varphi$$

continues to hold for all test-functions φ . Hörmander (1983, §IV.2) proves the uniqueness of such an extension.

It is also possible to provide a more explicit characterization of the convolution of distributions (Trèves 1967, Chapter 27). Suppose that it is T that has compact support. For any test function φ in $D(\mathbf{R}^n)$, consider the function

$$\psi(x) = \langle T, \tau_{-x} \varphi \rangle.$$

It can be readily shown that this defines a smooth function of x , which moreover has compact support. The convolution of S and T is defined by

$$\langle S * T, \varphi \rangle = \langle S, \psi \rangle.$$

This generalizes the classical notion of convolution of functions and is compatible with differentiation in the following sense:

$$\partial^\alpha (S * T) = (\partial^\alpha S) * T = S * (\partial^\alpha T).$$

This definition of convolution remains valid under less restrictive assumptions about S and T ; see for instance Gel'fand & Shilov (1966–1968, v. 1, pp. 103–104) and Benedetto (1997, Definition 2.5.8).

Distributions as derivatives of continuous functions

The formal definition of distributions exhibits them as a subspace of a very large space, namely the algebraic dual of $D(U)$ (or $S(\mathbf{R}^d)$ for tempered distributions). It is not immediately clear from the definition how exotic a distribution might be. To answer this question, it is instructive to see distributions built up from a smaller space, namely the space of continuous functions. Roughly, any distribution is locally a (multiple) derivative of a continuous function. A precise version of this result, given below, holds for distributions of compact support, tempered distributions, and general distributions. Generally speaking, no proper subset of the space of distributions contains all continuous functions and is closed under differentiation. This says that distributions are not particularly exotic objects; they are only as complicated as necessary.

Tempered distributions

If $f \in S'(\mathbf{R}^n)$ is a tempered distribution, then there exists a constant $C > 0$, and positive integers M and N such that for all Schwartz functions $\varphi \in S(\mathbf{R}^n)$

$$\langle f, \varphi \rangle \leq C \sum_{|\alpha| \leq N, |\beta| \leq M} \sup_{x \in \mathbf{R}^n} |x^\alpha D^\beta \varphi(x)| = C \sum_{|\alpha| \leq N, |\beta| \leq M} p_{\alpha, \beta}(\varphi).$$

This estimate along with some techniques from functional analysis can be used to show that there is a continuous slowly increasing function F and a multiindex α such that

$$f = D^\alpha F.$$

Compactly supported distributions

Let U be an open set, and K a compact subset of U . If f is a distribution supported on K , then there is a continuous function F compactly supported in U (possibly on a larger set than K itself) such that

$$f = D^\alpha F$$

for some multi-index α . This follows from the previously quoted result on tempered distributions by means of a localization argument.

Distributions with point support

If f has support at a single point $\{P\}$, then f is in fact a finite linear combination of distributional derivatives of the δ function at P . That is, there exists an integer m and complex constants a_α for multi indices $|\alpha| \leq m$ such that

$$f = \sum_{|\alpha| \leq m} a_\alpha D^\alpha (\tau_P \delta)$$

where τ_P is the translation operator.

General distributions

A version of the above theorem holds locally in the following sense (Rudin 1991). Let S be a distribution on U . Then one can find for every multi-index α a continuous function g_α such that

$$S = \sum_{\alpha} D^\alpha g_\alpha$$

and that any compact subset K of U intersects the supports of only finitely many g_α ; therefore, to evaluate the value of S for a given smooth function f compactly supported in U , we only need finitely many g_α ; hence the infinite sum above is well-defined as a distribution. If the distribution S is of finite order, then one can choose g_α in such a way that only finitely many of them are nonzero.

Using holomorphic functions as test functions

The success of the theory led to investigation of the idea of hyperfunction, in which spaces of holomorphic functions are used as test functions. A refined theory has been developed, in particular Mikio Sato's algebraic analysis, using sheaf theory and several complex variables. This extends the range of symbolic methods that can be made into rigorous mathematics, for example Feynman integrals.

Problem of multiplication

A possible limitation of the theory of distributions (and hyperfunctions) is that it is a purely linear theory, in the sense that the product of two distributions cannot consistently be defined (in general), as has been proved by Laurent Schwartz in the 1950s. For example, if p.v. $1/x$ is the distribution obtained by the Cauchy principal value

$$\left(p.v.\frac{1}{x}\right)[\phi] = \lim_{\epsilon \rightarrow 0^+} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx$$

for all $\phi \in S(\mathbf{R})$, and δ is the Dirac delta distribution then

$$(\delta \times x) \times p.v.\frac{1}{x} = 0$$

but

$$\delta \times \left(x \times p.v.\frac{1}{x}\right) = \delta$$

so the product of a distribution by a smooth function (which is always well defined) cannot be extended to an associative product on the space of distributions.

Thus, nonlinear problems cannot be posed in general and thus not solved within distribution theory alone. In the context of quantum field theory, however, solutions can be found. In more than two spacetime dimensions the problem is related to the regularization of divergences. Here Henri Epstein and Vladimir Glaser developed the mathematically rigorous (but extremely technical) *causal perturbation theory*. This does not solve the problem in other situations. Many other interesting theories are non linear, like for example Navier-Stokes equations of fluid dynamics.

In view of this, several not entirely satisfactory theories of **algebras** of generalized functions have been developed, among which Colombeau's (simplified) algebra is maybe the most popular in use today.

A simple solution of the multiplication problem is dictated by the path integral formulation of quantum mechanics. Since this is required to be equivalent to the Schrödinger theory of quantum mechanics which is invariant under coordinate transformations, this property must be shared by path integrals. This fixes all products of distributions as shown by Kleinert & Chervyakov (2001) The result is equivalent to what can be derived from dimensional regularization (Kleinert & Chervyakov 2000).

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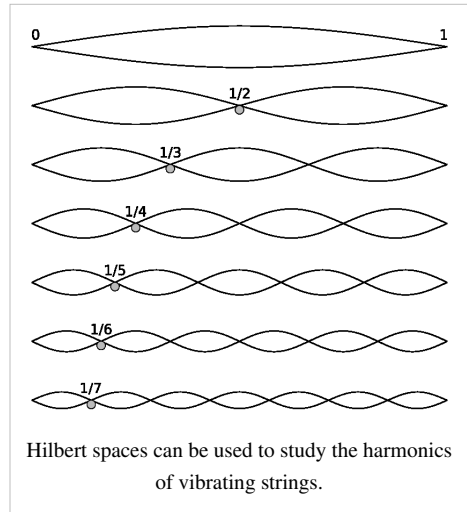
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- [2] <http://www.physik.fu-berlin.de/~kleinert/305/klch2.pdf>
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Hilbert space

The mathematical concept of a **Hilbert space**, named after David Hilbert, generalizes the notion of Euclidean space. It extends the methods of vector algebra and calculus from the two-dimensional Euclidean plane and three-dimensional space to spaces with any finite or infinite number of dimensions. A Hilbert space is an abstract vector space possessing the structure of an inner product that allows length and angle to be measured. Furthermore, Hilbert spaces are required to be complete, a property that stipulates the existence of enough limits in the space to allow the techniques of calculus to be used.

Hilbert spaces arise naturally and frequently in mathematics, physics, and engineering, typically as infinite-dimensional function spaces. The earliest Hilbert spaces were studied from this point of view in the first decade of the 20th century by David Hilbert, Erhard Schmidt, and Frigyes Riesz. They are indispensable tools in the theories of partial differential equations, quantum mechanics, Fourier analysis (which includes applications to signal processing and heat transfer) and ergodic theory which forms the mathematical underpinning of the study of thermodynamics. John von Neumann coined the term "Hilbert space" for the abstract concept underlying many of these diverse applications. The success of Hilbert space methods ushered in a very fruitful era for functional analysis. Apart from the classical Euclidean spaces, examples of Hilbert spaces include spaces of square-integrable functions, spaces of sequences, Sobolev spaces consisting of generalized functions, and Hardy spaces of holomorphic functions.

Geometric intuition plays an important role in many aspects of Hilbert space theory. Exact analogs of the Pythagorean theorem and parallelogram law hold in a Hilbert space. At a deeper level, perpendicular projection onto a subspace (the analog of "dropping the altitude" of a triangle) plays a significant role in optimization problems and other aspects of the theory. An element of a Hilbert space can be uniquely specified by its coordinates with respect to a set of coordinate axes (an orthonormal basis), in analogy with Cartesian coordinates in the plane. When that set of axes is countably infinite, this means that the Hilbert space can also usefully be thought of in terms of infinite sequences that are square-summable. Linear operators on a Hilbert space are likewise fairly concrete objects: in good cases, they are simply transformations that stretch the space by different factors in mutually perpendicular directions in a sense that is made precise by the study of their spectral theory.



Definition and illustration

Motivating example: Euclidean space

One of the most familiar examples of a Hilbert space is the Euclidean space consisting of three-dimensional vectors, denoted by $\mathbf{R}^3$, and equipped with the dot product. The dot product takes two vectors $\mathbf{x}$ and $\mathbf{y}$, and produces a real number $\mathbf{x} \cdot \mathbf{y}$. If $\mathbf{x}$ and $\mathbf{y}$ are represented in Cartesian coordinates, then the dot product is defined by

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

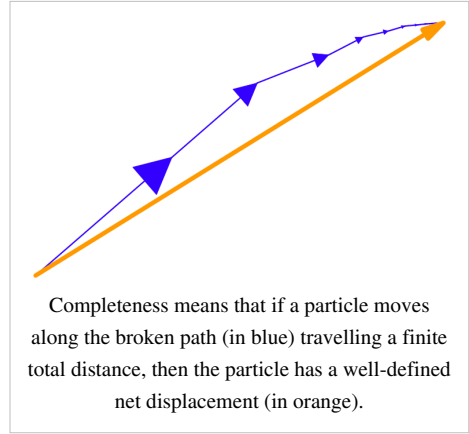
The dot product satisfies the properties:

1. It is symmetric in $\mathbf{x}$ and $\mathbf{y}$: $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.
2. It is linear in its first argument: $(a\mathbf{x}_1 + b\mathbf{x}_2) \cdot \mathbf{y} = a\mathbf{x}_1 \cdot \mathbf{y} + b\mathbf{x}_2 \cdot \mathbf{y}$ for any scalars a, b , and vectors $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{y}$.
3. It is positive definite: for all vectors $\mathbf{x}$, $\mathbf{x} \cdot \mathbf{x} \geq 0$ with equality if and only if $\mathbf{x} = 0$.

An operation on pairs of vectors that, like the dot product, satisfies these three properties is known as a (real) inner product. A vector space equipped with such an inner product is known as a (real) inner product space. Every finite-dimensional inner product space is also a Hilbert space. The basic feature of the dot product that connects it with Euclidean geometry is that it is related to both the length (or norm) of a vector, denoted $\|\mathbf{x}\|$, and to the angle θ between two vectors $\mathbf{x}$ and $\mathbf{y}$ by means of the formula

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta.$$

Multivariable calculus in Euclidean space relies on the ability to compute limits, and to have useful criteria for concluding that limits exist. A mathematical series



$$\sum_{n=0}^{\infty} \mathbf{x}_n$$

consisting of vectors in $\mathbf{R}^3$ is absolutely convergent provided that the sum of the lengths converges as an ordinary series of real numbers:^[1]

$$\sum_{k=0}^{\infty} \|\mathbf{x}_k\| < \infty.$$

Just as with a series of scalars, a series of vectors that converges absolutely also converges to some limit vector $\mathbf{L}$ in the Euclidean space, in the sense that

$$\left\| \mathbf{L} - \sum_{k=0}^N \mathbf{x}_k \right\| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

This property expresses the *completeness* of Euclidean space: that a series which converges absolutely also converges in the ordinary sense.

Definition

A **Hilbert space** H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.^[2] To say that H is a complex inner product space means that H is a complex vector space on which there is an inner product $\langle x, y \rangle$ associating a complex number to each pair of elements x, y of H that satisfies the following properties:

- $\langle y, x \rangle$ is the complex conjugate of $\langle x, y \rangle$:

$$\langle y, x \rangle = \overline{\langle x, y \rangle}.$$

- $\langle x, y \rangle$ is linear in its first argument.^[3] For all complex numbers a and b ,

$$\langle ax_1 + bx_2, y \rangle = a\langle x_1, y \rangle + b\langle x_2, y \rangle.$$

- The inner product is positive definite:

$$\langle x, x \rangle \geq 0$$

where the case of equality holds precisely when $x = 0$.

It follows from properties 1 and 2 that a complex inner product is antilinear in its second argument, meaning that

$$\langle x, ay_1 + by_2 \rangle = \bar{a}\langle x, y_1 \rangle + \bar{b}\langle x, y_2 \rangle.$$

A real inner product space is defined in the same way, except that H is a real vector space and the inner product takes real values. Such an inner product will be bilinear: that is, linear in each argument.

The norm defined by the inner product $\langle \bullet, \bullet \rangle$ is the real-valued function

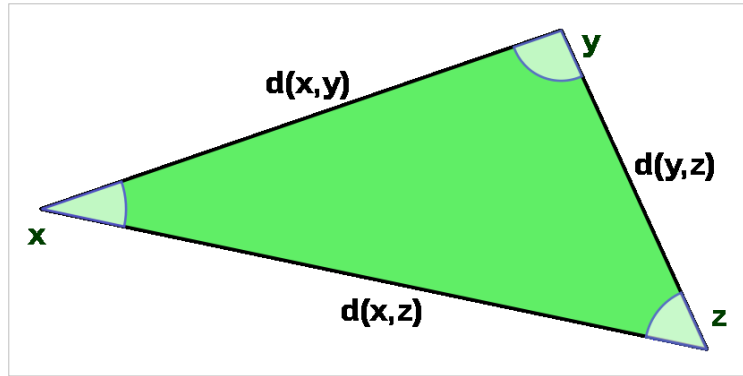
$$\|x\| = \sqrt{\langle x, x \rangle},$$

and the distance between two points x, y in H is defined in terms of the norm by

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}.$$

That this function is a distance function means (1) that it is symmetric in x and y , (2) that the distance between x and itself is zero, and otherwise the distance between x and y must be positive, and (3) that the triangle inequality holds, meaning that the length of one leg of a triangle xyz cannot exceed the sum of the lengths of the other two legs:

$$d(x, z) \leq d(x, y) + d(y, z).$$



This last property is ultimately a consequence of the more fundamental Cauchy–Schwarz inequality, which asserts

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

with equality if and only if x and y are linearly dependent.

Relative to a distance function defined in this way, any inner product space is a metric space, and sometimes is known as a **pre-Hilbert space**.^[4] A pre-Hilbert space is a Hilbert space if in addition it is complete. Completeness is expressed using a form of the Cauchy criterion for sequences in H : a pre-Hilbert space H is complete if every Cauchy sequence converges with respect to this norm to an element in the space. Completeness can be characterized by the following equivalent condition: if a series of vectors $\sum_{k=0}^{\infty} u_k$ converges absolutely in the sense that

$$\sum_{k=0}^{\infty} \|u_k\| < \infty,$$

then the series converges in H , in the sense that the partial sums converge to an element of H .

As a complete normed space, Hilbert spaces are by definition also Banach spaces. As such they are topological vector spaces, in which topological notions like the openness and closedness of subsets are well-defined. Of special importance is the notion of a closed linear subspace of a Hilbert space which, with the inner product induced by restriction, is also complete (being a closed set in a complete metric space) and therefore a Hilbert space in its own right.

Second example: sequence spaces

The sequence space ℓ^2 consists of all infinite sequences $\mathbf{z} = (z_1, z_2, \dots)$ of complex numbers such that the series

$$\sum_{n=1}^{\infty} |z_n|^2$$

converges. The inner product on ℓ^2 is defined by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{n=1}^{\infty} z_n \overline{w_n},$$

with the latter series converging as a consequence of the Cauchy–Schwarz inequality.

Completeness of the space holds provided that whenever a series of elements from ℓ^2 converges absolutely (in norm), then it converges to an element of ℓ^2 . The proof is basic in mathematical analysis, and permits mathematical series of elements of the space to be manipulated with the same ease as series of complex numbers (or vectors in a finite-dimensional Euclidean space).^[5]

History

Prior to the development of Hilbert spaces, other generalizations of Euclidean spaces were known to mathematicians and physicists. In particular, the idea of an abstract linear space had gained some traction towards the end of the 19th century:^[6] this is a space whose elements can be added together and multiplied by scalars (such as real or complex numbers) without necessarily identifying these elements with "geometric" vectors, such as position and momentum vectors in physical systems. Other objects studied by mathematicians at the turn of the 20th century, in particular spaces of sequences (including series) and spaces of functions,^[7] can naturally be thought of as linear spaces. Functions, for instance, can be added together or multiplied by constant scalars, and these operations obey the algebraic laws satisfied by addition and scalar multiplication of spatial vectors.

In the first decade of the 20th century, parallel developments led to the introduction of Hilbert spaces. The first of these was the observation, which arose during David Hilbert and Erhard Schmidt's study of integral equations,^[8] that two square-integrable real-valued functions f and g on an interval $[a, b]$ have an *inner product*

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx$$

which has many of the familiar properties of the Euclidean dot product. In particular, the idea of an orthogonal family of functions has meaning. Schmidt exploited the similarity of this inner product with the usual dot product to prove an analog of the spectral decomposition for an operator of the form

$$f(x) \mapsto \int_a^b K(x, y)f(y) dy$$

where K is a continuous function symmetric in x and y . The resulting eigenfunction expansion expresses the function K as a series of the form

$$K(x, y) = \sum_n \lambda_n \varphi_n(x) \varphi_n(y)$$



David Hilbert

where the functions φ_n are orthogonal in the sense that $\langle \varphi_n, \varphi_m \rangle = 0$ for all $n \neq m$. The individual terms in this series are sometimes referred to as elementary product solutions. However, there are eigenfunction expansions which fail to converge in a suitable sense to a square-integrable function: the missing ingredient, which ensures convergence, is completeness.^[9]

The second development was the Lebesgue integral, an alternative to the Riemann integral introduced by Henri Lebesgue in 1904.^[10] The Lebesgue integral made it possible to integrate a much broader class of functions. In 1907, Frigyes Riesz and Ernst Sigismund Fischer independently proved that the space L^2 of square Lebesgue-integrable functions is a complete metric space.^[11] As a consequence of the interplay between geometry and completeness, the 19th century results of Joseph Fourier, Friedrich Bessel and Marc-Antoine Parseval on trigonometric series easily carried over to these more general spaces, resulting in a geometrical and analytical apparatus now usually known as the Riesz-Fischer theorem.^[12]

Further basic results were proved in the early 20th century. For example, the Riesz representation theorem was independently established by Maurice Fréchet and Frigyes Riesz in 1907.^[13] John von Neumann coined the term *abstract Hilbert space* in his work on unbounded Hermitian operators.^[14] Although other mathematicians such as Hermann Weyl and Norbert Wiener had already studied particular Hilbert spaces in great detail, often from a physically-motivated point of view, von Neumann gave the first complete and axiomatic treatment of them.^[15] Von Neumann later used them in his seminal work on the foundations of quantum mechanics,^[16] and in his continued work with Eugene Wigner. The name "Hilbert space" was soon adopted by others, for example by Hermann Weyl in his book on quantum mechanics and the theory of groups.^[17]

The significance of the concept of a Hilbert space was underlined with the realization that it offers one of the best mathematical formulations of quantum mechanics.^[18] In short, the states of a quantum mechanical system are vectors in a certain Hilbert space, the observables are hermitian operators on that space, the symmetries of the system are unitary operators, and measurements are orthogonal projections. The relation between quantum mechanical symmetries and unitary operators provided an impetus for the development of the unitary representation theory of groups, initiated in the 1928 work of Hermann Weyl.^[17] On the other hand, in the early 1930s it became clear that certain properties of classical dynamical systems can be analyzed using Hilbert space techniques in the framework of ergodic theory.^[19]

The algebra of observables in quantum mechanics is naturally an algebra of operators defined on a Hilbert space, according to Werner Heisenberg's matrix mechanics formulation of quantum theory. Von Neumann began investigating operator algebras in the 1930s, as rings of operators on a Hilbert space. The kind of algebras studied by von Neumann and his contemporaries are now known as von Neumann algebras. In the 1940s, Israel Gelfand, Mark Naimark and Irving Segal gave a definition of a kind of operator algebras called C^* -algebras that on the one hand made no reference to an underlying Hilbert space, and on the other extrapolated many of the useful features of the operator algebras that had previously been studied. The spectral theorem for self-adjoint operators in particular that underlies much of the existing Hilbert space theory was generalized to C^* -algebras. These techniques are now basic in abstract harmonic analysis and representation theory.

Examples

Lebesgue spaces

Lebesgue spaces are function spaces associated to measure spaces (X, M, μ) , where X is a set, M is a σ -algebra of subsets of X , and μ is a countably additive measure on M . Let $L^2(X, \mu)$ be the space of those complex-valued measurable functions on X for which the Lebesgue integral of the square of the absolute value of the function is finite, i.e., for a function f in $L^2(X, \mu)$,

$$\int_X |f|^2 d\mu < \infty,$$

and where functions are identified if and only if they differ only on a set of measure zero.

The inner product of functions f and g in $L^2(X, \mu)$ is then defined as

$$\langle f, g \rangle = \int_X f(t) \overline{g(t)} d\mu(t).$$

For f and g in L^2 , this integral exists because of the Cauchy–Schwarz inequality, and defines an inner product on the space. Equipped with this inner product, L^2 is in fact complete.^[20] The Lebesgue integral is essential to ensure completeness: on domains of real numbers, for instance, not enough functions are Riemann integrable.^[21]

The Lebesgue spaces appear in many natural settings. The spaces $L^2(\mathbf{R})$ and $L^2([0,1])$ of square-integrable functions with respect to the Lebesgue measure on the real line and unit interval, respectively, are natural domains on which to define the Fourier transform and Fourier series. In other situations, the measure may be something other than the ordinary Lebesgue measure on the real line. For instance, if w is any positive measurable function, the space of all measurable functions f on the interval $[0,1]$ satisfying

$$\int_0^1 |f(t)|^2 w(t) dt < \infty$$

is called the weighted L^2 space $L^2_{w, \mu}([0,1])$, and w is called the weight function. The inner product is defined by

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} w(t) dt.$$

The weighted space $L^2_{w, \mu}([0,1])$ is identical with the Hilbert space $L^2([0,1], \mu)$ where the measure μ of a Lebesgue-measurable set A is defined by

$$\mu(A) = \int_A w(t) dt.$$

Weighted L^2 spaces like this are frequently used to study orthogonal polynomials, because different families of orthogonal polynomials are orthogonal with respect to different weighting functions.

Sobolev spaces

Sobolev spaces, denoted by H^s or $W^{s,2}$, are Hilbert spaces. These are a special kind of function space in which differentiation may be performed, but which (unlike other Banach spaces such as the Hölder spaces) support the structure of an inner product. Because differentiation is permitted, Sobolev spaces are a convenient setting for the theory of partial differential equations.^[22] They also form the basis of the theory of direct methods in the calculus of variations.^[23]

For s a non-negative integer and $\Omega \subset \mathbf{R}^n$, the Sobolev space $H^s(\Omega)$ contains L^2 functions whose weak derivatives of order up to s are also L^2 . The inner product in $H^s(\Omega)$ is

$$\langle f, g \rangle = \int_{\Omega} f(x) \overline{g(x)} dx + \int_{\Omega} Df \cdot D\overline{g}(x) dx + \dots + \int_{\Omega} D^s f(x) \cdot D^s \overline{g}(x) dx$$

where the dot indicates the dot product in the Euclidean space of partial derivatives of each order. Sobolev spaces can also be defined when s is not an integer.

Sobolev spaces are also studied from the point of view of spectral theory, relying more specifically on the Hilbert space structure. If Ω is a suitable domain, then one can define the Sobolev space $H^s(\Omega)$ as the space of Bessel potentials;^[24] roughly,

$$H^s(\Omega) = \{(1 - \Delta)^{-s/2} f \mid f \in L^2(\Omega)\}.$$

Here Δ is the Laplacian and $(1 - \Delta)^{-s/2}$ is understood in terms of the spectral mapping theorem. Apart from providing a workable definition of Sobolev spaces for non-integer s , this definition also has particularly desirable properties under the Fourier transform that make it ideal for the study of pseudodifferential operators. Using these methods on a compact Riemannian manifold, one can obtain for instance the Hodge decomposition which is the basis of Hodge theory.^[25]

Spaces of holomorphic functions

Hardy spaces

The Hardy spaces are function spaces, arising in complex analysis and harmonic analysis, whose elements are certain holomorphic functions in a complex domain.^[26] Let U denote the unit disc in the complex plane. Then the Hardy space $H^2(U)$ is defined to be the space of holomorphic functions f on U such that the means

$$M_r(f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$$

remain bounded for $r < 1$. The norm on this Hardy space is defined by

$$\|f\|_2 = \lim_{r \rightarrow 1} \sqrt{M_r(f)}.$$

Hardy spaces in the disc are related to Fourier series. A function f is in $H^2(U)$ if and only if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Thus $H^2(U)$ consists of those functions which are L^2 on the circle, and whose negative frequency Fourier coefficients vanish.

Bergman spaces

The Bergman spaces are another family of Hilbert spaces of holomorphic functions.^[27] Let D be a bounded open set in the complex plane (or a higher dimensional complex space) and let $L^{2,h}(D)$ be the space of holomorphic functions f in D that are also in $L^2(D)$ in the sense that

$$\|f\|^2 = \int_D |f(z)|^2 d\mu(z) < \infty,$$

where the integral is taken with respect to the Lebesgue measure in D . Clearly $L^{2,h}(D)$ is a subspace of $L^2(D)$; in fact, it is a closed subspace, and so a Hilbert space in its own right. This is a consequence of the estimate, valid on compact subsets K of D , that

$$\sup_{z \in K} |f(z)| \leq C_K \|f\|_2,$$

which in turn follows from Cauchy's integral formula. Thus convergence of a sequence of holomorphic functions in $L^2(D)$ implies also compact convergence, and so the limit function is also holomorphic. Another consequence of this inequality is that the linear functional that evaluates a function f at a point of D is actually continuous on $L^{2,h}(D)$. The Riesz representation theorem implies that the evaluation functional can be represented as an element of $L^{2,h}(D)$. Thus, for every $z \in D$, there is a function $\eta_z \in L^{2,h}(D)$ such that

$$f(z) = \int_D f(\zeta) \overline{\eta_z(\zeta)} d\mu(\zeta)$$

for all $f \in L^{2,h}(D)$. The integrand

$$K(\zeta, z) = \overline{\eta_z(\zeta)}$$

is known as the Bergman kernel of D . This integral kernel satisfies a reproducing property

$$f(z) = \int_D f(\zeta) K(\zeta, z) d\mu(\zeta).$$

A Bergman space is an example of a reproducing kernel Hilbert space, which is a Hilbert space of functions along with a kernel $K(\zeta, z)$ that verifies a reproducing property analogous to this one. The Hardy space $H^2(D)$ also admits a reproducing kernel, known as the Szegő kernel.^[28] Reproducing kernels are common in other areas of mathematics as well. For instance, in harmonic analysis the Poisson kernel is a reproducing kernel for the Hilbert space of square-integrable harmonic functions in the unit ball. That the latter is a Hilbert space at all is a consequence of the mean value theorem for harmonic functions.

Applications

Many of the applications of Hilbert spaces exploit the fact that Hilbert spaces support generalizations of simple geometric concepts like projection and change of basis from their usual finite dimensional setting. In particular, the spectral theory of continuous self-adjoint linear operators on a Hilbert space generalizes the usual spectral decomposition of a matrix, and this often plays a major role in applications of the theory to other areas of mathematics and physics.

Sturm–Liouville theory

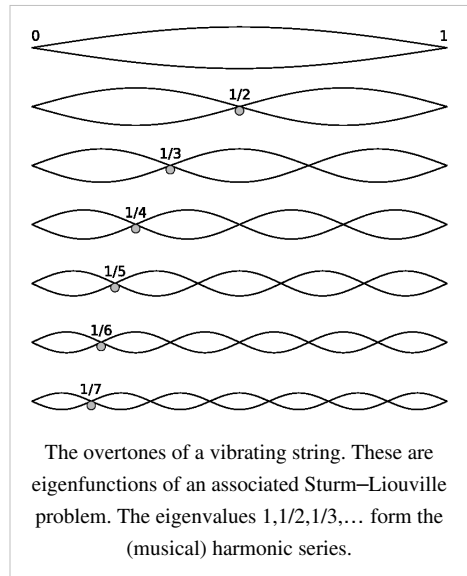
In the theory of ordinary differential equations, spectral methods on a suitable Hilbert space are used to study the behavior of eigenvalues and eigenfunctions of differential equations. For example, the Sturm–Liouville problem arises in the study of the harmonics of waves in a violin string or a drum, and is a central problem in ordinary differential equations.^[29] The problem is a differential equation of the form

$$-\frac{d}{dx} \left[p(x) \frac{dy}{dx} \right] + q(x)y = \lambda w(x)y$$

for an unknown function y on an interval $[a, b]$, satisfying general homogeneous Robin boundary conditions

$$\begin{cases} \alpha y(a) + \alpha' y'(a) = 0 \\ \beta y(b) + \beta' y'(b) = 0. \end{cases}$$

The functions p , q , and w are given in advance, and the problem is to find the function y and constants λ for which the equation has a solution. The problem only has solutions for certain values of λ , called eigenvalues of the system,



and this is a consequence of the spectral theorem for compact operators applied to the integral operator defined by the Green's function for the system. Furthermore, another consequence of this general result is that the eigenvalues λ of the system can be arranged in an increasing sequence tending to infinity.^[30]

Partial differential equations

Hilbert spaces form a basic tool in the study of partial differential equations.^[22] For many classes of partial differential equations, such as linear elliptic equations, it is possible to consider a generalized solution (known as a weak solution) by enlarging the class of functions. Many weak formulations involve the class of Sobolev functions, which is a Hilbert space. A suitable weak formulation reduces to a geometrical problem the analytic problem of finding a solution or, often what is more important, showing that a solution exists and is unique for given boundary data. For linear elliptic equations, one geometrical result that ensures unique solvability for a large class of problems is the Lax–Milgram theorem. This strategy forms the rudiment of the Galerkin method (a finite element method) for numerical solution of partial differential equations.^[31]

A typical example is the Poisson equation $-\Delta u = g$ with Dirichlet boundary conditions in a bounded domain Ω in $\mathbf{R}^2$. The weak formulation consists of finding a function u such that, for all continuously differentiable functions v in Ω vanishing on the boundary:

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} g v.$$

This can be recast in terms of the Hilbert space $H_0^1(\Omega)$ consisting of functions u such that u , along with its weak partial derivatives, are square integrable on Ω , and which vanish on the boundary. The question then reduces to finding u in this space such that for all v in this space

$$a(u, v) = b(v)$$

where a is a continuous bilinear form, and b is a continuous linear functional, given respectively by

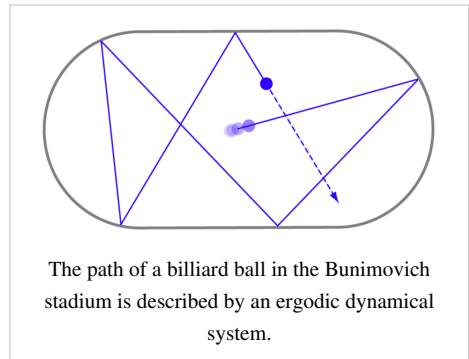
$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v, \quad b(v) = \int_{\Omega} g v.$$

Since the Poisson equation is elliptic, it follows from Poincaré's inequality that the bilinear form a is coercive. The Lax–Milgram theorem then ensures the existence and uniqueness of solutions of this equation.

Hilbert spaces allow for many elliptic partial differential equations to be formulated in a similar way, and the Lax–Milgram theorem is then a basic tool in their analysis. With suitable modifications, similar techniques can be applied to parabolic partial differential equations and certain hyperbolic partial differential equations.

Ergodic theory

The field of ergodic theory is the study of the long-term behavior of chaotic dynamical systems. The prototypical case of a field to which ergodic theory is applicable is that of thermodynamics in which, although the microscopic state of a system is extremely complicated—it is impossible to understand the ensemble of individual collisions between particles of matter—the average behavior over sufficiently long time intervals is tractable. The laws of thermodynamics are assertions about such average behavior. In particular, one formulation of the zeroth law of thermodynamics asserts that over sufficiently long timescales, the only functionally independent measurement that one can make of a thermodynamic system in equilibrium is its total energy, in the form of temperature.



An ergodic dynamical system is one for which, apart from the energy—measured by the Hamiltonian—there are no other functionally independent conserved quantities on the phase space. More explicitly, suppose that the energy E is fixed, and let Ω_E be the subset of the phase space consisting of all states of energy E (an energy surface), and let T_t denote the evolution operator on the phase space. The dynamical system is ergodic if there are no continuous non-constant functions on Ω_E such that

$$f(T_t w) = f(w)$$

for all w on Ω_E and all time t . Liouville's theorem implies that there exists a measure μ on the energy surface that is invariant under the time translation. As a result, time translation is a unitary transformation of the Hilbert space $L^2(\Omega_E, \mu)$ consisting of square-integrable functions on the energy surface Ω_E with respect to the inner product

$$\langle f, g \rangle_{L^2(\Omega_E, \mu)} = \int_E f \bar{g} d\mu.$$

The von Neumann mean ergodic theorem^[19] states the following:

- If U_t is a (strongly continuous) one-parameter semigroup of unitary operators on a Hilbert space H , and P is the orthogonal projection onto the space of common fixed points of U_t , $\{x \in H \mid U_t x = x \text{ for all } t > 0\}$, then

$$Px = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T U_t x dt.$$

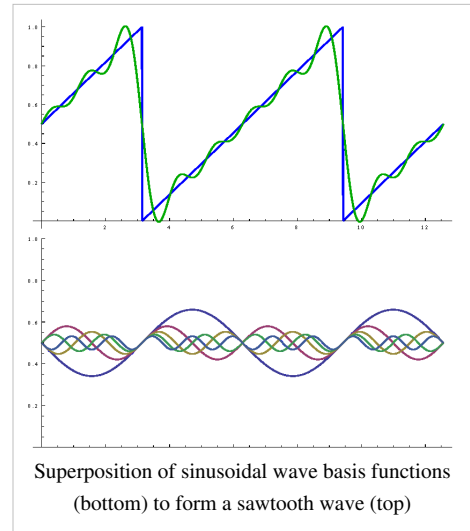
For an ergodic system, the fixed set of the time evolution consists only of the constant functions, so the ergodic theorem implies the following:^[32] for any function $f \in L^2(\Omega_E, \mu)$,

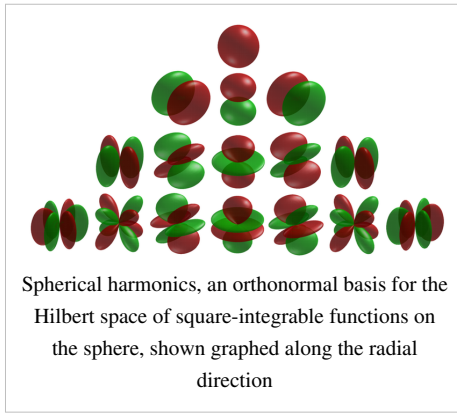
$$L^2\text{-}\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(T_t w) dt = \int_{\Omega_E} f(y) d\mu(y).$$

That is, the long time average of an observable f is equal to its expectation value over an energy surface.

Fourier analysis

One of the basic goals of Fourier analysis is to decompose a function into a (possibly infinite) linear combination of given basis functions: the associated Fourier series. The classical Fourier series associated to a function f defined on the interval $[0,1]$ is a series of the form





$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n \theta}$$

where

$$a_n = \int_0^1 f(\theta) e^{-2\pi i n \theta} d\theta.$$

The example of adding up the first few terms in a Fourier series for a sawtooth function is shown in the figure. The basis functions are sine waves with wavelengths λ/n (n =integer) shorter than the wavelength λ of the sawtooth itself (except for $n=1$, the *fundamental* wave). All basis functions have nodes at the nodes of the sawtooth, but all but the fundamental have additional nodes. The oscillation of the summed terms about the sawtooth is called the Gibbs phenomenon.

A significant problem in classical Fourier series asks in what sense the Fourier series converges, if at all, to the function f . Hilbert space methods provide one possible answer to this question.^[33] The functions $e_n(\theta) = e^{2\pi i n \theta}$ form an orthogonal basis of the Hilbert space $L^2([0,1])$. Consequently, any square-integrable function can be expressed as a series

$$f(\theta) = \sum_n a_n e_n(\theta), \quad a_n = \langle f, e_n \rangle$$

and, moreover, this series converges in the Hilbert space sense (that is, in the L^2 mean).

The problem can also be studied from the abstract point of view: every Hilbert space has an orthonormal basis, and every element of the Hilbert space can be written in a unique way as a sum of multiples of these basis elements. The coefficients appearing on these basis elements are sometimes known abstractly as the Fourier coefficients of the element of the space.^[34] The abstraction is especially useful when it is more natural to use different basis functions for a space such as $L^2([0,1])$. In many circumstances, it is desirable not to decompose a function into trigonometric functions, but rather into orthogonal polynomials or wavelets for instance,^[35] and in higher dimensions into spherical harmonics.^[36]

For instance, if e_n are any orthonormal basis functions of $L^2[0,1]$, then a given function in $L^2[0,1]$ can be approximated as a finite linear combination^[37]

$$f(x) \approx f_n(x) = a_1 e_1(x) + a_2 e_2(x) + \cdots + a_n e_n(x)$$

The coefficients $\{a_j\}$ are selected to make the magnitude of the difference $\|f - f_n\|^2$ as small as possible. Geometrically, the best approximation is the orthogonal projection of f onto the subspace consisting of all linear combinations of the $\{e_j\}$, and can be calculated by^[38]

$$a_j = \int_0^1 \overline{e_j(x)} f(x) dx.$$

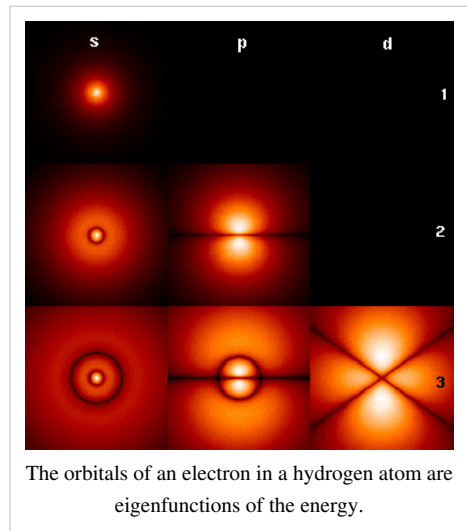
That this formula minimizes the difference $\|f - f_n\|^2$ is a consequence of Bessel's inequality and Parseval's formula.

In various applications to physical problems, a function can be decomposed into physically meaningful eigenfunctions of a differential operator (typically the Laplace operator): this forms the foundation for the spectral study of functions, in reference to the spectrum of the differential operator.^[39] A concrete physical application involves the problem of hearing the shape of a drum: given the fundamental modes of vibration that a drumhead is capable of producing, can one infer the shape of the drum itself?^[40] The mathematical formulation of this question involves the Dirichlet eigenvalues of the Laplace equation in the plane, that represent the fundamental modes of vibration in direct analogy with the integers that represent the fundamental modes of vibration of the violin string.

Spectral theory also underlies certain aspects of the Fourier transform of a function. Whereas Fourier analysis decomposes a function defined on a compact set into the discrete spectrum of the Laplacian (which corresponds to the vibrations of a violin string or drum), the Fourier transform of a function is the decomposition of a function defined on all of Euclidean space into its components in the continuous spectrum of the Laplacian. The Fourier transformation is also geometrical, in a sense made precise by the Plancherel theorem, that asserts that it is an isometry of one Hilbert space (the "time domain") with another (the "frequency domain"). This isometry property of the Fourier transformation is a recurring theme in abstract harmonic analysis, as evidenced for instance by the Plancherel theorem for spherical functions occurring in noncommutative harmonic analysis.

Quantum mechanics

In the mathematically rigorous formulation of quantum mechanics, developed by Paul Dirac^[41] and John von Neumann^[42], the possible states (more precisely, the pure states) of a quantum mechanical system are represented by unit vectors (called *state vectors*) residing in a complex separable Hilbert space, known as the state space, well defined up to a complex number of norm 1 (the phase factor). In other words, the possible states are points in the projectivization of a Hilbert space, usually called the complex projective space. The exact nature of this Hilbert space is dependent on the system; for example, the position and momentum states for a single non-relativistic spin zero particle is the space of all square-integrable functions, while the states for the spin of a single proton are unit elements of the two-dimensional complex Hilbert space of spinors. Each observable is represented by a self-adjoint linear operator acting on the state space. Each eigenstate of an observable corresponds to an eigenvector of the operator, and the associated eigenvalue corresponds to the value of the observable in that eigenstate.



The time evolution of a quantum state is described by the Schrödinger equation, in which the Hamiltonian, the operator corresponding to the total energy of the system, generates time evolution.

The inner product between two state vectors is a complex number known as a probability amplitude. During an ideal measurement of a quantum mechanical system, the probability that a system collapses from a given initial state to a particular eigenstate is given by the square of the absolute value of the probability amplitudes between the initial and final states. The possible results of a measurement are the eigenvalues of the operator—which explains the choice of self-adjoint operators, for all the eigenvalues must be real. The probability distribution of an observable in a given state can be found by computing the spectral decomposition of the corresponding operator.

For a general system, states are typically not pure, but instead are represented as statistical mixtures of pure states, or mixed states, given by density matrices: self-adjoint operators of trace one on a Hilbert space. Moreover, for general quantum mechanical systems, the effects of a single measurement can influence other parts of a system in a manner that is described instead by a positive operator valued measure. Thus the structure both of the states and observables

in the general theory is considerably more complicated than the idealization for pure states.

Heisenberg's uncertainty principle is represented by the statement that the operators corresponding to certain observables do not commute, and gives a specific form that the commutator must have.

Properties

Pythagorean identity

Two vectors u and v in a Hilbert space H are orthogonal when $\langle u, v \rangle = 0$. The notation for this is $u \perp v$. More generally, when S is a subset in H , the notation $u \perp S$ means that u is orthogonal to every element from S .

When u and v are orthogonal, one has

$$\|u + v\|^2 = \langle u + v, u + v \rangle = \langle u, u \rangle + 2\operatorname{Re}\langle u, v \rangle + \langle v, v \rangle = \|u\|^2 + \|v\|^2.$$

By induction on n , this is extended to any family $u_1, \dots, u_n$ of n orthogonal vectors,

$$\|u_1 + \dots + u_n\|^2 = \|u_1\|^2 + \dots + \|u_n\|^2.$$

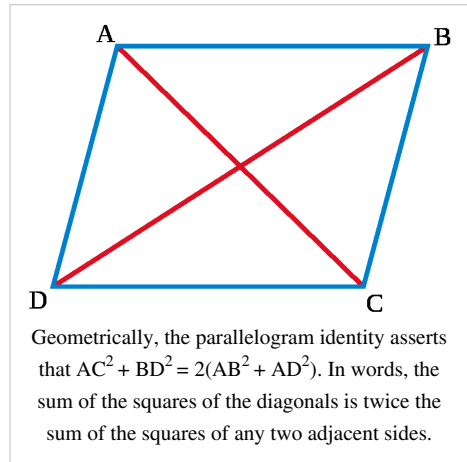
Whereas the Pythagorean identity as stated is valid in any inner product space, completeness is required for the extension of the Pythagorean identity to series. A series $\sum u_k$ of *orthogonal* vectors converges in H if and only if the series of squares of norms converges, and

$$\left\| \sum_{k=0}^{\infty} u_k \right\|^2 = \sum_{k=0}^{\infty} \|u_k\|^2.$$

Furthermore, the sum of a series of orthogonal vectors is independent of the order in which it is taken.

Parallelogram identity and polarization

By definition, every Hilbert space is also a Banach space. Furthermore, in every Hilbert space the following parallelogram identity holds:



$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Conversely, every Banach space in which the parallelogram identity holds is a Hilbert space, and the inner product is uniquely determined by the norm by the polarization identity.^[43] For real Hilbert spaces, the polarization identity is

$$\langle u, v \rangle = \frac{1}{4} (\|u + v\|^2 - \|u - v\|^2).$$

For complex Hilbert spaces, it is

$$\langle u, v \rangle = \frac{1}{4} (\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2).$$

The parallelogram law implies that any Hilbert space is a uniformly convex Banach space.^[44]

Best approximation

If C is a non-empty closed convex subset of a Hilbert space H and x a point in H , there exists a unique point $y \in C$ which minimizes the distance between x and points in C ,^[45]

$$y \in C, \quad \|x - y\| = \text{dist}(x, C) = \min\{\|x - z\| : z \in C\}.$$

This is equivalent to saying that there is a point with minimal norm in the translated convex set $D = C - x$. The proof consists in showing that every minimizing sequence $(d_n) \subset D$ is Cauchy (using the parallelogram identity) hence converges (using completeness) to a point in D that has minimal norm. More generally, this holds in any uniformly convex Banach space.^[46]

When this result is applied to a closed subspace F of H , it can be shown that the point $y \in F$ closest to x is characterized by^[47]

$$y \in F, \quad x - y \perp F.$$

This point y is the *orthogonal projection* of x onto F , and the mapping $P_F : x \rightarrow y$ is linear (see Orthogonal complements and projections). This result is especially significant in applied mathematics, especially numerical analysis, where it forms the basis of least squares methods.

In particular, when F is not equal to H , one can find a non-zero vector v orthogonal to F (select x not in F and $v = x - y$). A very useful criterion is obtained by applying this observation to the closed subspace F generated by a subset S of H .

A subset S of H spans a dense vector subspace if (and only if) the vector 0 is the sole vector $v \in H$ orthogonal to S .

Duality

The dual space H^* is the space of all continuous linear functions from the space H into the base field. It carries a natural norm, defined by

$$\|\varphi\| = \sup_{\|x\|=1, x \in H} |\varphi(x)|.$$

This norm satisfies the parallelogram law, and so the dual space is also an inner product space. The dual space is also complete, and so it is a Hilbert space in its own right.

The Riesz representation theorem affords a convenient description of the dual. To every element u of H , there is a unique element φ_u of H^* , defined by

$$\varphi_u(x) = \langle x, u \rangle.$$

The mapping $u \mapsto \varphi_u$ is an antilinear mapping from H to H^* . The Riesz representation theorem states that this mapping is an antilinear isomorphism.^[48] Thus to every element φ of the dual H^* there exists one and only one u_φ in H such that

$$\langle x, u_\varphi \rangle = \varphi(x)$$

for all $x \in H$. The inner product on the dual space H^* satisfies

$$\langle \varphi, \psi \rangle = \langle u_\psi, u_\varphi \rangle.$$

The reversal of order on the right-hand side restores linearity in φ from the antilinearity of u_φ . In the real case, the antilinear isomorphism from H to its dual is actually an isomorphism, and so real Hilbert spaces are naturally isomorphic to their own duals.

The representing vector u_φ is obtained in the following way. When $\varphi \neq 0$, the kernel $F = \ker \varphi$ is a closed vector subspace of H , not equal to H , hence there exists a non-zero vector v orthogonal to F . The vector u is a suitable scalar multiple λv of v . The requirement that $\varphi(v) = \langle v, u \rangle$ yields

$$u = \langle v, v \rangle^{-1} \overline{\varphi(v)} v.$$

This correspondence $\varphi \leftrightarrow u$ is exploited by the bra-ket notation popular in physics. It is common in physics to assume that the inner product, denoted by $\langle x|y\rangle$, is linear on the right,

$$\langle x|y\rangle = \langle y, x\rangle.$$

The result $\langle x|y\rangle$ can be seen as the action of the linear functional $\langle x|$ (the *bra*) on the vector $|y\rangle$ (the *ket*).

The Riesz representation theorem relies fundamentally not just on the presence of an inner product, but also on the completeness of the space. In fact, the theorem implies that the topological dual of any inner product space can be identified with its completion. An immediate consequence of the Riesz representation theorem is also that a Hilbert space H is reflexive, meaning that the natural map from H into its double dual space is an isomorphism.

Weakly convergent sequences

In a Hilbert space H , a sequence $\{x_n\}$ is weakly convergent to a vector $x \in H$ when

$$\lim_n \langle x_n, v \rangle = \langle x, v \rangle$$

for every $v \in H$.

For example, any orthonormal sequence $\{f_n\}$ converges weakly to 0, as a consequence of Bessel's inequality. Every weakly convergent sequence $\{x_n\}$ is bounded, by the uniform boundedness principle.

Conversely, every bounded sequence in a Hilbert space admits weakly convergent subsequences (Alaoglu's theorem).^[49] This fact may be used to prove minimization results for continuous convex functionals, in the same way that the Bolzano-Weierstrass theorem is used for continuous functions on $\mathbf{R}^d$. Among several variants, one simple statement is as follows:^[50]

If $f: H \rightarrow \mathbf{R}$ is a convex continuous function such that $f(x)$ tends to $+\infty$ when $\|x\|$ tends to ∞ , then f admits a minimum at some point $x_0 \in H$.

This fact (and its various generalizations) are fundamental for direct methods in the calculus of variations. Minimization results for convex functionals are also a direct consequence of the slightly more abstract fact that closed bounded convex subsets in a Hilbert space H are weakly compact, since H is reflexive. The existence of weakly convergent subsequences is a special case of the Eberlein-Šmulian theorem.

Banach space properties

Any general property of Banach spaces continues to hold for Hilbert spaces. The open mapping theorem states that a continuous surjective linear transformation from one Banach space to another is an open mapping meaning that it sends open sets to open sets. A corollary is the bounded inverse theorem, that a continuous and bijective linear function from one Banach space to another is an isomorphism (that is, a continuous linear map whose inverse is also continuous). This theorem is considerably simpler to prove in the case of Hilbert spaces than in general Banach spaces.^[51] The open mapping theorem is equivalent to the closed graph theorem, which asserts that a function from one Banach space to another is continuous if and only if its graph is a closed set.^[52] In the case of Hilbert spaces, this is basic in the study of unbounded operators (see closed operator).

The (geometrical) Hahn-Banach theorem asserts that a closed convex set can be separated from any point outside it by means of a hyperplane of the Hilbert space. This is an immediate consequence of the best approximation property: if y is the element of a closed convex set F closest to x , then the separating hyperplane is the plane perpendicular to the segment xy passing through its midpoint.^[53]

The distortion problem on Hilbert space asks whether or not every real valued Lipschitz function f defined on the sphere of a separable and infinite dimensional Hilbert space H stabilizes on the sphere of an infinite dimensional subspace, i.e. whether there is a real number $a \in \mathbf{R}$ so that for every $\delta > 0$ there is an infinite dimensional subspace Y of H , so that $|a - f(y)| < \delta$, for all $y \in Y$, with $\|y\|=1$. This problem was solved negatively by E. Odell and Th. Schlumprecht (1994).

Operators on Hilbert spaces

Bounded operators

The continuous linear operators $A : H_1 \rightarrow H_2$ from a Hilbert space H_1 to a second Hilbert space H_2 are *bounded* in the sense that they map bounded sets to bounded sets. Conversely, if an operator is bounded, then it is continuous. The space of such bounded linear operators has a norm, the operator norm given by

$$\|A\| = \sup \{ \|Ax\| : \|x\| \leq 1 \}.$$

The sum and the composite of two bounded linear operators is again bounded and linear. For y in H_2 , the map that sends $x \in H_1$ to $\langle Ax, y \rangle$ is linear and continuous, and according to the Riesz representation theorem can therefore be represented in the form

$$\langle x, A^*y \rangle = \langle Ax, y \rangle$$

for some vector A^*y in H_1 . This defines another bounded linear operator $A^* : H_2 \rightarrow H_1$, the adjoint of A . One can see that $A^{**} = A$.

The set $B(H)$ of all bounded linear operators on H , together with the addition and composition operations, the norm and the adjoint operation, is a C^* -algebra, which is a type of operator algebra.

An element A of $B(H)$ is called *self-adjoint* or *Hermitian* if $A^* = A$. If A is Hermitian and $\langle Ax, x \rangle \geq 0$ for every x , then A is called *non-negative*, written $A \geq 0$; if equality holds only when $x = 0$, then A is called *positive*. The set of self adjoint operators admits a partial order, in which $A \geq B$ if $A - B \geq 0$. If A has the form B^*B for some B , then A is non-negative; if B is invertible, then A is positive. A converse is also true in the sense that, for a non-negative operator A , there exists a unique non-negative square root B such that

$$A = B^2 = B^*B.$$

In a sense made precise by the spectral theorem, self-adjoint operators can usefully be thought of as operators that are "real". An element A of $B(H)$ is called *normal* if $A^*A = AA^*$. Normal operators decompose into the sum of a self-adjoint operators and an imaginary multiple of a self adjoint operator

$$A = \frac{A + A^*}{2} + i \frac{(A - A^*)}{2i}$$

that commute with each other. Normal operators can also usefully be thought of in terms of their real and imaginary parts.

An element U of $B(H)$ is called *unitary* if U is invertible and its inverse is given by U^* . This can also be expressed by requiring that U be onto and $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all x and y in H . The unitary operators form a group under composition, which is the isometry group of H .

An element of $B(H)$ is *compact* if it sends bounded sets to relatively compact sets. Equivalently, a bounded operator T is compact if, for any bounded sequence $\{x_k\}$, the sequence $\{Tx_k\}$ has a convergent subsequence. Many integral operators are compact, and in fact define a special class of operators known as Hilbert–Schmidt operators that are especially important in the study of integral equations. Fredholm operators are those which differ from a compact operator by a multiple of the identity, and are equivalently characterized as operators with a finite dimensional kernel and cokernel. The index of a Fredholm operator T is defined by

$$\text{index } T = \dim \ker T - \dim \text{coker } T.$$

The index is homotopy invariant, and plays a deep role in differential geometry via the Atiyah–Singer index theorem.

Unbounded operators

Unbounded operators are also tractable in Hilbert spaces, and have important applications to quantum mechanics.^[54] An unbounded operator T on a Hilbert space H is defined to be a linear operator whose domain $D(T)$ is a linear subspace of H . Often the domain $D(T)$ is a dense subspace of H , in which case T is known as a densely-defined operator.

The adjoint of a densely defined unbounded operator is defined in essentially the same manner as for bounded operators. Self-adjoint unbounded operators play the role of the *observables* in the mathematical formulation of quantum mechanics. Examples of self-adjoint unbounded operators on the Hilbert space $L^2(\mathbf{R})$ are:^[55]

- A suitable extension of the differential operator

$$(Af)(x) = i \frac{d}{dx} f(x),$$

where i is the imaginary unit and f is a differentiable function of compact support.

- The multiplication-by- x operator:

$$(Bf)(x) = xf(x).$$

These correspond to the momentum and position observables, respectively. Note that neither A nor B is defined on all of H , since in the case of A the derivative need not exist, and in the case of B the product function need not be square integrable. In both cases, the set of possible arguments form dense subspaces of $L^2(\mathbf{R})$.

Constructions

Direct sums

Two Hilbert spaces H_1 and H_2 can be combined into another Hilbert space, called the (orthogonal) direct sum,^[56] and denoted

$$H_1 \oplus H_2,$$

consisting of the set of all ordered pairs (x_1, x_2) where $x_i \in H_i$, $i = 1, 2$, and inner product defined by

$$\langle (x_1, x_2), (y_1, y_2) \rangle_{H_1 \oplus H_2} = \langle x_1, y_1 \rangle_{H_1} + \langle x_2, y_2 \rangle_{H_2}.$$

More generally, if H_i is a family of Hilbert spaces indexed by $i \in I$, then the direct sum of the H_i , denoted

$$\bigoplus_{i \in I} H_i$$

consists of the set of all indexed families

$$x = (x_i \in H_i | i \in I) \in \prod_{i \in I} H_i$$

in the Cartesian product of the H_i such that

$$\sum_{i \in I} \|x_i\|^2 < \infty.$$

The inner product is defined by

$$\langle x, y \rangle = \sum_{i \in I} \langle x_i, y_i \rangle_{H_i}.$$

Each of the H_i is included as a closed subspace in the direct sum of all of the H_i . Moreover, the H_i are pairwise orthogonal. Conversely, if there is a system of closed subspaces V_i , $i \in I$, in a Hilbert space H which are pairwise orthogonal and whose union is dense in H , then H is canonically isomorphic to the direct sum of V_i . In this case, H is called the internal direct sum of the V_i . A direct sum (internal or external) is also equipped with a family of orthogonal projections E_i onto the i th direct summand H_i . These projections are bounded, self-adjoint, idempotent operators which satisfy the orthogonality condition

$$E_i E_j = 0, \quad i \neq j.$$

The spectral theorem for compact self-adjoint operators on a Hilbert space H states that H splits into an orthogonal direct sum of the eigenspaces of an operator, and also gives an explicit decomposition of the operator as a sum of projections onto the eigenspaces. The direct sum of Hilbert spaces also appears in quantum mechanics as the Fock space of a system containing a variable number of particles, where each Hilbert space in the direct sum corresponds to an additional degree of freedom for the quantum mechanical system. In representation theory, the Peter-Weyl theorem guarantees that any unitary representation of a compact group on a Hilbert space splits as the direct sum of finite-dimensional representations.

Tensor products

If H_1 and H_2 , then one defines an inner product on the (ordinary) tensor product as follows. On simple tensors, let

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle.$$

This formula then extends by sesquilinearity to an inner product on $H_1 \otimes H_2$. The Hilbertian tensor product of H_1 and H_2 , sometimes denoted by $H_1 \widehat{\otimes} H_2$, is the Hilbert space obtained by completing $H_1 \otimes H_2$ for the metric associated to this inner product.^[57]

An example is provided by the Hilbert space $L^2([0, 1])$. The Hilbertian tensor product of two copies of $L^2([0, 1])$ is isometrically and linearly isomorphic to the space $L^2([0, 1]^2)$ of square-integrable functions on the square $[0, 1]^2$. This isomorphism sends a simple tensor $f_1 \otimes f_2$ to the function

$$(s, t) \mapsto f_1(s) f_2(t)$$

on the square.

This example is typical in the following sense.^[58] Associated to every simple tensor product $x_1 \otimes x_2$ is the rank one operator

$$x^* \in H_1^* \rightarrow x^*(x_1) x_2$$

from the (continuous) dual H_1^* to H_2 . This mapping defined on simple tensors extends to a linear identification between $H_1 \otimes H_2$ and the space of finite rank operators from H_1^* to H_2 . This extends to a linear isometry of the Hilbertian tensor product $H_1 \widehat{\otimes} H_2$ with the Hilbert space $HS(H_1^*, H_2)$ of Hilbert-Schmidt operators from H_1^* to H_2 .

Orthonormal bases

The notion of an orthonormal basis from linear algebra generalizes over to the case of Hilbert spaces.^[59] In a Hilbert space H , an orthonormal basis is a family $\{e_k\}_{k \in B}$ of elements of H satisfying the conditions:

1. *Orthogonality*: Every two different elements of B are orthogonal: $\langle e_k, e_j \rangle = 0$ for all k, j in B with $k \neq j$.
2. *Normalization*: Every element of the family has norm 1: $\|e_k\| = 1$ for all k in B .
3. *Completeness*: The linear span of the family $e_k, k \in B$, is dense in H .

A system of vectors satisfying the first two conditions is called an orthonormal system or an orthonormal set (or an orthonormal sequence if B is countable). Such a system is always linearly independent. Completeness of an orthonormal system of vectors of a Hilbert space can be equivalently restated as:

$$\text{if } \langle v, e_k \rangle = 0 \text{ for all } k \in B \text{ and some } v \in H \text{ then } v = 0.$$

This is related to the fact that the only vector orthogonal to a dense linear subspace is the zero vector, for if S is any orthonormal set and v is orthogonal to S , then v is orthogonal to the closure of the linear span of S , which is the whole space.

Examples of orthonormal bases include:

- the set $\{(1,0,0), (0,1,0), (0,0,1)\}$ forms an orthonormal basis of $\mathbf{R}^3$ with the dot product;

- the sequence $\{f_n : n \in \mathbb{Z}\}$ with $f_n(x) = \exp(2\pi i n x)$ forms an orthonormal basis of the complex space $L^2([0,1])$;

In the infinite-dimensional case, an orthonormal basis will not be a basis in the sense of linear algebra; to distinguish the two, the latter basis is also called a Hamel basis. That the span of the basis vectors is dense implies that every vector in the space can be written as the sum of an infinite series, and the orthogonality implies that this decomposition is unique.

Sequence spaces

The space ℓ^2 of square-summable sequences of complex numbers has an orthonormal basis

$$e_1 = (1, 0, 0, \dots)$$

$$e_2 = (0, 1, 0, \dots)$$

$\vdots$

More generally, if B is any set, then one can form a Hilbert space of sequences with index set B , defined by

$$\ell^2(B) = \left\{ x : B \xrightarrow{x} \mathbb{C} \mid \sum_{b \in B} |x(b)|^2 < \infty \right\}.$$

The summation over B is here defined by

$$\sum_{b \in B} |x(b)|^2 = \sup \sum_{n=1}^N |x(b_n)|^2$$

the supremum being taken over all finite subsets of B . It follows that, in order for this sum to be finite, every element of $\ell^2(B)$ has only countably many nonzero terms. This space becomes a Hilbert space with the inner product

$$\langle x, y \rangle = \sum_{b \in B} x(b) \overline{y(b)}$$

for all x and y in $\ell^2(B)$. Here the sum also has only countably many nonzero terms, and is unconditionally convergent by the Cauchy–Schwarz inequality.

An orthonormal basis of $\ell^2(B)$ is indexed by the set B , given by

$$e_b(b') = \begin{cases} 1 & \text{if } b = b' \\ 0 & \text{otherwise.} \end{cases}$$

Bessel's inequality and Parseval's formula

Let $f_1, \dots, f_n$ be a finite orthonormal system in H . For an arbitrary vector x in H , let

$$y = \sum_{j=1}^n \langle x, f_j \rangle f_j.$$

Then $\langle x, f_k \rangle = \langle y, f_k \rangle$ for every $k = 1, \dots, n$. It follows that $x - y$ is orthogonal to each f_k , hence $x - y$ is orthogonal to y . Using the Pythagorean identity twice, it follows that

$$\|x\|^2 = \|x - y\|^2 + \|y\|^2 \geq \|y\|^2 = \sum_{j=1}^n |\langle x, f_j \rangle|^2.$$

Let $\{f_i\}$, $i \in I$, be an arbitrary orthonormal system in H . Applying the preceding inequality to every finite subset J of I gives the *Bessel inequality*^[60]

$$\sum_{i \in I} |\langle x, f_i \rangle|^2 \leq \|x\|^2, \quad x \in H$$

(according to the definition of the sum of an arbitrary family of non-negative real numbers).

Geometrically, Bessel's inequality implies that the orthogonal projection of x onto the linear subspace spanned by the f_i has norm that does not exceed that of x . In two dimensions, this is the assertion that the length of the leg of a right

triangle may not exceed the length of the hypotenuse.

Bessel's inequality is a stepping stone to the more powerful Parseval identity which governs the case when Bessel's inequality is actually an equality. If $\{e_k\}_{k \in B}$ is an orthonormal basis of H , then every element x of H may be written as

$$x = \sum_{k \in B} \langle x, e_k \rangle e_k.$$

Even if B is uncountable, Bessel's inequality guarantees that the expression is well-defined and consists only of countably many nonzero terms. This sum is called the *Fourier expansion* of x , and the individual coefficients $\langle x, e_k \rangle$ are the *Fourier coefficients* of x . Parseval's formula is then

$$\|x\|^2 = \sum_{k \in B} |\langle x, e_k \rangle|^2.$$

Conversely, if $\{e_k\}$ is an orthonormal set such that Parseval's identity holds for every x , then $\{e_k\}$ is an orthonormal basis.

Hilbert dimension

As a consequence of Zorn's lemma, every Hilbert space admits an orthonormal basis; furthermore, any two orthonormal bases of the same space have the same cardinality, called the Hilbert dimension of the space.^[61] For instance, since $\ell^2(B)$ has an orthonormal basis indexed by B , its Hilbert dimension is the cardinality of B (which may be a finite integer, or a countable or uncountable cardinal number).

As a consequence of Parseval's identity, if $\{e_k\}_{k \in B}$ is an orthonormal basis of H , then the map $\Phi : H \rightarrow \ell^2(B)$ defined by $\Phi(x) = (\langle x, e_k \rangle)_{k \in B}$ is an isometric isomorphism of Hilbert spaces: it is a bijective linear mapping such that

$$\langle \Phi(x), \Phi(y) \rangle_{\ell^2(B)} = \langle x, y \rangle_H$$

for all x and y in H . The cardinal number of B is the Hilbert dimension of H . Thus every Hilbert space is isometrically isomorphic to a sequence space $\ell^2(B)$ for some set B .

Separable spaces

A Hilbert space is separable if and only if it admits a countable orthonormal basis. All infinite-dimensional separable Hilbert spaces are therefore isometrically isomorphic to ℓ^2 .

In the past, Hilbert spaces were often required to be separable as part of the definition.^[62] Most spaces used in physics are separable, and since these are all isomorphic to each other, one often refers to any infinite-dimensional separable Hilbert space as "the Hilbert space" or just "Hilbert space".^[63] Even in quantum field theory, most of the Hilbert spaces are in fact separable, as stipulated by the Wightman axioms. However, it is sometimes argued that non-separable Hilbert spaces are also important in quantum field theory, roughly because the systems in the theory possess an infinite number of degrees of freedom and any infinite Hilbert tensor product (of spaces of dimension greater than one) is non-separable.^[64] For instance, a bosonic field can be naturally thought of as an element of a tensor product whose factors represent harmonic oscillators at each point of space. From this perspective, the natural state space of a boson might seem to be a non-separable space.^[64] However, it is only a small separable subspace of the full tensor product that can contain physically meaningful fields (on which the observables can be defined). Another non-separable Hilbert space models the state of an infinite collection of particles in an unbounded region of space. An orthonormal basis of the space is indexed by the density of the particles, a continuous parameter, and since the set of possible densities is uncountable, the basis is not countable.^[64]

Orthogonal complements and projections

If S is a subset of a Hilbert space H , the set of vectors orthogonal to S is defined by

$$S^\perp = \{x \in H : \langle x, s \rangle = 0 \ \forall s \in S\}.$$

$S^\perp$ is a closed subspace of H and so forms itself a Hilbert space. If V is a closed subspace of H , then $V^\perp$ is called the *orthogonal complement* of V . In fact, every x in H can then be written uniquely as $x = v + w$, with v in V and w in $V^\perp$. Therefore, H is the internal Hilbert direct sum of V and $V^\perp$.

The linear operator $P_V : H \rightarrow H$ which maps x to v is called the *orthogonal projection* onto V . There is a natural one-to-one correspondence between the set of all closed subspaces of H and the set of all bounded self-adjoint operators P such that $P^2 = P$. Specifically,

Theorem. The orthogonal projection P_V is a self-adjoint linear operator on H of norm ≤ 1 with the property $P_V^2 = P_V$. Moreover, any self-adjoint linear operator E such that $E^2 = E$ is of the form P_V , where V is the range of E . For every x in H , $P_V(x)$ is the unique element v of V which minimizes the distance $\|x - v\|$.

This provides the geometrical interpretation of $P_V(x)$: it is the best approximation to x by elements of V .^[65]

An operator P such that $P = P^2 = P^*$ is called an orthogonal projection. The orthogonal projection P_V onto a closed subspace V of H is the adjoint of the inclusion mapping

$$i_V : V \rightarrow H,$$

meaning that

$$\langle i_V x, y \rangle = \langle x, P_V y \rangle$$

for all $x \in H$ and $y \in V$. Projections P_U and P_V are called mutually orthogonal if $P_U P_V = 0$. This is equivalent to U and V being orthogonal as subspaces of H . As a result, the sum of the two projections P_U and P_V is only a projection if U and V are orthogonal to each other, and in that case $P_U + P_V = P_{U+V}$. The composite $P_U P_V$ is generally not a projection; in fact, the composite is a projection if and only if the two projections commute, and in that case $P_U P_V = P_{U \cap V}$.

The operator norm of a projection P onto a non-zero closed subspace is equal to one:

$$\|P\| = \sup_{x \in H, x \neq 0} \frac{\|Px\|}{\|x\|} = 1.$$

Every closed subspace V of a Hilbert space is therefore the image of an operator P of norm one such that $P^2 = P$. In fact this property characterizes Hilbert spaces.^[66]

- A Banach space of dimension higher than 2 is (isometrically) a Hilbert space if and only if, to every closed subspace V , there is an operator P_V of norm one whose image is V such that $P_V^2 = P_V$.

While this result characterizes the metric structure of a Hilbert space, the structure of a Hilbert space as a topological vector space can itself be characterized in terms of the presence of complementary subspaces.^[67]

- A Banach space X is topologically and linearly isomorphic to a Hilbert space if and only if, to every closed subspace V , there is a closed subspace W such that X is equal to the internal direct sum $V \oplus W$.

The orthogonal complement satisfies some more elementary results. It is a monotone function in the sense that if $U \subset V$, then $V^\perp \subseteq U^\perp$ with equality holding if and only if V is contained in the closure of U . This result is a special case of the Hahn–Banach theorem. The closure of a subspace can be completely characterized in terms of the orthogonal complement: If V is a subspace of H , then the closure of V is equal to $V^{\perp\perp}$. The orthogonal complement is thus a Galois connection on the partial order of subspaces of a Hilbert space. In general, the orthogonal complement of a sum of subspaces is the intersection of the orthogonal complements.^[68] $(\sum_i V_i)^\perp = \bigcap_i V_i^\perp$. If the V_i are in addition closed, then $\overline{\sum_i V_i} = (\bigcap_i V_i)^\perp$.

Spectral theory

There is a well-developed spectral theory for self-adjoint operators in a Hilbert space, that is roughly analogous to the study of symmetric matrices over the reals or self-adjoint matrices over the complex numbers.^[69] In the same sense, one can obtain a "diagonalization" of a self-adjoint operator as a suitable sum (actually an integral) of orthogonal projection operators.

The spectrum of an operator T , denoted $\sigma(T)$ is the set of complex numbers λ such that $T - \lambda$ lacks a continuous inverse. If T is bounded, then the spectrum is always a compact set in the complex plane, and lies inside the disc $|z| \leq \|T\|$. If T is self-adjoint, then the spectrum is real. In fact, it is contained in the interval $[m, M]$ where

$$m = \inf_{\|x\|=1} \langle Tx, x \rangle, \quad M = \sup_{\|x\|=1} \langle Tx, x \rangle.$$

Moreover, m and M are both actually contained within the spectrum.

The eigenspaces of an operator T are given by

$$H_\lambda = \ker(T - \lambda).$$

Unlike with finite matrices, not every element of the spectrum of T must be an eigenvalue: the linear operator $T - \lambda$ may only lack an inverse because it is not surjective. Elements of the spectrum of an operator in the general sense are known as *spectral values*. Since spectral values need not be eigenvalues, the spectral decomposition is often more subtle than in finite dimensions.

However, the spectral theorem of a self-adjoint operator T takes a particularly simple form if, in addition, T is assumed to be a compact operator. The spectral theorem for compact self-adjoint operators states:^[70]

- A compact self-adjoint operator T has only countably (or finitely) many spectral values. The spectrum of T has no limit point in the complex plane except possibly zero. The eigenspaces of T decompose H into an orthogonal direct sum:

$$H = \bigoplus_{\lambda \in \sigma(T)} H_\lambda.$$

Moreover, if E_λ denotes the orthogonal projection onto the eigenspace H_λ , then

$$T = \sum_{\lambda \in \sigma(T)} \lambda E_\lambda$$

where the sum converges with respect to the norm on $B(H)$.

This theorem plays a fundamental role in the theory of integral equations, as many integral operators are compact, in particular those that arise from Hilbert-Schmidt operators.

The general spectral theorem for self-adjoint operators involves a kind of operator-valued Riemann–Stieltjes integral, rather than an infinite summation.^[71] The *spectral family* associated to T associates to each real number λ an operator E_λ , which is the projection onto the nullspace of the operator $(T - \lambda)^+$, where the positive part of a self-adjoint operator is defined by

$$A^+ = \frac{1}{2} (\sqrt{A^2} + A).$$

The operators E_λ are monotone increasing relative to the partial order defined on self-adjoint operators; the eigenvalues correspond precisely to the jump discontinuities. One has the spectral theorem, which asserts

$$T = \int_{\mathbb{R}} \lambda dE_\lambda.$$

The integral is understood as a Riemann–Stieltjes integral, convergent with respect to the norm on $B(H)$. In particular, one has the ordinary scalar-valued integral representation

$$\langle Tx, y \rangle = \int_{\mathbb{R}} \lambda d\langle E_\lambda x, y \rangle.$$

A somewhat similar spectral decomposition holds for normal operators, although because the spectrum may now contain non-real complex numbers, the operator-valued Stieltjes measure dE_λ must instead be replaced by a resolution of the identity.

A major application of spectral methods is the spectral mapping theorem, which allows one to apply to a self-adjoint operator T any continuous complex function f defined on the spectrum of T by forming the integral

$$f(T) = \int_{\sigma(T)} f(\lambda) dE_\lambda.$$

The resulting continuous functional calculus has applications in particular to pseudodifferential operators.^[72]

The spectral theory of *unbounded* self-adjoint operators is only marginally more difficult than for bounded operators. The spectrum of an unbounded operator is defined in precisely the same way as for bounded operators: λ is a spectral value if the resolvent operator

$$R_\lambda = (T - \lambda)^{-1}$$

fails to be a well-defined continuous operator. The self-adjointness of T still guarantees that the spectrum is real. Thus the essential idea of working with unbounded operators is to look instead at the resolvent R_λ where λ is non-real. This is a *bounded* normal operator, which admits a spectral representation that can then be transferred to a spectral representation of T itself. A similar strategy is used, for instance, to study the spectrum of the Laplace operator: rather than address the operator directly, one instead looks at an associated resolvent such as a Riesz potential or Bessel potential.

A precise version of the spectral theorem which holds in this case is:^[73]

Given a densely-defined self-adjoint operator T on a Hilbert space H , there corresponds a unique resolution of the identity E on the Borel sets of $\mathbf{R}$, such that

$$\langle Tx, y \rangle = \int_{\mathbf{R}} \lambda dE_{x,y}(\lambda)$$

for all $x \in D(T)$ and $y \in H$. The spectral measure E is concentrated on the spectrum of T .

There is also a version of the spectral theorem that applies to unbounded normal operators.

Notes

- [1] Marsden 1974, §2.8
- [2] The mathematical material in this section can be found in any good textbook on functional analysis, such as Dieudonné (1960), Hewitt & Stromberg (1965), Reed & Simon (1980) or Rudin (1980).
- [3] In some conventions, inner products are linear in their second arguments instead.
- [4] Dieudonné 1960, §6.2
- [5] Dieudonné 1960
- [6] Largely from the work of Hermann Grassmann, at the urging of August Ferdinand Möbius (Boyer & Merzbach 1991, pp. 584–586). The first modern axiomatic account of abstract vector spaces ultimately appeared in Giuseppe Peano's 1888 account (Grattan-Guinness 2000, §5.2.2; O'Connor & Robertson 1996).
- [7] A detailed account of the history of Hilbert spaces can be found in Bourbaki 1987.
- [8] Schmidt 1908
- [9] Titchmarsh 1946, §IX.1
- [10] Lebesgue 1904. Further details on the history of integration theory can be found in Bourbaki (1987) and Saks (2005).
- [11] Bourbaki 1987.
- [12] Dunford & Schwartz 1958, §IV.16
- [13] In Dunford & Schwartz (1958, §IV.16), the result that every linear functional on $L^2[0,1]$ is represented by integration is jointly attributed to Fréchet (1907) and Riesz (1907). The general result, that the dual of a Hilbert space is identified with the Hilbert space itself, can be found in Riesz (1934).
- [14] von Neumann 1929.
- [15] Kline 1972, p. 1092
- [16] Hilbert, Nordheim & von Neumann 1927.
- [17] Weyl 1931.

- [18] Prugovečki 1981, pp. 1–10.
- [19] von Neumann 1932
- [20] Halmos 1957, Section 42.
- [21] Hewitt & Stromberg 1965.
- [22] Bers, John & Schechter 1981.
- [23] Giusti 2003.
- [24] Stein 1970
- [25] Details can be found in Warner (1983).
- [26] A general reference on Hardy spaces is the book Duren (1970).
- [27] Krantz 2002, §1.4
- [28] Krantz 2002, §1.5
- [29] Young 1987, Chapter 9.
- [30] The eigenvalues of the Fredholm kernel are $1/\lambda$, which tend to zero.
- [31] More detail on finite element methods from this point of view can be found in Brenner & Scott (2005).
- [32] Reed & Simon 1980
- [33] A treatment of Fourier series from this point of view is available, for instance, in Rudin (1987) or Folland (2009).
- [34] Halmos 1957, §5
- [35] Bachman, Narici & Beckenstein 2000
- [36] Stein & Weiss 1971, §IV.2.
- [37] Lancos 1988, pp. 212–213
- [38] Lanczos 1988, Equation 4-3.10
- [39] The classic reference for spectral methods is Courant & Hilbert 1953. A more up-to-date account is Reed & Simon 1975.
- [40] Kac 1966
- [41] Dirac 1930
- [42] von Neumann 1955
- [43] Young 1988, p. 23.
- [44] Clarkson 1936.
- [45] Rudin 1987, Theorem 4.10
- [46] Dunford & Schwartz 1958, II.4.29
- [47] Rudin 1987, Theorem 4.11
- [48] Weidmann 1980, Theorem 4.8
- [49] Weidmann 1980, §4.5
- [50] Buttazzo, Giaquinta & Hildebrandt 1998, Theorem 5.17
- [51] Halmos 1982, Problem 52, 58
- [52] Rudin 1973
- [53] Trèves 1967, Chapter 18
- [54] See Prugovečki (1981), Reed & Simon (1980, Chapter VIII) and Folland (1989).
- [55] Prugovečki 1981, III, §1.4
- [56] Dunford & Schwartz 1958, IV.4.17-18
- [57] Weidmann 1980, §3.4
- [58] Kadison & Ringrose 1983, Theorem 2.6.4
- [59] Dunford & Schwartz 1958, §IV.4.
- [60] For the case of finite index sets, see, for instance, Halmos 1957, §5. For infinite index sets, see Weidmann 1980, Theorem 3.6.
- [61] Levitan 2001. Many authors, such as Dunford & Schwartz (1958, §IV.4), refer to this just as the dimension. Unless the Hilbert space is finite dimensional, this is not the same thing as its dimension as a linear space (the cardinality of a Hamel basis).
- [62] Prugovečki 1981, I, §4.2
- [63] von Neumann (1955) defines a Hilbert space via a countable Hilbert basis, which amounts to an isometric isomorphism with ℓ^2 . The convention still persists in most rigorous treatments of quantum mechanics; see for instance Sobrino 1996, Appendix B.
- [64] Streater & Wightman 1964, pp. 86–87
- [65] Young 1988, Theorem 15.3
- [66] Kakutani 1939
- [67] Lindenstrauss & Tzafriri 1971
- [68] Halmos 1957, §12
- [69] A general account of spectral theory in Hilbert spaces can be found in Riesz & Sz Nagy (1990). A more sophisticated account in the language of C^* -algebras is in Rudin (1973) or Kadison & Ringrose (1997)
- [70] See, for instance, Riesz & Sz Nagy (1990, Chapter VI) or Weidmann 1980, Chapter 7. This result was already known to Schmidt (1907) in the case of operators arising from integral kernels.
- [71] Riesz & Sz Nagy 1990, §§107–108
- [72] Shubin 1987

[73] Rudin 1973, Theorem 13.30.

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External links

- Hilbert Space at Mathworld (<http://mathworld.wolfram.com/HilbertSpace.html>)
- 245B, notes 5: Hilbert spaces (<http://terrytao.wordpress.com/2009/01/17/254a-notes-5-hilbert-spaces/>) by Terence Tao

Von Neumann algebra

In mathematics, a **von Neumann algebra** or **W*-algebra** is a *-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology and contains the identity operator. They were originally introduced by John von Neumann, motivated by the study of single operators, group representations, ergodic theory and quantum mechanics. His double commutant theorem shows that the analytic definition is equivalent to a purely algebraic definition as an algebra of symmetries.

Two basic examples of von Neumann algebras are as follows. The ring $L^\infty(\mathbf{R})$ of essentially bounded measurable functions on the real line is a commutative von Neumann algebra, which acts by pointwise multiplication on the Hilbert space $L^2(\mathbf{R})$ of square integrable functions. The algebra $B(H)$ of all bounded operators on a Hilbert space H is a von Neumann algebra, non-commutative if the Hilbert space has dimension at least 2.

Von Neumann algebras were first studied by von Neumann (1929); he and Francis Murray developed the basic theory, under the original name of **rings of operators**, in a series of papers written in the 1930s and 1940s (F.J. Murray & J. von Neumann 1936, 1937, 1943; J. von Neumann 1938, 1940, 1943, 1949), reprinted in the collected works of von Neumann (1961).

Introductory accounts of von Neumann algebras are given in the online notes of Jones (2003) and Wassermann (1991) and the books by Dixmier (1981), Schwartz (1967), Blackadar (2005) and Sakai (1971). The three volume work by Takesaki (1979) gives an encyclopedic account of the theory. The book by Connes (1994) discusses more advanced topics.

Definitions

There are three common ways to define von Neumann algebras.

The first and most common way is to define them as weakly closed *-algebras of bounded operators (on a Hilbert space) containing the identity. In this definition the weak (operator) topology can be replaced by many other common topologies including the strong, ultrastrong or ultraweak operator topologies. The *-algebras of bounded operators that are closed in the norm topology are C*-algebras, so in particular any von Neumann algebra is a C*-algebra.

The second definition is that a von Neumann algebra is a subset of the bounded operators closed under * and equal to its double commutant, or equivalently the commutant of some subset closed under *. The von Neumann double commutant theorem (von Neumann 1929) says that the first two definitions are equivalent.

The first two definitions describe a von Neumann algebras concretely as a set of operators acting on some given Hilbert space. Sakai (1971) showed that von Neumann algebras can also be defined abstractly as C*-algebras that have a predual; in other words the von Neumann algebra, considered as a Banach space, is the dual of some other Banach space called the predual. The predual of a von Neumann algebra is in fact unique up to isomorphism. Some authors use "von Neumann algebra" for the algebras together with a Hilbert space action, and "W*-algebra" for the abstract concept, so a von Neumann algebra is a W*-algebra together with a Hilbert space and a suitable faithful unital action on the Hilbert space. The concrete and abstract definitions of a von Neumann algebra are similar to the concrete and abstract definitions of a C*-algebra, which can be defined either as norm-closed *-algebras of operators on a Hilbert space, or as Banach *-algebras such that $\|a^*\| = \|a\|$ and $\|a^*a\| = \|a\|^2$.

Terminology

Some of the terminology in von Neumann algebra theory can be confusing, and the terms often have different meanings outside the subject.

- A **factor** is a von Neumann algebra with trivial center, i.e. a center consisting only of scalar operators.
- A **finite** von Neumann algebra is one which is the direct integral of finite factors. Similarly, **properly infinite** von Neumann algebras are the direct integral of properly infinite factors.
- A von Neumann algebra that acts on a separable Hilbert space is called **separable**. Note that such algebras are rarely separable in the norm topology.
- The von Neumann algebra **generated** by a set of bounded operators on a Hilbert space is the smallest von Neumann algebra containing all those operators.
- The **tensor product** of two von Neumann algebras acting on two Hilbert spaces is defined to be the von Neumann algebra generated by their algebraic tensor product, considered as operators on the Hilbert space tensor product of the Hilbert spaces.

By forgetting about the topology on a von Neumann algebra, we can consider it a (unital) $*$ -algebra, or just a ring. Von Neumann algebras are semihereditary: every finitely generated submodule of a projective module is itself projective. There have been several attempts to axiomatize the underlying rings of von Neumann algebras, including Baer $*$ -rings and AW $*$ algebras. The $*$ -algebra of affiliated operators of a finite von Neumann algebra is a von Neumann regular ring. (The von Neumann algebra itself is in general not von Neumann regular.)

Commutative von Neumann algebras

Main article: Abelian von Neumann algebra

The relationship between commutative von Neumann algebras and measure spaces is analogous to that between commutative C^* -algebras and locally compact Hausdorff spaces. Every commutative von Neumann algebra is isomorphic to $L^\infty(X)$ for some measure space (X, μ) and conversely, for every σ -finite measure space X , the $*$ algebra $L^\infty(X)$ is a von Neumann algebra.

Due to this analogy, the theory of von Neumann algebras has been called noncommutative measure theory, while the theory of C^* -algebras is sometimes called noncommutative topology (Connes 1994).

Projections

Operators E in a von Neumann algebra for which $E = EE = E^*$ are called **projections**; they are exactly the operators which give an orthogonal projection of H onto some closed subspace. A subspace of the Hilbert space H is said to **belong to** the von Neumann algebra M if it is the image of some projection in M . Informally these are the closed subspaces that can be described using elements of M , or that M "knows" about. The closure of the image of any operator in M , or the kernel of any operator in M belong to M , and the closure of the image of any subspace belonging to M under an operator of M also belongs to M . There is a 1:1 correspondence between projections of M and subspaces that belong to it.

The basic theory of projections was worked out by Murray & von Neumann (1936). Two subspaces belonging to M are called (**Murray-von Neumann**) **equivalent** if there is a partial isometry mapping the first isomorphically onto the other that is an element of the von Neumann algebra (informally, if M "knows" that the subspaces are isomorphic). This induces a natural equivalence relation on projections by defining E to be equivalent to F if the corresponding subspaces are equivalent, or in other words if there is a partial isometry of H that maps the image of E isometrically to the image of F and is an element of the von Neumann algebra. Another way of stating this is that E is equivalent to F if $E = uu^*$ and $F = u^*u$ for some partial isometry u in M .

The equivalence relation $\sim$ thus defined is additive in the following sense: Suppose $E_1 \sim F_1$ and $E_2 \sim F_2$. If $E_1 \perp E_2$ and $F_1 \perp F_2$, then $E_1 + E_2 \sim F_1 + F_2$. This is not true in general if one requires unitary equivalence in the definition of

$\sim$, i.e. if we say E is equivalent to F if $u^*Eu = F$ for some unitary u .

The subspaces belonging to M are partially ordered by inclusion, and this induces a partial order $\leq$ of projections. There is also a natural partial order on the set of *equivalence classes* of projections, induced by the partial order $\leq$ of projections. If M is a factor, $\leq$ is a total order on equivalence classes of projections, described in the section on traces below.

A projection (or subspace belonging to M) E is said to be *finite* if there is no projection $F < E$ that is equivalent to E . For example, all finite-dimensional projections (or subspaces) are finite (since isometries between Hilbert spaces leave the dimension fixed), but the identity operator on an infinite-dimensional Hilbert space is not finite in the von Neumann algebra of all bounded operators on it, since it is isometrically isomorphic to a proper subset of itself. However it is possible for infinite dimensional subspaces to be finite.

Orthogonal projections are noncommutative analogues of indicator functions in $L^\infty(\mathbf{R})$. $L^\infty(\mathbf{R})$ is the $\|\cdot\|_\infty$ -closure of the subspace generated by the indicator functions. Similarly, a von Neumann algebra is generated by its projections; this is a consequence of the spectral theorem for self-adjoint operators.

Factors

A von Neumann algebra N whose center consists only of multiples of the identity operator is called a **factor**. von Neumann (1949) showed that every von Neumann algebra on a separable Hilbert space is isomorphic to a direct integral of factors. This decomposition is essentially unique. Thus, the problem of classifying isomorphism classes of von Neumann algebras on separable Hilbert spaces can be reduced to that of classifying isomorphism classes of factors.

Murray & von Neumann (1936) showed that every factor has one of 3 types as described below. The type classification can be extended to von Neumann algebras that are not factors, and a von Neumann algebra is of type X if it can be decomposed as a direct integral of type X factors; for example, every commutative von Neumann algebra has type I_1 . Every von Neumann algebra can be written uniquely as a sum of von Neumann algebras of types I, II, and III.

There are several other ways to divide factors into classes that are sometimes used:

- A factor is called **discrete** (or occasionally **tame**) if it has type I, and **continuous** (or occasionally **wild**) if it has type II or III.
- A factor is called **semifinite** if it has type I or II, and **purely infinite** if it has type III.
- A factor is called **finite** if the projection 1 is finite and **properly infinite** otherwise. Factors of types I and II may be either finite or properly infinite, but factors of type III are always properly infinite.

Type I factors

A factor is said to be of **type I** if there is a minimal projection $E \neq 0$, i.e. a projection E such that there is no other projection F with $0 < F < E$. Any factor of type I is isomorphic to the von Neumann algebra of *all* bounded operators on some Hilbert space; since there is one Hilbert space for every cardinal number, isomorphism classes of factors of type I correspond exactly to the cardinal numbers. Since many authors consider von Neumann algebras only on separable Hilbert spaces, it is customary to call the bounded operators on a Hilbert space of finite dimension n a factor of type I_n , and the bounded operators on a separable infinite-dimensional Hilbert space, a factor of type I_∞ .

Type II factors

A factor is said to be of **type II** if there are no minimal projections but there are non-zero finite projections. This implies that every projection E can be halved in the sense that there are equivalent projections F and G such that $E = F + G$. If the identity operator in a type II factor is finite, the factor is said to be of type II_1 ; otherwise, it is said to be of type II_∞ . The best understood factors of type II are the hyperfinite type II_1 factor and the hyperfinite type II_∞ factor, found by Murray & von Neumann (1936). These are the unique hyperfinite factors of types II_1 and II_∞ ; there are an uncountable number of other factors of these types that are the subject of intensive study. Murray & von Neumann (1937) proved the fundamental result that a factor of type II_1 has a unique finite tracial state, and the set of traces of projections is $[0,1]$.

A factor of type II_∞ has a semifinite trace, unique up to rescaling, and the set of traces of projections is $[0,\infty]$. The set of real numbers λ such that there is an automorphism rescaling the trace by a factor of λ is called the **fundamental group** of the type II_∞ factor.

The tensor product of a factor of type II_1 and an infinite type I factor has type II_∞ , and conversely any factor of type II_∞ can be constructed like this. The **fundamental group** of a type II_1 factor is defined to be the fundamental group of its tensor product with the infinite (separable) factor of type I. For many years it was an open problem to find a type II factor whose fundamental group was not the group of all positive reals, but Connes then showed that the von Neumann group algebra of a countable discrete group with Kazhdan's property T (the trivial representation is isolated in the dual space), such as $\text{SL}_3(\mathbf{Z})$, has a countable fundamental group. Subsequently Sorin Popa showed that the fundamental group can be trivial for certain groups, including the semidirect product of $\mathbf{Z}^2$ by $\text{SL}_2(\mathbf{Z})$.

An example of a type II_1 factor is the von Neumann group algebra of a countable infinite discrete group such that every non-trivial conjugacy class is infinite. McDuff (1969) found an uncountable family of such groups with non-isomorphic von Neumann group algebras, thus showing the existence of uncountably many different separable type II_1 factors.

Type III factors

Lastly, **type III** factors are factors that do not contain any nonzero finite projections at all. In their first paper Murray & von Neumann (1936) were unable decide whether or not they existed; the first examples were later found by von Neumann (1940). Since the identity operator is always infinite in those factors, they were sometimes called type III_∞ in the past, but recently that notation has been superseded by the notation III_λ , where λ is a real number in the interval $[0,1]$. More precisely, if the Connes spectrum (of its modular group) is 1 then the factor is of type III_0 , if the Connes spectrum is all integral powers of λ for $0 < \lambda < 1$, then the type is III_λ , and if the Connes spectrum is all positive reals then the type is III_1 . (The Connes spectrum is a closed subgroup of the positive reals, so these are the only possibilities.) The only trace on type III factors takes value ∞ on all non-zero positive elements, and any two non-zero projections are equivalent. At one time type III factors were considered to be intractable objects, but Tomita–Takesaki theory has led to a good structure theory. In particular, any type III factor can be written in a canonical way as the crossed product of a type II_∞ factor and the real numbers.

The predual

Any von Neumann algebra M has a **predual** M_* , which is the Banach space of all ultraweakly continuous linear functionals on M . As the name suggests, M is (as a Banach space) the dual of its predual. The predual is unique in the sense that any other Banach space whose dual is M is canonically isomorphic to M_* . Sakai (1971) showed that the existence of a predual characterizes von Neumann algebras among C^* algebras.

The definition of the predual given above seems to depend on the choice of Hilbert space that M acts on, as this determines the ultraweak topology. However the predual can also be defined without using the Hilbert space that M acts on, by defining it to be the space generated by all positive **normal** linear functionals on M . (Here "normal" means that it preserves suprema when applied to increasing nets of self adjoint operators; or equivalently to increasing sequences of projections.)

The predual M_* is a closed subspace of the dual M^* (which consists of all norm-continuous linear functionals on M) but is generally smaller. The proof that M_* is (usually) not the same as M^* is nonconstructive and uses the axiom of choice in an essential way; it is very hard to exhibit explicit elements of M^* that are not in M_* . For example, exotic positive linear forms on the von Neumann algebra $\ell^\infty(Z)$ are given by free ultrafilters; they correspond to exotic $*$ -homomorphisms into C and describe the Stone-Cech compactification of Z .

Examples:

1. The predual of the von Neumann algebra $L^\infty(\mathbf{R})$ of essentially bounded functions on $\mathbf{R}$ is the Banach space $L^1(\mathbf{R})$ of integrable functions. The dual of $L^\infty(\mathbf{R})$ is strictly larger than $L^1(\mathbf{R})$. For example, a functional on $L^\infty(\mathbf{R})$ that extends the Dirac measure δ_0 on the closed subspace of bounded continuous functions $C_b^0(\mathbf{R})$ cannot be represented as a function in $L^1(\mathbf{R})$.
2. The predual of the von Neumann algebra $B(H)$ of bounded operators on a Hilbert space H is the Banach space of all trace class operators with the trace norm $\|A\| = \text{Tr}(|A|)$. The Banach space of trace class operators is itself the dual of the C^* -algebra of compact operators (which is not a von Neumann algebra).

Weights, states, and traces

Weights and their special cases states and traces are discussed in detail in (Takesaki 1979).

- A **weight** ω on a von Neumann algebra is a linear map from the set of positive elements (those of the form a^*a) to $[0, \infty]$.
- A **positive linear functional** is a weight with $\omega(1)$ finite (or rather the extension of ω to the whole algebra by linearity).
- A **state** is a weight with $\omega(1)=1$.
- A **trace** is a weight with $\omega(aa^*)=\omega(a^*a)$ for all a .
- A **tracial state** is a trace with $\omega(1)=1$.

Any factor has a trace such that the trace of a non-zero projection is non-zero and the trace of a projection is infinite if and only if the projection is infinite. Such a trace is unique up to rescaling. For factors that are separable or finite, two projections are equivalent if and only if they have the same trace. The type of a factor can be read off from the possible values of this trace as follows:

- Type I_n : $0, x, 2x, \dots, nx$ for some positive x (usually normalized to be $1/n$ or 1).
- Type I_∞ : $0, x, 2x, \dots, \infty$ for some positive x (usually normalized to be 1).
- Type II_1 : $[0, x]$ for some positive x (usually normalized to be 1).
- Type II_∞ : $[0, \infty]$.
- Type III: $0, \infty$.

If a von Neumann algebra acts on a Hilbert space containing a norm 1 vector v , then the functional $a \rightarrow (av, v)$ is a normal state. This construction can be reversed to give an action on a Hilbert space from a normal state: this is the

GNS construction for normal states.

Modules over a factor

Given an abstract separable factor, one can ask for a classification of its modules, meaning the separable Hilbert spaces that it acts on. The answer is given as follows: every such module H can be given an M -dimension $\dim_M(H)$ (not its dimension as a complex vector space) such that modules are isomorphic if and only if they have the same M -dimension. The M -dimension is additive, and a module is isomorphic to a subspace of another module if and only if it has smaller or equal M -dimension.

A module is called **standard** if it has a cyclic separating vector. Each factor has a standard representation, which is unique up to isomorphism. The standard representation has an antilinear involution J such that $JMJ = M'$. For finite factors the standard module is given by the GNS construction applied to the unique normal tracial state and the M -dimension is normalized so that the standard module has M -dimension 1, while for infinite factors the standard module is the module with M -dimension equal to ∞ .

The possible M -dimensions of modules are given as follows:

- Type I_n (n finite): The M -dimension can be any of $0/n, 1/n, 2/n, 3/n, \dots, \infty$. The standard module has M -dimension 1 (and complex dimension n^2 .)
- Type I_∞ : The M -dimension can be any of $0, 1, 2, 3, \dots, \infty$. The standard representation of $B(H)$ is $H \otimes H$; its M -dimension is ∞ .
- Type II_1 : The M -dimension can be anything in $[0, \infty]$. It is normalized so that the standard module has M -dimension 1. The M -dimension is also called the **coupling constant** of the module H .
- Type II_∞ : The M -dimension can be anything in $[0, \infty]$. There is in general no canonical way to normalize it; the factor may have outer automorphisms multiplying the M -dimension by constants. The standard representation is the one with M -dimension ∞ .
- Type III: The M -dimension can be 0 or ∞ . Any two non-zero modules are isomorphic, and all non-zero modules are standard.

Amenable von Neumann algebras

Connes (1976) and others proved that the following conditions on a von Neumann algebra M on a separable Hilbert space H are all equivalent:

- M is **hyperfinite** or **AFD** or **approximately finite dimensional** or **approximately finite**: this means the algebra contains an ascending sequence of finite dimensional subalgebras with dense union. (Warning: some authors use "hyperfinite" to mean "AFD and finite".)
- M is **amenable**: this means that the derivations of M with values in a normal dual Banach bimodule are all inner.
- M has Schwartz's **property P**: for any bounded operator T on H the weak operator closed convex hull of the elements uTu^* contains an element commuting with M .
- M is **semidiscrete**: this means the identity map from M to M is a weak pointwise limit of completely positive maps of finite rank.
- M has **property E** or the **Hakeda-Tomiyama extension property**: this means that there is a projection of norm 1 from bounded operators on H to M' .
- M is **injective**: any completely positive linear map from any self adjoint closed subspace containing 1 of any unital C^* -algebra A to M can be extended to a completely positive map from A to M .

There is no generally accepted term for the class of algebras above; Connes has suggested that **amenable** should be the standard term.

The amenable factors have been classified: there is a unique one of each of the types I_n , I_∞ , II_1 , II_∞ , III_λ , for $0 < \lambda \leq 1$, and the ones of type III_0 correspond to certain ergodic flows. (For type III_0 calling this a classification is a little misleading, as it is known that there is no easy way to classify the corresponding ergodic flows.) The ones of type I and II_1 were classified by Murray & von Neumann (1943), and the remaining ones were classified by Connes (1976), except for the type III_1 case which was completed by Haagerup.

All amenable factors can be constructed using the **group-measure space construction** of Murray and von Neumann for a single ergodic transformation. In fact they are precisely the factors arising as crossed products by free ergodic actions of Z or Z_n on abelian von Neumann algebras $L^\infty(X)$. Type I factors occur when the measure space X is atomic and the action transitive. When X is diffuse or non-atomic, it is equivalent to $[0,1]$ as a measure space. Type II factors occur when X admits an equivalent finite (II_1) or infinite (II_∞) measure, invariant under Z . Type III factors occur in the remaining cases where there is no invariant measure, but only an invariant measure class: these factors are called **Krieger factors**.

Tensor products of von Neumann algebras

The Hilbert space tensor product of two Hilbert spaces is the completion of their algebraic tensor product. One can define a tensor product of von Neumann algebras (a completion of the algebraic tensor product of the algebras considered as rings), which is again a von Neumann algebra, and act on the tensor product of the corresponding Hilbert spaces. The tensor product of two finite algebras is finite, and the tensor product of an infinite algebra and a non-zero algebra is infinite. The type of the tensor product of two von Neumann algebras (I, II, or III) is the maximum of their types. The **commutation theorem for tensor products** states that

$$(M \otimes N)' = M' \otimes N'$$

(where M' denotes the commutant of M).

The tensor product of an infinite number of von Neumann algebras, if done naively, is usually a ridiculously large non-separable algebra. Instead von Neumann (1938) showed that one should choose a state on each of the von Neumann algebras, use this to define a state on the algebraic tensor product, which can be used to produce a Hilbert space and a (reasonably small) von Neumann algebra. Araki & Woods (1968) studied the case where all the factors are finite matrix algebras; these factors are called **Araki-Woods factors** or **ITPFI factors** (ITPFI stands for "infinite tensor product of finite type I factors"). The type of the infinite tensor product can vary dramatically as the states are changed; for example, the infinite tensor product of an infinite number of type I_2 factors can have any type depending on the choice of states. In particular Powers (1967) found an uncountable family of non-isomorphic hyperfinite type III_λ factors for $0 < \lambda < 1$, called **Powers factors**, by taking an infinite tensor product of type I_2 factors, each with the state given by $x \mapsto \text{Tr} \begin{pmatrix} \frac{1}{\lambda+1} & 0 \\ 0 & \frac{\lambda}{\lambda+1} \end{pmatrix} x$.

All hyperfinite von Neumann algebras not of type III_0 are isomorphic to Araki-Woods factors, but there are uncountably many of type III_0 that are not.

Bimodules and subfactors

A **bimodule** (or correspondence) is a Hilbert space H with module actions of two commuting von Neumann algebras. Bimodules have a much richer structure than that of modules. Any bimodule over two factors always gives a subfactor since one of the factors is always contained in the commutant of the other. There is also a subtle relative tensor product operation due to Connes on bimodules. The theory of subfactors, initiated by Vaughan Jones, reconciles these two seemingly different points of view.

Bimodules are also important for the von Neumann group algebra M of a discrete group Γ . Indeed if V is any unitary representation of Γ , then, regarding Γ as the diagonal subgroup of $\Gamma \times \Gamma$, the corresponding induced representation on $l^2(\Gamma, V)$ is naturally a bimodule for two commuting copies of M . Important representation

theoretic properties of Γ can be formulated entirely in terms of bimodules and therefore make sense for the von Neumann algebra itself. For example Connes and Jones gave a definition of an analogue of Kazhdan's Property T for von Neumann algebras in this way.

Non-amenable factors

Von Neumann algebras of type I are always amenable, but for the other types there are an uncountable number of different non-amenable factors, which seem very hard to classify, or even distinguish from each other. Nevertheless Voiculescu has shown that the class of non-amenable factors coming from the group-measure space construction is **disjoint** from the class coming from group von Neumann algebras of free groups. Later Narutaka Ozawa proved that group von Neumann algebras of hyperbolic groups yield prime type II_1 factors, i.e. ones that cannot be factored as tensor products of type II_1 factors, a result first proved by Leeming Ge for free group factors using Voiculescu's free entropy. Popa's work on fundamental groups of non-amenable factors represents another significant advance. The theory of factors "beyond the hyperfinite" is rapidly expanding at present, with many new and surprising results; it has close links with rigidity phenomena in geometric group theory and ergodic theory.

Examples

- The essentially bounded functions on a σ -finite measure space form a commutative (type I_1) von Neumann algebra acting on the L^2 functions. For certain non- σ -finite measure spaces, usually considered pathological, $L^\infty(X)$ is not a von Neumann algebra; for example, the σ -algebra of measurable sets might be the countable-cocountable algebra on an uncountable set.
- The bounded operators on any Hilbert space form a von Neumann algebra, indeed a factor, of type I.
- If we have any unitary representation of a group G on a Hilbert space H then the bounded operators commuting with G form a von Neumann algebra G' , whose projections correspond exactly to the closed subspaces of H invariant under G . Equivalent subrepresentations correspond to equivalent projections in G' . The double commutant G'' of G is also a von Neumann algebra.
- The **von Neumann group algebra** of a discrete group G is the algebra of all bounded operators on $H = \ell^2(G)$ commuting with the action of G on H through right multiplication. One can show that this is the von Neumann algebra generated by the operators corresponding to multiplication from the left with an element $g \in G$. It is a factor (of type II_1) if every non-trivial conjugacy class of G is infinite (for example, a non-abelian free group), and is the hyperfinite factor of type II_1 if in addition G is a union of finite subgroups (for example, the group of all permutations of the integers fixing all but a finite number of elements).
- The tensor product of two von Neumann algebras, or of a countable number with states, is a von Neumann algebra as described in the section above.
- The crossed product of a von Neumann algebra by a discrete (or more generally locally compact) group can be defined, and is a von Neumann algebra. Special cases are the **group-measure space construction** of Murray and von Neumann and **Krieger factors**.
- The von Neumann algebras of a measurable equivalence relation and a measurable groupoid can be defined. These examples generalise von Neumann group algebras and the group-measure space construction.

Applications

Von Neumann algebras have found applications in diverse areas of mathematics like knot theory, statistical mechanics, Quantum field theory, Local quantum physics, Free probability, Noncommutative geometry, representation theory, geometry, and probability.

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C*-algebra

C*-algebras (pronounced "C-star") are an important area of research in functional analysis, a branch of mathematics. The prototypical example of a C*-algebra is a complex algebra A of linear operators on a complex Hilbert space with two additional properties:

- A is a topologically closed set in the norm topology of operators.
- A is closed under the operation of taking adjoints of operators.

It is generally believed that C*-algebras were first considered primarily for their use in quantum mechanics to model algebras of physical observables. This line of research began with Werner Heisenberg's matrix mechanics and in a more mathematically developed form with Pascual Jordan around 1933. Subsequently John von Neumann attempted to establish a general framework for these algebras which culminated in a series of papers on rings of operators. These papers considered a special class of C*-algebras which are now known as von Neumann algebras.

Around 1943, the work of Israel Gelfand and Mark Naimark yielded an abstract characterisation of C*-algebras making no reference to operators.

C*-algebras are now an important tool in the theory of unitary representations of locally compact groups, and are also used in algebraic formulations of quantum mechanics.

Abstract characterization

We begin with the abstract characterization of C*-algebras given in the 1943 paper by Gelfand and Naimark.

A C*-algebra, A , is a Banach algebra over the field of complex numbers, together with a map, $*$: $A \rightarrow A$, called an involution. The image of an element x of A under the involution is written x^* . Involution has the following properties:

- For all x, y in A :

$$\begin{aligned}(x + y)^* &= x^* + y^* \\ (xy)^* &= y^* x^*\end{aligned}$$

- For every λ in $\mathbb{C}$ and every x in A :

$$(\lambda x)^* = \bar{\lambda} x^*.$$

- For all x in A

$$(x^*)^* = x$$

- The **C*-identity** holds for all x in A :

$$\|x^* x\| = \|x\| \|x^*\|.$$

Note that the C* identity is equivalent to: for all x in A :

$$\|xx^*\| = \|x\| \|x^*\|.$$

This relation is equivalent to $\|xx^*\| = \|x\|^2$, which is sometimes called the B*-identity. For history behind the names C*- and B*-algebras, see the history section below.

The C*-identity is a very strong requirement. For instance, together with the spectral radius formula, it implies the C*-norm is uniquely determined by the algebraic structure:

$$\|x\|^2 = \|x^* x\| = \sup\{|\lambda| : x^* x - \lambda 1 \text{ is not invertible}\}.$$

A bounded linear map, $\pi : A \rightarrow B$, between C*-algebras A and B is called a ***-homomorphism** if

- For x and y in A

$$\pi(xy) = \pi(x)\pi(y)$$

- For x in A

$$\pi(x^*) = \pi(x)^*$$

In the case of C*-algebras, any *-homomorphism π between C*-algebras is non-expansive, i.e. bounded with norm ≤ 1 . Furthermore, an injective *-homomorphism between C*-algebras is isometric. These are consequences of the C*-identity.

A bijective *-homomorphism π is called a **C*-isomorphism**, in which case A and B are said to be **isomorphic**.

Some history: B*-algebras and C*-algebras

The term B*-algebra was introduced by C. E. Rickart in 1946 to describe Banach *-algebras that satisfy the condition

- $\|xx^*\| = \|x\|^2$ for all x in the given B*-algebra. (B*-condition)

This condition automatically implies that the *-involution is isometric, that is, $\|x^*\| = \|x\|$. Hence $\|xx^*\| = \|x\| \|x^*\|$, and therefore, a B*-algebra is a C*-algebra. Conversely, the C*-condition implies the B*-condition. This is nontrivial, and can be proved without using the condition $\|x\| = \|x^*\|$. (For details, see R. S. Doran, V. A. Belfi, *Characterizations of C*-Algebras --- the Gelfand-Naimark Theorems*, CRC, 1986.)

For these reasons, the term B*-algebra is rarely used in current terminology, and has been replaced by the term 'C* algebra'.

The term C*-algebra was introduced by I. E. Segal in 1947 to describe norm-closed subalgebras of $B(H)$, namely, the space of bounded operators on some Hilbert space H . 'C' stood for 'closed'.

Examples

Finite-dimensional C*-algebras

The algebra $M_n(\mathbb{C})$ of n -by- n matrices over $\mathbb{C}$ becomes a C*-algebra if we consider matrices as operators on the Euclidean space, $\mathbb{C}^n$, and use the operator norm $\|\cdot\|$ on matrices. The involution is given by the conjugate transpose. More generally, one can consider finite direct sums of matrix algebras. In fact, all finite dimensional C*-algebras are of this form. The self-adjoint requirement means finite-dimensional C*-algebras are semisimple, from which fact one can deduce the following theorem of Artin–Wedderburn type:

Theorem. A finite-dimensional C*-algebra, A , is canonically isomorphic to a finite direct sum

$$A = \bigoplus_{e \in \min A} Ae$$

where $\min A$ is the set of minimal nonzero self-adjoint central projections of A .

Each C*-algebra, Ae , is isomorphic (in a noncanonical way) to the full matrix algebra $M_{\dim(e)}(\mathbb{C})$. The finite family indexed on $\min A$ given by $\{\dim(e)\}_e$ is called the *dimension vector* of A . This vector uniquely determines the isomorphism class of a finite-dimensional C*-algebra.

C*-algebras of operators

The prototypical example of a C*-algebra is the algebra $B(H)$ of bounded (equivalently continuous) linear operators defined on a complex Hilbert space H ; here x^* denotes the adjoint operator of the operator $x : H \rightarrow H$. In fact, every C*-algebra, A , is *-isomorphic to a norm-closed adjoint closed subalgebra of $B(H)$ for a suitable Hilbert space, H ; this is the content of the Gelfand–Naimark theorem.

Commutative C*-algebras

Let X be a locally compact Hausdorff space. The space $C_0(X)$ of complex-valued continuous functions on X that *vanish at infinity* (defined in the article on local compactness) form a commutative C*-algebra $C_0(X)$ under pointwise multiplication and addition. The involution is pointwise conjugation. $C_0(X)$ has a multiplicative unit element if and only if X is compact. As does any C*-algebra, $C_0(X)$ has an approximate identity. In the case of $C_0(X)$ this is immediate: consider the directed set of compact subsets of X , and for each compact K let f_K be a function of compact support which is identically 1 on K . Such functions exist by the Tietze extension theorem which applies to locally compact Hausdorff spaces. $\{f_K\}_K$ is an approximate identity.

The Gelfand representation states that every commutative C*-algebra is *-isomorphic to the algebra $C_0(X)$, where X is the space of characters equipped with the weak* topology. Furthermore if $C_0(X)$ is isomorphic to $C_0(Y)$ as C*-algebras, it follows that X and Y are homeomorphic. This characterization is one of the motivations for the noncommutative topology and noncommutative geometry programs.

C*-algebras of compact operators

Let H be a separable infinite-dimensional Hilbert space. The algebra $K(H)$ of compact operators on H is a norm closed subalgebra of $B(H)$. It is also closed under involution; hence it is a C*-algebra.

Concrete C*-algebras of compact operators admit a characterization similar to Wedderburn's theorem for finite dimensional C*-algebras.

Theorem. If A is a C*-subalgebra of $K(H)$, then there exists Hilbert spaces $\{H_i\}_{i \in I}$ such that A is isomorphic to the following direct sum

$$\bigoplus_{i \in I} K(H_i),$$

where the (C*-)direct sum consists of elements (T_i) of the Cartesian product $\prod K(H_i)$ with $\|T_i\| \rightarrow 0$.

Though $K(H)$ does not have an identity element, a sequential approximate identity for $K(H)$ can be easily displayed. To be specific, H is isomorphic to the space of square summable sequences ℓ^2 ; we may assume that

$$H = \ell^2.$$

For each natural number n let H_n be the subspace of sequences of ℓ^2 which vanish for indices

$$k \geq n$$

and let

$$e_n$$

be the orthogonal projection onto H_n . The sequence $\{e_n\}_n$ is an approximate identity for $K(H)$.

$K(H)$ is a two-sided closed ideal of $B(H)$. For separable Hilbert spaces, it is the unique ideal. The quotient of $B(H)$ by $K(H)$ is the Calkin algebra.

C*-enveloping algebra

Given a B*-algebra A with an approximate identity, there is a unique (up to C*-isomorphism) C*-algebra $E(A)$ and *-morphism π from A into $E(A)$ which is universal, that is, every other B*-morphism $\pi' : A \rightarrow B$ factors uniquely through π . The algebra $E(A)$ is called the **C*-enveloping algebra** of the B*-algebra A .

Of particular importance is the C*-algebra of a locally compact group G . This is defined as the enveloping C*-algebra of the group algebra of G . The C*-algebra of G provides context for general harmonic analysis of G in the case G is non-abelian. In particular, the dual of a locally compact group is defined to be the primitive ideal space of the group C*-algebra. See spectrum of a C*-algebra.

von Neumann algebras

von Neumann algebras, known as W* algebras before the 1960s, are a special kind of C*-algebra. They are required to be closed in the weak operator topology, which is weaker than the norm topology. Their study is a specialized area of functional analysis.

Properties of C*-algebras

C*-algebras have a large number of properties that are technically convenient. These properties can be established by use the continuous functional calculus or by reduction to commutative C*-algebras. In the latter case, we can use the fact that the structure of these is completely determined by the Gelfand isomorphism.

- The set of elements of a C*-algebra A of the form x^*x forms a closed convex cone. This cone is identical to the elements of the form $x x^*$. Elements of this cone are called *non-negative* (or sometimes *positive*, even though this terminology conflicts with its use for elements of $\mathbf{R}$.)
- The set of self-adjoint elements of a C*-algebra A naturally has the structure of a partially ordered vector space; the ordering is usually denoted $\geq$. In this ordering, a self-adjoint element x of A satisfies $x \geq 0$ if and only if the spectrum of x is non-negative. Two self-adjoint elements x and y of A satisfy $x \geq y$ if $x - y \geq 0$.
- Any C*-algebra A has an approximate identity. In fact, there is a directed family $\{e_\lambda\}_{\lambda \in I}$ of self-adjoint elements of A such that

$$x e_\lambda \rightarrow x$$

$$0 \leq e_\lambda \leq e_\mu \leq 1 \quad \text{whenever } \lambda \leq \mu.$$

In case A is separable, A has a sequential approximate identity. More generally, A will have a sequential approximate identity if and only if A contains a **strictly positive element**, i.e. a positive element h such that hAh is dense in A .

- Using approximate identities, one can show that the algebraic quotient of a C*-algebra by a closed proper two-sided ideal, with the natural norm, is a C*-algebra.
- Similarly, a closed two-sided ideal of a C*-algebra is itself a C*-algebra.

Type for C*-algebras

A C*-algebra A is of type I if and only if for all non-degenerate representations π of A the von Neumann algebra $\pi(A)''$ (that is, the bicommutant of $\pi(A)$) is a type I von Neumann algebra. In fact it is sufficient to consider only factor representations, i.e. representations π for which $\pi(A)''$ is a factor.

A locally compact group is said to be of type I if and only if its group C*-algebra is type I.

However, if a C*-algebra has non-type I representations, then by results of James Glimm it also has representations of type II and type III. Thus for C*-algebras and locally compact groups, it is only meaningful to speak of type I and non type I properties.

C*-algebras and quantum field theory

In quantum field theory, one typically describes a physical system with a C*-algebra A with unit element; the self-adjoint elements of A (elements x with $x^* = x$) are thought of as the *observables*, the measurable quantities, of the system. A *state* of the system is defined as a positive functional on A (a $\mathbb{C}$ -linear map $\varphi : A \rightarrow \mathbb{C}$ with $\varphi(u^* u) \geq 0$ for all $u \in A$) such that $\varphi(1) = 1$. The expected value of the observable x , if the system is in state φ , is then $\varphi(x)$.

See Local quantum physics.

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Kac-Moody algebra

In mathematics, a **Kac–Moody algebra** (named for Victor Kac and Robert Moody, who independently discovered them) is a Lie algebra, usually infinite-dimensional, that can be defined by generators and relations through a generalized Cartan matrix. These algebras form a generalization of finite-dimensional semisimple Lie algebras, and many properties related to the structure of a Lie algebra such as its root system, irreducible representations, and connection to flag manifolds have natural analogues in the Kac–Moody setting. A class of Kac–Moody algebras called **affine Lie algebras** is of particular importance in mathematics and theoretical physics, especially conformal field theory and the theory of exactly solvable models. Kac discovered an elegant proof of certain combinatorial identities, Macdonald identities, which is based on the representation theory of affine Kac–Moody algebras. Garland and Lepowski demonstrated that Rogers-Ramanujan identities can be derived in a similar fashion.

Definition

A Kac–Moody algebra is given by the following:

1. An $n \times n$ generalized Cartan matrix $C = (c_{ij})$ of rank r .
2. A vector space $\mathfrak{h}$ over the complex numbers of dimension $2n - r$.
3. A set of n linearly independent elements α_i of $\mathfrak{h}$ and a set of n linearly independent elements α_i^* of the dual space, such that $\alpha_i^*(\alpha_j) = c_{ij}$. The α_i are known as **coroots**, while the α_i^* are known as **roots**.

The Kac–Moody algebra is the Lie algebra $\mathfrak{g}$ defined by generators e_i and f_i and the elements of $\mathfrak{h}$ and relations

- $[e_i, f_i] = \alpha_i$
- $[e_i, f_j] = 0$ for $i \neq j$
- $[e_i, x] = \alpha_i^*(x)e_i$, for $x \in \mathfrak{h}$
- $[f_i, x] = -\alpha_i^*(x)f_i$, for $x \in \mathfrak{h}$
- $[x, x'] = 0$ for $x, x' \in \mathfrak{h}$
- $\text{ad}(e_i)^{1-c_{ij}}(e_j) = 0$
- $\text{ad}(f_i)^{1-c_{ij}}(f_j) = 0$

where $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$, $\text{ad}(x)(y) = [x, y]$ is the adjoint representation of $\mathfrak{g}$.

A real (possibly infinite-dimensional) Lie algebra is also considered a Kac–Moody algebra if its complexification is a Kac–Moody algebra.

Interpretation

$\mathfrak{h}$ is a Cartan subalgebra of the Kac–Moody algebra.

If g is an element of the Kac–Moody algebra such that

$$\forall x \in \mathfrak{h}, [g, x] = \omega(x)g$$

where ω is an element of $\mathfrak{h}^*$, then g is said to have weight ω . The Kac–Moody algebra can be diagonalized into weight eigenvectors. The Cartan subalgebra $\mathfrak{h}$ has weight zero, e_i has weight α_i^* and f_i has weight $-\alpha_i^*$. If the Lie bracket of two weight eigenvectors is nonzero, then its weight is the sum of their weights. The condition $[e_i, f_j] = 0$ for $i \neq j$ simply means the α_i^* are simple roots.

Types of Kac–Moody algebras

Properties of a Kac–Moody algebra are controlled by the algebraic properties of its generalized Cartan matrix C . In order to classify Kac–Moody algebras, it is enough to consider the case of an *indecomposable* matrix C , that is, assume that there is no decomposition of the set of indices I into a disjoint union of non-empty subsets I_1 and I_2 such that $C_{ij} = 0$ for all i in I_1 and j in I_2 . Any decomposition of the generalized Cartan matrix leads to the direct sum decomposition of the corresponding Kac–Moody algebra:

$$\mathfrak{g}(C) \simeq \mathfrak{g}(C_1) \oplus \mathfrak{g}(C_2),$$

where the two Kac–Moody algebras in the right hand side are associated with the submatrices of C corresponding to the index sets I_1 and I_2 .

An important subclass of Kac–Moody algebras corresponds to *symmetrizable* generalized Cartan matrices C , which can be decomposed as DS , where D is a diagonal matrix with positive integer entries and S is a symmetric matrix. Under the assumptions that C is symmetrizable and indecomposable, the Kac–Moody algebras are divided into three classes:

- A positive definite matrix S gives rise to a finite-dimensional simple Lie algebra.
- A positive semidefinite matrix S gives rise to an infinite-dimensional Kac–Moody algebra of **affine type**, or an affine Lie algebra.
- An indefinite matrix S gives rise to a Kac–Moody algebra of **indefinite type**.
- Since the diagonal entries of C and S are positive, S cannot be negative definite or negative semidefinite.

Symmetrizable indecomposable generalized Cartan matrices of finite and affine type have been completely classified. They correspond to Dynkin diagrams and affine Dynkin diagrams. Very little is known about the Kac–Moody algebras of indefinite type. Among those, the main focus has been on the (generalized) Kac–Moody algebras of **hyperbolic type**, for which the matrix S is indefinite, but for each proper subset of I , the corresponding submatrix is positive definite or positive semidefinite. Such matrices have rank at most 10 and have also been completely determined.

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Spectral theory

In mathematics, **spectral theory** is an inclusive term for theories extending the eigenvector and eigenvalue theory of a single square matrix to a much broader theory of the structure of operators in a variety of mathematical spaces.^[1] It is a result of studies of linear algebra and the solutions of systems of linear equations and their generalizations.^[2] The theory is connected to that of analytic functions because the spectral properties of an operator are related to analytic functions of the spectral parameter.^[3]

Mathematical background

The name *spectral theory* was introduced by David Hilbert in his original formulation of Hilbert space theory, which was cast in terms of quadratic forms in infinitely many variables. The original spectral theorem was therefore conceived as a version of the theorem on principal axes of an ellipsoid, in an infinite-dimensional setting. The later discovery in quantum mechanics that spectral theory could explain features of atomic spectra was therefore fortuitous.

There have been three main ways to formulate spectral theory, all of which retain their usefulness. After Hilbert's initial formulation, the later development of abstract Hilbert space and the spectral theory of a single normal operator on it did very much go in parallel with the requirements of physics; particularly at the hands of von Neumann.^[4] The further theory built on this to include Banach algebras, which can be given abstractly. This development leads to the Gelfand representation, which covers the commutative case, and further into non-commutative harmonic analysis.

The difference can be seen in making the connection with Fourier analysis. The Fourier transform on the real line is in one sense the spectral theory of differentiation *qua* differential operator. But for that to cover the phenomena one has already to deal with generalized eigenfunctions (for example, by means of a rigged Hilbert space). On the other hand it is simple to construct a group algebra, the spectrum of which captures the Fourier transform's basic properties, and this is carried out by means of Pontryagin duality.

One can also study the spectral properties of operators on Banach spaces. For example, compact operators on Banach spaces have many spectral properties similar to that of matrices.

Physical background

The background in the physics of vibrations has been explained in this way:^[5]

“Spectral theory is connected with the investigation of localized vibrations of a variety of different objects, from atoms and molecules in chemistry to obstacles in acoustic waveguides. These vibrations have frequencies, and the issue is to decide when such localized vibrations occur, and how to go about computing the frequencies. This is a very complicated problem since every object has not only a fundamental tone but also a complicated series of overtones, which vary radically from one body to another.”

The mathematical theory is not dependent on such physical ideas on a technical level, but there are examples of mutual influence (see for example Mark Kac's question *Can you hear the shape of a drum?*). Hilbert's adoption of the term "spectrum" has been attributed to an 1897 paper of Wilhelm Wirtinger on Hill's equation (by Jean Dieudonné), and it was taken up by his students during the first decade of the twentieth century, among them Erhard Schmidt and Hermann Weyl. The conceptual basis for Hilbert space was developed from Hilbert's ideas by Erhard Schmidt and Frigyes Riesz.^[6] ^[7] It was almost twenty years later, when quantum mechanics was formulated in terms of the Schrödinger equation, that the connection was made to atomic spectra; a connection with the mathematical physics of vibration had been suspected before, as remarked by Henri Poincaré, but rejected for simple quantitative reasons, absent an explanation of the Balmer series.^[8] The later discovery in quantum mechanics that spectral theory could explain features of atomic spectra was therefore fortuitous, rather than being an object of Hilbert's spectral theory.

A definition of spectrum

Consider a bounded linear transformation T defined everywhere over a general Banach space. We form the transformation:

$$R_\zeta = (\zeta I - T)^{-1} .$$

Here I is the identity operator and ζ is a complex number. The *inverse* of an operator T , that is T^{-1} , is defined by:

$$T T^{-1} = T^{-1} T = I .$$

If the inverse exists, T is called *regular*. If it does not exist, T is called *singular*.

With these definitions, the *resolvent set* of T is the set of all complex numbers ζ such that R_ζ exists and is bounded. This set often is denoted as $\varrho(T)$. The *spectrum* of T is the set of all complex numbers ζ such that R_ζ fails to exist or is unbounded. Often the spectrum of T is denoted by $\sigma(T)$. The function R_ζ for all ζ in $\varrho(T)$ (that is, wherever R_ζ exists) is called the resolvent of T . The *spectrum* of T is therefore the complement of the *resolvent set* of T in the complex plane.^[9] Every eigenvalue of T belongs to $\sigma(T)$, but $\sigma(T)$ may contain non-eigenvalues.^[10]

This definition applies to a Banach space, but of course other types of space exist as well, for example, topological vector spaces include Banach spaces, but can be more general.^{[11] [12]} On the other hand, Banach spaces include Hilbert spaces, and it is these spaces that find the greatest application and the richest theoretical results.^[13] With suitable restrictions, much can be said about the structure of the spectra of transformations in a Hilbert space. In particular, for self-adjoint operators, the spectrum lies on the real line and (in general) is a spectral combination of a point spectrum of discrete eigenvalues and a continuous spectrum.^[14]

What is spectral theory, roughly speaking?

In functional analysis and linear algebra the spectral theorem establishes conditions under which an operator can be expressed in simple form as a sum of simpler operators. As a full rigorous presentation is not appropriate for this article, we take an approach that avoids much of the rigor and satisfaction of a formal treatment with the aim of being more comprehensible to a non-specialist.

This topic is easiest to describe by introducing the bra-ket notation of Dirac for operators.^{[15] [16]} As an example, a very particular linear operator L might be written as a dyadic product:^{[17] [18]}

$$L = |k_1\rangle\langle b_1| ,$$

in terms of the "bra" $\langle b_1$ and the "ket" $|k_1\rangle$. A function f is described by a *ket* as $|f\rangle$. The function $f(x)$ defined on the coordinates $(x_1, x_2, x_3, \dots)$ is denoted as:

$$f(x) = \langle x, f \rangle ,$$

and the magnitude of f by:

$$||f||^2 = \langle f, f \rangle = \int dx \langle f, x \rangle \langle x, f \rangle = \int dx f^*(x) f(x) ,$$

where the notation '*' denotes a complex conjugate. This inner product choice defines a very specific inner product space, restricting the generality of the arguments that follow.^[13]

The effect of L upon a function f is then described as:

$$L|f\rangle = |k_1\rangle\langle b_1| f \rangle$$

expressing the result that the effect of L on f is to produce a new function $|k_1\rangle$ multiplied by the inner product represented by $\langle b_1|f\rangle$.

A more general linear operator L might be expressed as:

$$L = \lambda_1 |e_1\rangle\langle f_1| + \lambda_2 |e_2\rangle\langle f_2| + \lambda_3 |e_3\rangle\langle f_3| + \dots ,$$

where the $\{\lambda_i\}$ are scalars and the $\{|e_i\rangle\}$ are a basis and the $\{\langle f_i|\}$ a reciprocal basis for the space. The relation between the basis and the reciprocal basis is described, in part, by:

$$\langle f_i | e_j \rangle = \delta_{ij}$$

If such a formalism applies, the $\{ \lambda_i \}$ are eigenvalues of L and the functions $\{ |e_i\rangle \}$ are eigenfunctions of L . The eigenvalues are in the *spectrum* of L .^[19]

Some natural questions are: under what circumstances does this formalism work, and for what operators L are expansions in series of other operators like this possible? Can any function f be expressed in terms of the eigenfunctions (are they a complete set) and under what circumstances does a point spectrum or a continuous spectrum arise? How do the formalisms for infinite dimensional spaces and finite dimensional spaces differ, or do they differ? Can these ideas be extended to a broader class of spaces? Answering such questions is the realm of spectral theory and requires considerable background in functional analysis and matrix algebra.

Resolution of the identity

This section continues in the rough and ready manner of the above section using the bra-ket notation, and glossing over the many important and fascinating details of a rigorous treatment.^[20] A rigorous mathematical treatment may be found in various references.^[21]

Using the bra-ket notation of the above section, the identity operator may be written as:

$$I = \sum_{i=1}^n |e_i\rangle \langle f_i|$$

where it is supposed as above that $\{ |e_i\rangle \}$ are a basis and the $\{ \langle f_i| \}$ a reciprocal basis for the space satisfying the relation:

$$\langle f_i | e_j \rangle = \delta_{ij}.$$

This expression of the identity operation is called a *representation* or a *resolution* of the identity.^{[20], [21]} This formal representation satisfies the basic property of the identity:

$$I^n = I$$

valid for every positive integer n .

Applying the resolution of the identity to any function in the space $|\psi\rangle$, one obtains:

$$I|\psi\rangle = |\psi\rangle = \sum_{i=1}^n |e_i\rangle \langle f_i|\psi\rangle = \sum_{i=1}^n c_i |e_i\rangle$$

which is the generalized Fourier expansion of ψ in terms of the basis functions $\{ e_i \}$.^[22]

Given some operator equation of the form:

$$O|\psi\rangle = |h\rangle$$

with h in the space, this equation can be solved in the above basis through the formal manipulations:

$$\begin{aligned} O|\psi\rangle &= \sum_{i=1}^n c_i (O|e_i\rangle) = \sum_{i=1}^n |e_i\rangle \langle f_i|h\rangle, \\ \langle f_j|O|\psi\rangle &= \sum_{i=1}^n c_i \langle f_j|O|e_i\rangle = \sum_{i=1}^n \langle f_j|e_i\rangle \langle f_i|h\rangle = \langle f_j|h\rangle, \quad \forall j \end{aligned}$$

which converts the operator equation to a matrix equation determining the unknown coefficients c_j in terms of the generalized Fourier coefficients $\langle f_j|h\rangle$ of h and the matrix elements $O_{ji} = \langle f_j|O|e_i\rangle$ of the operator O .

The role of spectral theory arises in establishing the nature and existence of the basis and the reciprocal basis. In particular, the basis might consist of the eigenfunctions of some linear operator L :

$$L|e_i\rangle = \lambda_i |e_i\rangle;$$

with the $\{ \lambda_i \}$ the eigenvalues of L from the spectrum of L . Then the resolution of the identity above provides the dyad expansion of L :

$$LI = L = \sum_{i=1}^n L|e_i\rangle\langle f_i| = \sum_{i=1}^n \lambda_i |e_i\rangle\langle f_i|.$$

Resolvent operator

Using spectral theory, the resolvent operator R :

$$R = (\lambda - L)^{-1},$$

can be evaluated in terms of the eigenfunctions and eigenvalues of L , and the Green's function corresponding to L can be found.

Applying R to some arbitrary function in the space, say φ ,

$$R|\varphi\rangle = (\lambda - L)^{-1}|\varphi\rangle = \sum_{i=1}^n \frac{1}{\lambda - \lambda_i} |e_i\rangle\langle f_i, \varphi|.$$

This function has poles in the complex λ -plane at each eigenvalue of L . Thus, using the calculus of residues:

$$\frac{1}{2\pi i} \oint_C d\lambda (\lambda - L)^{-1} |\varphi\rangle = -\sum_{i=1}^n |e_i\rangle\langle f_i, \varphi| = -|\varphi\rangle,$$

where the line integral is over a contour C that includes all the eigenvalues of L .

Suppose our functions are defined over some coordinates $\{x_j\}$, that is:

$$\langle x, \varphi \rangle = \varphi(x_1, x_2, \dots),$$

where the bra-kets corresponding to $\{x_j\}$ satisfy:^[23]

$$\langle x, y \rangle = \delta(x - y),$$

and where $\delta(x - y) = \delta(x_1 - y_1, x_2 - y_2, x_3 - y_3, \dots)$ is the Dirac delta function.^[24]

Then:

$$\begin{aligned} \left\langle x, \frac{1}{2\pi i} \oint_C d\lambda (\lambda - L)^{-1} \varphi \right\rangle &= \frac{1}{2\pi i} \oint_C d\lambda \langle x, (\lambda - L)^{-1} \varphi \rangle \\ &= \frac{1}{2\pi i} \oint_C d\lambda \int dy \langle x, (\lambda - L)^{-1} y \rangle \langle y, \varphi \rangle \end{aligned}$$

The function $G(x, y; \lambda)$ defined by:

$$\begin{aligned} G(x, y; \lambda) &= \langle x, (\lambda - L)^{-1} y \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \langle x, e_i \rangle \langle f_i, (\lambda - L)^{-1} e_j \rangle \langle f_j, y \rangle \\ &= \sum_{i=1}^n \frac{\langle x, e_i \rangle \langle f_i, y \rangle}{\lambda - \lambda_i} \\ &= \sum_{i=1}^n \frac{e_i(x) f_i^*(y)}{\lambda - \lambda_i}, \end{aligned}$$

is called the *Green's function* for operator L , and satisfies:^[25]

$$\frac{1}{2\pi i} \oint_C d\lambda G(x, y; \lambda) = -\sum_{i=1}^n \langle x, e_i \rangle \langle f_i, y \rangle = -\langle x, y \rangle = -\delta(x - y).$$

Operator equations

Consider the operator equation:

$$(O - \lambda I) |\psi\rangle = |h\rangle;$$

in terms of coordinates:

$$\int dy \langle x, (O - \lambda I) y \rangle \langle y, \psi \rangle = h(x).$$

A particular case is $\lambda = 0$.

The Green's function of the previous section is:

$$\langle y, G(\lambda)z \rangle = \langle y, (O - \lambda I)^{-1}z \rangle = G(y, z; \lambda),$$

and satisfies:

$$\int dy \langle x, (O - \lambda I) y \rangle \langle y, G(\lambda)z \rangle = \int dy \langle x, (O - \lambda I) y \rangle \langle y, (O - \lambda I)^{-1}z \rangle = \langle x, z \rangle = \delta(x - z).$$

Using this Green's function property:

$$\int dy \langle x, (O - \lambda I) y \rangle G(y, z; \lambda) = \delta(x - z).$$

Then, multiplying both sides of this equation by $h(z)$ and integrating:

$$\int dz h(z) \int dy \langle x, (O - \lambda I) y \rangle G(y, z; \lambda) = \int dy \langle x, (O - \lambda I) y \rangle \int dz h(z) G(y, z; \lambda) = h(x),$$

which suggests the solution is:

$$\psi(x) = \int dz h(z) G(x, z; \lambda).$$

That is, the function $\psi(x)$ satisfying the operator equation is found if we can find the spectrum of O , and construct G , for example by using:

$$G(x, z; \lambda) = \sum_{i=1}^n \frac{e_i(x) f_i^*(z)}{\lambda - \lambda_i}.$$

There are many other ways to find G , of course.^[26] See the articles on Green's functions and on Fredholm integral equations. It must be kept in mind that the above mathematics is purely formal, and a rigorous treatment involves some pretty sophisticated mathematics, including a good background knowledge of functional analysis, Hilbert spaces, distributions and so forth. Consult these articles and the references for more detail.

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Quantum Field Theory, SUSY, Quantum Geometry and Quantum Algebraic Topology

Quantum electrodynamics

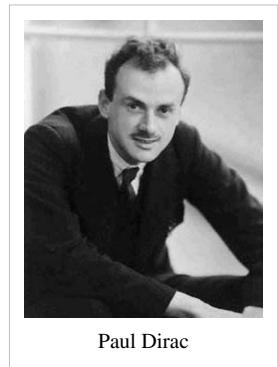
Quantum electrodynamics (QED) is the relativistic quantum field theory of electrodynamics. In essence, it describes how light and matter interact and is the first theory where full agreement between quantum mechanics and special relativity is achieved. QED mathematically describes all phenomena involving electrically charged particles interacting by means of exchange of photons and represents the quantum counterpart of classical electrodynamics giving a complete account of matter and light interaction. One of the founding fathers of QED, Richard Feynman, has called it "the jewel of physics" for its extremely accurate predictions of quantities like the anomalous magnetic moment of the electron, and the Lamb shift of the energy levels of hydrogen.^[1]

In technical terms, QED can be described as a perturbation theory of the electromagnetic quantum vacuum.

History

The first formulation of a quantum theory describing radiation and matter interaction is due to Paul Adrien Maurice Dirac, who, during 1920, was first able to compute the coefficient of spontaneous emission of an atom.^[2]

Dirac described the quantization of the electromagnetic field as an ensemble of harmonic oscillators with the introduction of the concept of creation and annihilation operators of particles. In the following years, with contributions from Wolfgang Pauli, Eugene Wigner, Pascual Jordan, Werner Heisenberg and an elegant formulation of quantum electrodynamics due to Enrico Fermi,^[3] physicists came to believe that, in principle, it would be possible to perform any computation for any physical process involving photons and charged particles. However, further studies by Felix Bloch with Arnold Nordsieck,^[4] and Victor Weisskopf,^[5] in 1937 and 1939, revealed that such computations were reliable only at a first order of perturbation theory, a problem already pointed out by Robert Oppenheimer.^[6] At higher orders in the series infinities emerged, making such computations meaningless and casting serious doubts on the internal consistency of the theory itself. With no solution for this problem known at the time, it appeared that a fundamental incompatibility existed between special relativity and quantum mechanics .



Paul Dirac



Hans Bethe

Difficulties with the theory increased through the end of 1940. Improvements in microwave technology made it possible to take more precise measurements of the shift of the levels of a hydrogen atom,^[7] now known as the Lamb shift and magnetic moment of the electron.^[8] These experiments unequivocally exposed discrepancies which the theory was unable to explain.

A first indication of a possible way out was given by Hans Bethe. In 1947, while he was traveling by train to reach Schenectady from New York,^[9] after giving a talk at the conference at Shelter Island on the subject, Bethe completed the first non-relativistic computation of the shift of the lines of the hydrogen atom as measured by Lamb and

Retherford.^[10] Despite the limitations of the computation, agreement was excellent. The idea was simply to attach infinities to corrections at mass and charge that were actually fixed to a finite value by experiments. In this way, the infinities get absorbed in those constants and yield a finite result in good agreement with experiments. This procedure was named renormalization.

Based on Bethe's intuition and fundamental papers on the subject by Sin-Itiro Tomonaga,^[11] Julian Schwinger,^{[12] [13]} Richard Feynman^{[14] [15] [16]} and Freeman Dyson^{[17] [18]}, it was finally possible to get fully covariant formulations that were finite at any order in a perturbation series of quantum electrodynamics. Sin-Itiro Tomonaga, Julian Schwinger and Richard Feynman were jointly awarded with a Nobel prize in physics in 1965 for their work in this area.^[19] Their contributions, and those of Freeman Dyson, were about covariant and gauge invariant formulations of quantum electrodynamics that allow computations of observables at any order of perturbation theory. Feynman's mathematical technique, based on his diagrams, initially seemed very different from the field-theoretic, operator-based approach of Schwinger and Tomonaga, but Freeman Dyson later showed that the two approaches were equivalent.^[17] Renormalization, the need to attach a physical meaning at certain divergences appearing in the theory through integrals, has subsequently become one of the fundamental aspects of quantum field theory and has come to be seen as a criterion for a theory's general acceptability. Even though renormalization works very well in practice, Feynman was never entirely comfortable with its mathematical validity, even referring to renormalization as a "shell game" and "hocus pocus".^[20]



Shelter Island Conference group photo (Courtesy of Archives, National Academy of Sciences).



Feynman (center) and Oppenheimer (left) at Los Alamos.

QED has served as the model and template for all subsequent quantum field theories. One such subsequent theory is quantum chromodynamics, which began in the early 1960s and attained its present form in the 1975 work by H. David Politzer, Sidney Coleman, David Gross and Frank Wilczek. Building on the pioneering work of Schwinger, Gerald Guralnik, Dick Hagen, and Tom Kibble,^{[21] [22]} Peter Higgs, Jeffrey Goldstone, and others, Sheldon Glashow, Steven Weinberg and Abdus Salam independently showed how the weak nuclear force and quantum electrodynamics could be merged into a single electroweak force.

Feynman's view of quantum electrodynamics

Introduction

Near the end of his life, Richard P. Feynman gave a series of lectures on QED intended for the lay public. These lectures were transcribed and published as Feynman (1985), *QED: The strange theory of light and matter*,^{[1] [20]} a classic non-mathematical exposition of QED from the point of view articulated below.

The key components of Feynman's presentation of QED are three basic actions.

- A photon goes from one place and time to another place and time.
- An electron goes from one place and time to another place and time.
- An electron emits or absorbs a photon at a certain place and time.

These actions are represented in a form of visual shorthand by the three basic elements of Feynman diagrams: a wavy line for the photon, a straight line for the electron and a junction of two straight lines and a wavy one for a vertex representing emission or absorption of a photon by an electron. These may all be seen in the adjacent diagram.

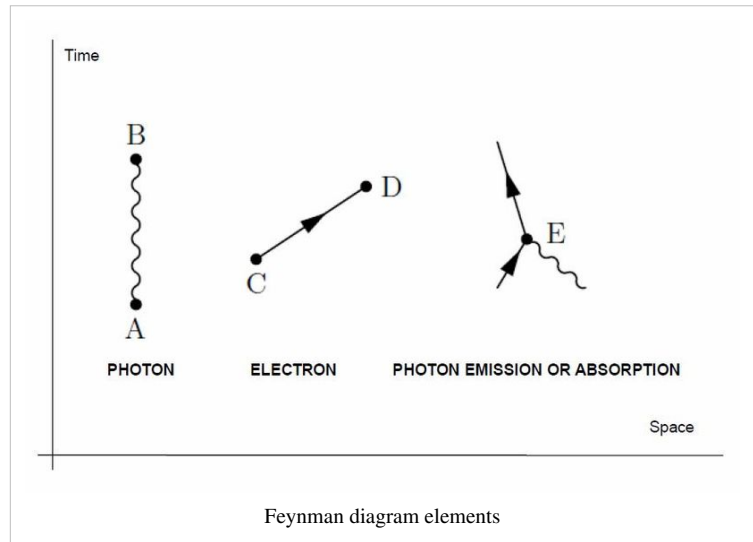
It is important not to over-interpret these diagrams. Nothing is implied about *how* a particle gets from one point to another. The diagrams do *not* imply that the particles are moving in straight or curved lines. They do *not* imply that the particles are moving with

fixed speeds. The fact that the photon is often represented, by convention, by a wavy line and not a straight one does *not* imply that it is thought that it is more wavelike than is an electron. The images are just symbols to represent the actions above: photons and electrons do, somehow, move from point to point and electrons, somehow, emit and absorb photons. We do not know how these things happen, but the theory tells us about the probabilities of these things happening. Trajectory is a meaningless concept in quantum mechanics.

As well as the visual shorthand for the actions Feynman introduces another kind of shorthand for the numerical quantities which tell us about the probabilities. If a photon moves from one place and time – in shorthand, A – to another place and time – shorthand, B – the associated quantity is written in Feynman's shorthand as $P(A \text{ to } B)$. The similar quantity for an electron moving from C to D is written $E(C \text{ to } D)$. The quantity which tells us about the probability for the emission or absorption of a photon he calls 'j'. This is related to, but not the same as, the measured electron charge 'e'.

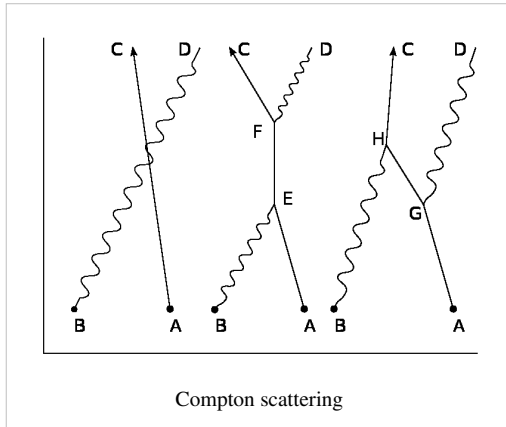
QED is based on the assumption that complex interactions of many electrons and photons can be represented by fitting together a suitable collection of the above three building blocks, and then using the probability-quantities to calculate the probability of any such complex interaction. It turns out that the basic idea of QED can be communicated while making the assumption that the quantities mentioned above are just our everyday probabilities. (A simplification of Feynman's book.) Later on this will be corrected to include specifically quantum mathematics, following Feynman.

The basic rules of probabilities that will be used are that a) if an event can happen in a variety of different ways then its probability is the **sum** of the probabilities of the possible ways and b) if a process involves a number of independent subprocesses then its probability is the **product** of the component probabilities.



Basic constructions

Suppose we start with one electron at a certain place and time (this place and time being given the arbitrary label A) and a photon at another place and time (given the label B). A typical question from a physical standpoint is: 'What is the probability of finding an electron at C (another place and a later time) and a photon at D (yet another place and time)?'. The simplest process to achieve this end is for the electron to move from A to C (an elementary action) and that the photon moves from B to D (another elementary action). From a knowledge of the probabilities of each of these subprocesses – $E(A \text{ to } C)$ and $P(B \text{ to } D)$ – then we would expect to calculate the probability of both happening by multiplying them, using rule b) above. This gives a simple estimated answer to our question.



But there are other ways in which the end result could come about. The electron might move to a place and time E where it absorbs the photon; then move on before emitting another photon at F; then move on to C where it is detected, while the new photon moves on to D. The probability of this complex process can again be calculated by knowing the probabilities of each of the individual actions: three electron actions, two photon actions and two vertexes – one emission and one absorption. We would expect to find the total probability by multiplying the probabilities of each of the actions, for any chosen positions of E and F. We then, using rule a) above, have to add up all these probabilities for all the alternatives for E and F. (This is not elementary in practice, and

involves integration.) But there is another possibility: that is that the electron first moves to G where it emits a photon which goes on to D, while the electron moves on to H, where it absorbs the first photon, before moving on to C. Again we can calculate the probability of these possibilities (for all points G and H). We then have a better estimation for the total probability by adding the probabilities of these two possibilities to our original simple estimate. Incidentally the name given to this process of a photon interacting with an electron in this way is Compton Scattering.

There are an *infinite number* of other intermediate processes in which more and more photons are absorbed and/or emitted. For each of these possibilities there is a Feynman diagram describing it. This implies a complex computation for the resulting probabilities, but provided it is the case that the more complicated the diagram the less it contributes to the result, it is only a matter of time and effort to find as accurate an answer as one wants to the original question. This is the basic approach of QED. To calculate the probability of **any** interactive process between electrons and photons it is a matter of first noting, with Feynman diagrams, all the possible ways in which the process can be constructed from the three basic elements. Each diagram involves some calculation involving definite rules to find the associated probability.

That basic scaffolding remains when one moves to a quantum description but some conceptual changes are requested. One is that whereas we might expect in our everyday life that there would be some constraints on the points to which a particle can move, that is **not** true in full quantum electrodynamics. There is a certain possibility of an electron or photon at A moving as a basic action to *any other place and time in the universe*. That includes places that could only be reached at speeds greater than that of light and also *earlier times*. (An electron moving backwards in time can be viewed as a positron moving forward in time.)

Probability amplitudes

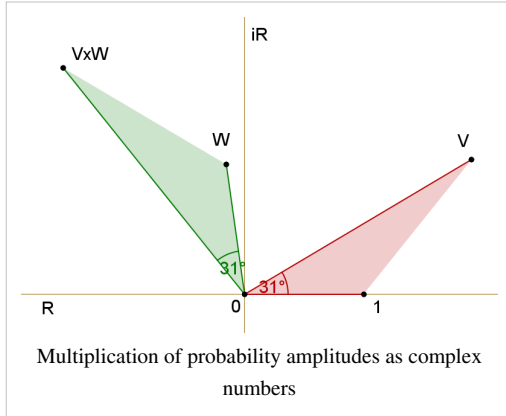
Quantum mechanics introduces an important change on the way probabilities are computed. It has been found that the quantities which we have to use to represent the probabilities are not the usual real numbers we use for probabilities in our everyday world, but complex numbers which are called probability amplitudes. Feynman avoids exposing the reader to the mathematics of complex numbers by using a simple but accurate representation of them as arrows on a piece of paper or screen. (These must not be confused with the arrows of Feynman diagrams which are actually simplified representations in two dimensions of a relationship between points in three dimensions of space and one of time.) The amplitude-arrows are fundamental to the description of the world given by quantum theory. No satisfactory reason has been given for *why* they are needed. But pragmatically we have to accept that they are an essential part of our description of all quantum phenomena. They are related to our everyday ideas of probability by the simple rule that the probability of an event is the **square** of the length of the corresponding amplitude-arrow. So, for a given process, if two probability amplitudes, $\mathbf{v}$ and $\mathbf{w}$, are involved, the probability of the process will be given either by

$$P = |\mathbf{v} + \mathbf{w}|^2$$

or

$$P = |\mathbf{v} \times \mathbf{w}|^2.$$

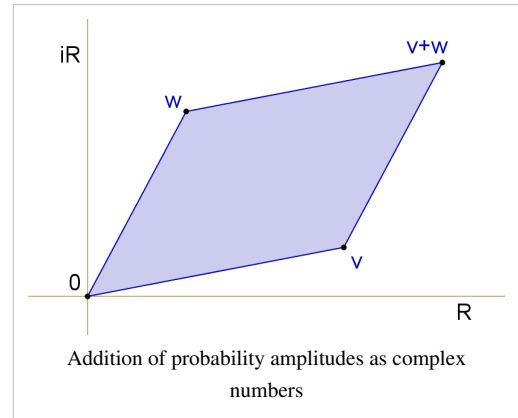
The rules as regards adding or multiplying, however, are the same as above. But where you would expect to add or multiply probabilities, instead you add or multiply probability amplitudes that now are complex numbers.



Addition and multiplication are familiar operations in the theory of complex numbers and are given in the figures. The sum is found as follows. Let the start of the second arrow be at the end of the first. The sum is then a third arrow that goes directly from the start of the first to the end of the second. The product of two arrows is an arrow whose length is the product of the two lengths. The direction of the product is found by adding the angles that each of the two have been turned through relative to a reference direction: that gives the angle that the product is turned relative to the reference direction.

That change, from probabilities to probability amplitudes, complicates the mathematics without changing the basic approach. But that change is still not quite enough because it fails to take into account the fact that both photons and electrons can be polarized, which is to say that their orientation in space and time have to be taken into account. Therefore $P(A \text{ to } B)$ actually consists of 16 complex numbers, or probability amplitude arrows. There are also some minor changes to do with the quantity " j ", which may have to be rotated by a multiple of 90° for some polarizations, which is only of interest for the detailed bookkeeping.

Associated with the fact that the electron can be polarized is another small necessary detail which is connected with the fact that an electron is a Fermion and obeys Fermi-Dirac statistics. The basic rule is that if we have the probability amplitude for a given complex process involving more than one electron, then when we include (as we always must) the complementary Feynman diagram in which we just exchange two electron events, the resulting amplitude is the reverse – the negative – of the first. The simplest case would be two electrons starting at A and B ending at C and D. The amplitude would be calculated as the "difference", $E(A \text{ to } B) \times E(C \text{ to } D) - E(A \text{ to } C) \times E(B \text{ to } D)$, where we would expect, from our everyday idea of probabilities, that it would be a sum.



Propagators

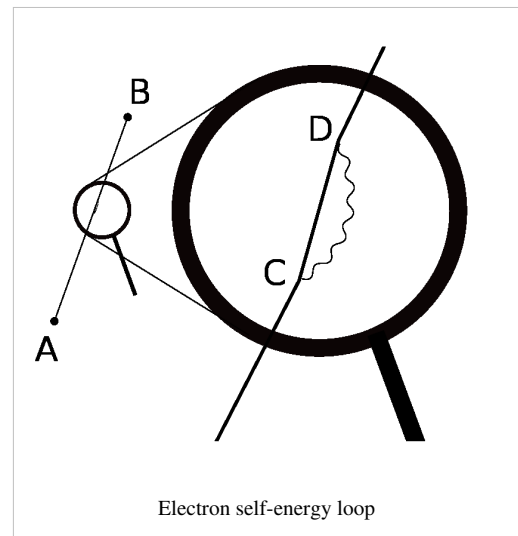
Finally, one has to compute $P(A \text{ to } B)$ and $E(C \text{ to } D)$ corresponding to the probability amplitudes for the photon and the electron respectively. These are essentially the solutions of the Dirac Equation which describes the behavior of the electron's probability amplitude and the Klein-Gordon equation which describes the behavior of the photon's probability amplitude. These are called Feynman propagators. The translation to a notation commonly used in the standard literature is as follows:

$$P(A \text{ to } B) \rightarrow D_F(x_B - x_A), \quad E(C \text{ to } D) \rightarrow S_F(x_D - x_C)$$

where a shorthand symbol such as x_A stands for the four real numbers which give the time and position in three dimensions of the point labeled A.

Mass renormalization

A problem arose historically which held up progress for twenty years: although we start with the assumption of three basic "simple" actions, the rules of the game say that if we want to calculate the probability amplitude for an electron to get from A to B we must take into account **all** the possible ways: all possible Feynman diagrams with those end points. Thus there will be a way in which the electron travels to C, emits a photon there and then absorbs it again at D before moving on to B. Or it could do this kind of thing twice, or more. In short we have a fractal-like situation in which if we look closely at a line it breaks up into a collection of "simple" lines, each of which, if looked at closely, are in turn composed of "simple" lines, and so on *ad infinitum*. This is a very difficult situation to handle. If adding that detail only altered things slightly then it would not have been too bad, but disaster struck when it was found that the simple correction mentioned above led to *infinite* probability amplitudes. In time this problem was "fixed" by the technique of renormalization (see below and the article on mass renormalization). However, Feynman himself remained unhappy about it, calling it a "dippy process".^[20]



Conclusions

Within the above framework physicists were then able to calculate to a high degree of accuracy some of the properties of electrons, such as the anomalous magnetic dipole moment. However, as Feynman points out, it fails totally to explain why particles such as the electron have the masses they do. "There is no theory that adequately explains these numbers. We use the numbers in all our theories, but we don't understand them – what they are, or where they come from. I believe that from a fundamental point of view, this is a very interesting and serious problem."^[23]

Mathematics

Mathematically, QED is an abelian gauge theory with the symmetry group $U(1)$. The gauge field, which mediates the interaction between the charged spin-1/2 fields, is the electromagnetic field. The QED Lagrangian for a spin-1/2 field interacting with the electromagnetic field is given by the real part of

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

where

γ^μ are Dirac matrices;

ψ a bispinor field of spin-1/2 particles (e.g. electron-positron field);

$\bar{\psi} \equiv \psi^\dagger \gamma_0$, called "psi-bar", is sometimes referred to as Dirac adjoint;

$D_\mu = \partial_\mu + ieA_\mu + ieB_\mu$ is the gauge covariant derivative;

e is the coupling constant, equal to the electric charge of the bispinor field;

A_μ is the covariant four-potential of the electromagnetic field generated by the electron itself;

B_μ is the external field imposed by external source;

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor.

Equations of motion

To begin, substituting the definition of D into the Lagrangian gives us:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - e\bar{\psi}\gamma_\mu(A^\mu + B^\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Next, we can substitute this Lagrangian into the Euler-Lagrange equation of motion for a field:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad (2)$$

to find the field equations for QED.

The two terms from this Lagrangian are then:

$$\begin{aligned} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) &= \partial_\mu (i\bar{\psi}\gamma^\mu) \\ \frac{\partial \mathcal{L}}{\partial \psi} &= -e\bar{\psi}\gamma_\mu(A^\mu + B^\mu) - m\bar{\psi}. \end{aligned}$$

Substituting these two back into the Euler-Lagrange equation (2) results in:

$$i\partial_\mu \bar{\psi}\gamma^\mu + e\bar{\psi}\gamma_\mu(A^\mu + B^\mu) + m\bar{\psi} = 0$$

with complex conjugate:

$$i\gamma^\mu \partial_\mu \psi - e\gamma_\mu(A^\mu + B^\mu)\psi - m\psi = 0.$$

Bringing the middle term to the right-hand side transforms this second equation into:

$$i\gamma^\mu \partial_\mu \psi - m\psi = e\gamma_\mu(A^\mu + B^\mu)\psi.$$

The left-hand side is like the original Dirac equation and the right-hand side is the interaction with the electromagnetic field.

One further important equation can be found by substituting the Lagrangian into another Euler-Lagrange equation, this time for the field, A^μ :

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0. \quad (3)$$

The two terms this time are:

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} \right) = \partial_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -e \bar{\psi} \gamma^\mu \psi$$

and these two terms, when substituted back into (3) give us:

$$\partial_\nu F^{\nu\mu} = e \bar{\psi} \gamma^\mu \psi.$$

Interaction picture

This theory can be straightforwardly quantized treating bosonic and fermionic sectors as free. This permits to build a set of asymptotic states to start a computation of the probability amplitudes for different processes. In order to be able to do so, we have to compute an evolution operator that, for a given initial state, will give a final state in such a way to have

$$M_{fi} = \langle f | U | i \rangle.$$

This technique is also known as the S-Matrix. Evolution operator is obtained in the interaction picture where time evolution is given by the interaction Hamiltonian. So, from equations above is

$$V = e \int d^3x \bar{\psi} \gamma^\mu \psi A_\mu$$

and so, one has

$$U = T \exp \left[-\frac{i}{\hbar} \int_{t_0}^t dt' V(t') \right]$$

being T the time ordering operator. This evolution operator has only a meaning as a series and what we get here is a perturbation series with a development parameter being fine structure constant. This series is named Dyson series.

Feynman diagrams

Despite the conceptual clarity of this Feynman approach to QED, almost no textbooks follow him in their presentation. When performing calculations it is much easier to work with the Fourier transforms of the propagators. Quantum physics considers particle's momenta rather than their positions, and it is convenient to think of particles as being created or annihilated when they interact. Feynman diagrams then *look* the same, but the lines have different interpretations. The electron line represents an electron with a given energy and momentum, with a similar interpretation of the photon line. A vertex diagram represents the annihilation of one electron and the creation of another together with the absorption or creation of a photon, each having specified energies and momenta.

Using Wick theorem on the terms of the Dyson series, all the terms of the S-matrix for quantum electrodynamics can be computed through the technique of Feynman diagrams. In this case rules for drawing are the following

$$\alpha \longrightarrow \beta \quad \rightarrow \quad \left(\frac{i}{\not{p} - m + i\varepsilon} \right)_{\beta\alpha}$$

$$\mu \text{ wavy line } \nu \quad \rightarrow \quad \frac{-i\eta_{\mu\nu}}{p^2 + i\varepsilon}$$

$$\begin{array}{c} \beta \\ \nearrow \\ \alpha \end{array} \text{ wavy line } \mu \quad \rightarrow \quad -ie\gamma_{\beta\alpha}^{\mu} (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3).$$

$$\text{Incoming fermion: } \alpha \longrightarrow \bullet \quad \rightarrow \quad u_{\alpha}(\vec{p}, s)$$

$$\text{Incoming antifermion: } \alpha \longleftarrow \bullet \quad \rightarrow \quad \bar{v}_{\alpha}(\vec{p}, s)$$

$$\text{Outgoing fermion: } \bullet \longrightarrow \alpha \quad \rightarrow \quad \bar{u}_{\alpha}(\vec{p}, s)$$

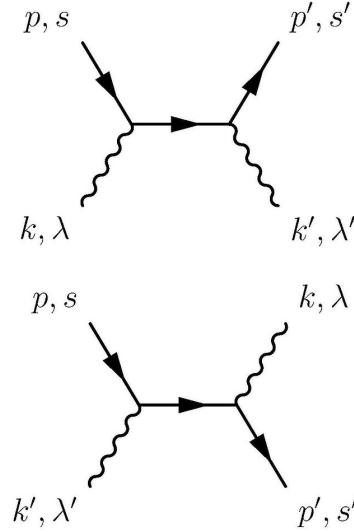
$$\text{Outgoing antifermion: } \bullet \longleftarrow \alpha \quad \rightarrow \quad v_{\alpha}(p, s)$$

$$\text{Incoming photon: } \mu \text{ wavy line } \bullet \quad \rightarrow \quad \epsilon_{\mu}(\vec{k}, \lambda)$$

$$\text{Outgoing photon: } \bullet \text{ wavy line } \mu \quad \rightarrow \quad \epsilon_{\mu}(\vec{k}, \lambda)^*$$

To these rules we must add a further one for closed loops that implies an integration on momenta $\int d^4p/(2\pi)^4$.

From them, computations of probability amplitudes are straightforwardly given. An example is Compton scattering, with an electron and a photon undergoing elastic scattering. Feynman diagrams are in this case



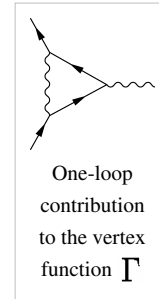
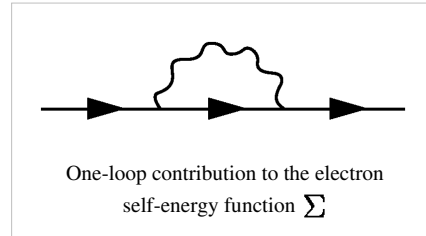
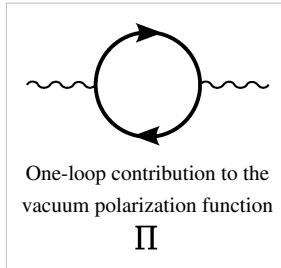
and so we are able to get the corresponding amplitude at the first order of a perturbation series for S-matrix:

$$M_{fi} = (ie)^2 \bar{u}(\vec{p}', s') \not{\epsilon}'(\vec{k}', \lambda')^* \frac{\not{p} + \not{k} + m_e}{(p+k)^2 - m_e^2} \not{\epsilon}(\vec{k}, \lambda) u(\vec{p}, s) + (ie)^2 \bar{u}(\vec{p}', s') \not{\epsilon}(\vec{k}, \lambda) \frac{\not{p} - \not{k}' + m_e}{(p-k')^2 - m_e^2} \not{\epsilon}'(\vec{k}', \lambda')$$

from which we are able to compute the cross section for this scattering.

Renormalizability

Higher order terms can be straightforwardly computed for the evolution operator but these terms display diagrams containing the following simpler ones



that, being closed loops, imply the presence of diverging integrals having no mathematical meaning. To overcome this difficulty, a technique like renormalization has been devised, producing finite results in very close agreement with experiments. It is important to note that a criterion for theory being meaningful after renormalization is that the number of diverging diagrams is finite. In this case the theory is said **renormalizable**. The reason for this is that to get observables renormalized one needs a finite number of constants to maintain the predictive value of the theory untouched. This is exactly the case of quantum electrodynamics displaying just three diverging diagrams. This procedure gives observables in very close agreement with experiment as seen e.g. for electron gyromagnetic ratio.

Renormalizability has become an essential criterion for a quantum field theory to be considered as a viable one. All the theories describing fundamental interactions, except gravitation whose quantum counterpart is presently under very active research, are renormalizable theories.

Nonconvergence of series

An argument by Freeman Dyson shows that the radius of convergence of the perturbation series in QED is zero.^[24] The basic argument goes as follows: if the coupling constant were negative, this would be equivalent to the Coulomb force constant being negative. This would "reverse" the electromagnetic interaction so that *like* charges would *attract* and *unlike* charges would *repel*. This would render the vacuum unstable against decay into a cluster of electrons on one side of the universe and a cluster of positrons on the other side of the universe. Because the theory is sick for any negative value of the coupling constant, the series do not converge, but are an asymptotic series. This can be taken as a need for a new theory, a problem with perturbation theory, or ignored by taking a "shut-up-and-calculate" approach.

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Journals

- Dudley, J.M., and Kwan, A.M. (1996) "Richard Feynman's popular lectures on quantum electrodynamics: The 1979 Robb Lectures at Auckland University," *American Journal of Physics* 64: 694-698.

External links

- Feynman's Nobel Prize lecture describing the evolution of QED and his role in it (<http://nobelprize.org/physics/laureates/1965/feynman-lecture.html>)
 - Feynman's New Zealand lectures on QED for non-physicists (<http://www.vega.org.uk/video/subseries/8>)
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Quantum field theory

Quantum field theory (QFT)^[1] provides a theoretical framework for constructing quantum mechanical models of systems classically parametrized (represented) by an infinite number of dynamical degrees of freedom, that is, fields and (in a condensed matter context) many-body systems. It is the natural and quantitative language of particle physics and condensed matter physics. Most theories in modern particle physics, including the Standard Model of elementary particles and their interactions, are formulated as relativistic quantum field theories. Quantum field theories are used in many contexts, elementary particle physics being the most vital example, where the particle count/number going into a reaction fluctuates and changes, differing from the count/number going out, for example, and for the description of critical phenomena and quantum phase transitions, such as in the BCS theory of superconductivity, also see phase transition, quantum phase transition, critical phenomena. Quantum field theory is thought by many to be the unique and correct outcome of combining the rules of quantum mechanics with special relativity.

In perturbative quantum field theory, the forces between particles are mediated by other particles. The electromagnetic force between two electrons is caused by an exchange of photons. Intermediate vector bosons mediate the weak force and gluons mediate the strong force. There is currently no complete quantum theory of the remaining fundamental force, gravity, but many of the proposed theories postulate the existence of a graviton particle that mediates it. These force-carrying particles are virtual particles and, by definition, cannot be detected while carrying the force, because such detection will imply that the force is not being carried. In addition, the notion of "force mediating particle" comes from perturbation theory, and thus does not make sense in a context of bound states.

In QFT photons are not thought of as 'little billiard balls', they are considered to be field quanta - necessarily chunked ripples in a field, or "excitations," that 'look like' particles. Fermions, like the electron, can also be described as ripples/excitations in a field, where each kind of fermion has its own field. In summary, the classical visualisation of "everything is particles and fields," in quantum field theory, resolves into "everything is particles," which then resolves into "everything is fields." In the end, particles are regarded as excited states of a field (field quanta). The gravitational field and the electromagnetic field are the only two fundamental fields in Nature that have infinite range and a corresponding classical low-energy limit, which greatly diminishes and hides their "particle-like" excitations. Albert Einstein, in 1905, attributed "particle-like" and discrete exchanges of momenta and energy, characteristic of "field quanta," to the electromagnetic field. Originally, his principal motivation was to explain the thermodynamics of radiation. Although it is often claimed that the photoelectric and Compton effects require a quantum description of the EM field, this is now understood to be untrue, and proper proof of the quantum nature of radiation is now taken up into modern quantum optics as in the antibunching effect^[2]. The word "photon" was coined in 1926 by the great physical chemist Gilbert Newton Lewis (see also the articles photon antibunching and laser).

The "low-energy limit" of the correct quantum field-theoretic description of the electromagnetic field, quantum electrodynamics, is believed to become James Clerk Maxwell's 1864 theory, although the "classical limit" of quantum electrodynamics has not been as widely explored as that of quantum mechanics. Presumably, the as yet unknown correct quantum field-theoretic treatment of the gravitational field will become and "look exactly like" Einstein's general theory of relativity in the "low-energy limit." Indeed, quantum field theory itself is quite possibly the low-energy-effective-field-theory limit of a more fundamental theory such as superstring theory. Compare in this context the article effective field theory.

History

Quantum field theory originated in the 1920s from the problem of creating a quantum mechanical theory of the electromagnetic field. In 1925 Werner Heisenberg, Max Born, and Pascual Jordan constructed such a theory by expressing the field's internal degrees of freedom as an infinite set of harmonic oscillators and by employing the usual procedure for quantizing those oscillators. This theory assumed that no electric charges or currents were present, and today would be called a free field theory. The first reasonably complete theory of quantum electrodynamics, which included both the electromagnetic field and electrically charged matter (specifically, electrons) as quantum mechanical objects, was created by Paul Dirac in 1927.^[3] This quantum field theory could be used to model important processes such as the emission of a photon by an electron dropping into a quantum state of lower energy, a process in which the *number of particles changes* — one atom in the initial state becomes an atom plus a photon in the final state. It is now understood that the ability to describe such processes is one of the most important features of quantum field theory.

It was evident from the beginning that a proper quantum treatment of the electromagnetic field had to somehow incorporate Einstein's relativity theory, which had after all grown out of the study of classical electromagnetism. This need to *put together relativity and quantum mechanics* was the second major motivation in the development of quantum field theory. Pascual Jordan and Wolfgang Pauli showed in 1928 that quantum fields could be made to behave in the way predicted by special relativity during coordinate transformations (specifically, they showed that the field commutators were Lorentz invariant). A further boost for quantum field theory came with the discovery of the Dirac equation, which was originally formulated and interpreted as a single-particle equation analogous to the Schrödinger equation, but unlike the Schrödinger equation, the Dirac equation satisfies both Lorentz invariance, that is, the requirements of special relativity, and the rules of quantum mechanics. The Dirac equation accommodated the spin-1/2 value of the electron and accounted for its magnetic moment as well as giving accurate predictions for the spectra of hydrogen. The attempted interpretation of the Dirac equation as a single-particle equation could not be maintained long, however, and finally it was shown that several of its undesirable properties (such as negative-energy states) could be made sense of by reformulating and reinterpreting the Dirac equation as a true field equation, in this case for the quantized "Dirac field" or the "electron field", with the "negative-energy solutions" pointing to the existence of anti-particles. This work was performed first by Dirac himself with the invention of hole theory 1930 and also by Wendell Furry, Robert Oppenheimer, Vladimir Fock, and others. Schrödinger, during the same period that he discovered his famous equation in 1926, also independently found the relativistic generalization of it known as the Klein-Gordon equation but dismissed it since, without spin, it predicted impossible properties for the hydrogen spectrum. See Oskar Klein, Walter Gordon. All relativistic wave equations that describe spin-zero particles are said to be of the Klein-Gordon type.

A subtle and careful analysis in 1933 and later in 1950 by Niels Bohr and Leon Rosenfeld showed that there is a fundamental limitation on the ability to simultaneously measure the electric and magnetic field strengths that enter into the description of charges in interaction with radiation, imposed by the uncertainty principle, which must apply to all canonically conjugate quantities. This limitation is crucial for the successful formulation and interpretation of a quantum field theory of photons and electrons (quantum electrodynamics), and indeed, any perturbative quantum field theory. The analysis of Bohr and Rosenfeld explains fluctuations in the values of the electromagnetic field that differ from the classically "allowed" values distant from the sources of the field. Their analysis was crucial to showing that the limitations and physical implications of the uncertainty principle apply to all dynamical systems, whether fields or material particles. Their analysis also convinced most people that any notion of returning to a fundamental description of nature based on classical field theory, such as what Einstein aimed at with his numerous and failed attempts at a classical unified field theory, was simply out of the question.

The third thread in the development of quantum field theory was the need to *handle the statistics of many-particle systems* consistently and with ease. In 1927 Jordan tried to extend the canonical quantization of fields to the many-body wave functions of identical particles, a procedure that is sometimes called second quantization. In 1928,

Jordan and Eugene Wigner found that the quantum field describing electrons, or other fermions, had to be expanded using anti-commuting creation and annihilation operators due to the Pauli exclusion principle. This thread of development was incorporated into many-body theory and strongly influenced condensed matter physics and nuclear physics.

Despite its early successes quantum field theory was plagued by several serious theoretical difficulties. Basic physical quantities, such as the self-energy of the electron, the energy shift of electron states due to the presence of the electromagnetic field, gave infinite, divergent contributions — a nonsensical result — when computed using the perturbative techniques available in the 1930s and most of the 1940s. The electron self-energy problem was already a serious issue in the classical electromagnetic field theory, where the attempt to attribute to the electron a finite size or extent (the classical electron-radius) led immediately to the question of what non-electromagnetic stresses would need to be invoked, which would presumably hold the electron together against the Coulomb repulsion of its finite-sized "parts". The situation was dire, and had certain features that reminded many of the "Rayleigh-Jeans difficulty". What made the situation in the 1940s so desperate and gloomy, however, was the fact that the correct ingredients (the second-quantized Maxwell-Dirac field equations) for the theoretical description of interacting photons and electrons were well in place, and no major conceptual change was needed analogous to that which was necessitated by a finite and physically sensible account of the radiative behavior of hot objects, as provided by the Planck radiation law.

This "divergence problem" was solved in the case of quantum electrodynamics during the late 1940s and early 1950s by Hans Bethe, Tomonaga, Schwinger, Feynman, and Dyson, through the procedure known as renormalization. Great progress was made after realizing that ALL infinities in quantum electrodynamics are related to two effects: the self-energy of the electron/positron and vacuum polarization. Renormalization concerns the business of paying very careful attention to just what is meant by, for example, the very concepts "charge" and "mass" as they occur in the pure, non-interacting field-equations. The "vacuum" is itself polarizable and, hence, populated by virtual particle (on shell and off shell) pairs, and, hence, is a seething and busy dynamical system in its own right. This was a critical step in identifying the source of "infinities" and "divergences". The "bare mass" and the "bare charge" of a particle, the values that appear in the free-field equations (non-interacting case), are abstractions that are simply not realized in experiment (in interaction). What we measure, and hence, what we must take account of with our equations, and what the solutions must account for, are the "renormalized mass" and the "renormalized charge" of a particle. That is to say, the "shifted" or "dressed" values these quantities must have when due care is taken to include all deviations from their "bare values" is dictated by the very nature of quantum fields themselves.

The first approach that bore fruit is known as the "interaction representation," (see the article Interaction picture) a Lorentz covariant and gauge-invariant generalization of time-dependent perturbation theory used in ordinary quantum mechanics, and developed by Tomonaga and Schwinger, generalizing earlier efforts of Dirac, Fock and Podolsky. Tomonaga and Schwinger invented a relativistically covariant scheme for representing field commutators and field operators intermediate between the two main representations of a quantum system, the Schrödinger and the Heisenberg representations (see the article on quantum mechanics). Within this scheme, field commutators at separated points can be evaluated in terms of "bare" field creation and annihilation operators. This allows for keeping track of the time-evolution of both the "bare" and "renormalized", or perturbed, values of the Hamiltonian and expresses everything in terms of the coupled, gauge invariant "bare" field-equations. Schwinger gave the most elegant formulation of this approach. The next and most famous development is due to Feynman, who, with his brilliant rules for assigning a "graph"/"diagram" to the terms in the scattering matrix (See S-Matrix Feynman diagrams). These directly corresponded (through the Schwinger-Dyson equation) to the measurable physical processes (cross sections, probability amplitudes, decay widths and lifetimes of excited states) one needs to be able to calculate. This revolutionized how quantum field theory calculations are carried-out in practice.

Two classic text-books from the 1960s, J.D. Bjorken and S.D. Drell, *Relativistic Quantum Mechanics* (1964) and J.J. Sakurai, *Advanced Quantum Mechanics* (1967), thoroughly developed the Feynman graph expansion techniques

using physically intuitive and practical methods following from the correspondence principle, without worrying about the technicalities involved in deriving the Feynman rules from the superstructure of quantum field theory itself. Although both Feynman's heuristic and pictorial style of dealing with the infinities, as well as the formal methods of Tomonaga and Schwinger, worked extremely well, and gave spectacularly accurate answers, the true analytical nature of the question of "renormalizability", that is, whether ANY theory formulated as a "quantum field theory" would give finite answers, was not worked-out till much later, when the urgency of trying to formulate finite theories for the strong and electro-weak (and gravitational interactions) demanded its solution.

Renormalization in the case of QED was largely fortuitous due to the smallness of the coupling constant, the fact that the coupling has no dimensions involving mass, the so-called fine structure constant, and also the zero-mass of the gauge boson involved, the photon, rendered the small-distance/high-energy behavior of QED manageable. Also, electromagnetic processes are very "clean" in the sense that they are not badly suppressed/damped and/or hidden by the other gauge interactions. By 1958 Sidney Drell observed: "Quantum electrodynamics (QED) has achieved a status of peaceful coexistence with its divergences...."

The unification of the electromagnetic force with the weak force encountered initial difficulties due to the lack of accelerator energies high enough to reveal processes beyond the Fermi interaction range. Additionally, a satisfactory theoretical understanding of hadron substructure had to be developed, culminating in the quark model.

In the case of the strong interactions, progress concerning their short-distance/high-energy behavior was much slower and more frustrating. For strong interactions with the electro-weak fields, there were difficult issues regarding the strength of coupling, the mass generation of the force carriers as well as their non-linear, self interactions. Although there has been theoretical progress toward a grand unified quantum field theory incorporating the electro-magnetic force, the weak force and the strong force, empirical verification is still pending. Superunification, incorporating the gravitational force, is still very speculative, and is under intensive investigation by many of the best minds in contemporary theoretical physics. Gravitation is a tensor field description of a spin-2 gauge-boson, the "graviton", and is further discussed in the articles on general relativity and quantum gravity.

From the point of view of the techniques of (four-dimensional) quantum field theory, and as the numerous and heroic efforts to formulate a consistent quantum gravity theory by some very able minds attests, gravitational quantization was, and is still, the reigning champion for bad behavior. There are problems and frustrations stemming from the fact that the gravitational coupling constant has dimensions involving inverse powers of mass, and as a simple consequence, it is plagued by badly behaved (in the sense of perturbation theory) non-linear and violent self-interactions. Gravity, basically, gravitates, which in turn...gravitates...and so on, (i.e., gravity is itself a source of gravity,...) thus creating a nightmare at all orders of perturbation theory. Also, gravity couples to all energy equally strongly, as per the equivalence principle, so this makes the notion of ever really "switching-off", "cutting-off" or separating, the gravitational interaction from other interactions ambiguous and impossible since, with gravitation, we are dealing with the very structure of space-time itself. (See general covariance and, for a modest, yet highly non-trivial and significant interplay between (QFT) and gravitation (spacetime), see the article Hawking radiation and references cited therein. Also quantum field theory in curved spacetime).

Thanks to the somewhat brute-force, clanky and heuristic methods of Feynman, and the elegant and abstract methods of Tomonaga/Schwinger, from the period of early renormalization, we do have the modern theory of quantum electrodynamics (QED). It is still the most accurate physical theory known, the prototype of a successful quantum field theory. Beginning in the 1950s with the work of Yang and Mills, as well as Ryoyu Utiyama, following the previous lead of Weyl and Pauli, deep explorations illuminated the types of symmetries and invariances any field theory must satisfy. QED, and indeed, all field theories, were generalized to a class of quantum field theories known as gauge theories. Quantum electrodynamics is the most famous example of what is known as an Abelian gauge theory. It relies on the symmetry group $U(1)$ and has one massless gauge field, the $U(1)$ gauge symmetry, dictating the form of the interactions involving the electromagnetic field, with the photon being the gauge boson. That symmetries dictate, limit and necessitate the form of interaction between particles is the essence of the "gauge theory

revolution." Yang and Mills formulated the first explicit example of a non-Abelian gauge theory, Yang-Mills theory, with an attempted explanation of the strong interactions in mind. The strong interactions were then (incorrectly) understood in the mid-1950s, to be mediated by the pi-mesons, the particles predicted by Hideki Yukawa in 1935, based on his profound reflections concerning the reciprocal connection between the mass of any force-mediating particle and the range of the force it mediates. This was allowed by the uncertainty principle. The 1960s and 1970s saw the formulation of a gauge theory now known as the Standard Model of particle physics, which systematically describes the elementary particles and the interactions between them.

The electroweak interaction part of the standard model was formulated by Sheldon Glashow in the years 1958-60 with his discovery of the $SU(2) \times U(1)$ group structure of the theory. Steven Weinberg and Abdus Salam brilliantly invoked the Anderson-Higgs mechanism for the generation of the W's and Z masses (the intermediate vector boson(s) responsible for the weak interactions and neutral-currents) and keeping the mass of the photon zero. The Goldstone/Higgs idea for generating mass in gauge theories was sparked in the late 1950s and early 1960s when a number of theoreticians (including Yoichiro Nambu, Steven Weinberg, Jeffrey Goldstone, François Englert, Robert Brout, G. S. Guralnik, C. R. Hagen, Tom Kibble and Philip Warren Anderson) noticed a possibly useful analogy to the (spontaneous) breaking of the $U(1)$ symmetry of electromagnetism in the formation of the BCS ground-state of a superconductor. The gauge boson involved in this situation, the photon, behaves as though it has acquired a finite mass. There is a further possibility that the physical vacuum (ground-state) does not respect the symmetries implied by the "unbroken" electroweak Lagrangian (see the article Electroweak interaction for more details) from which one arrives at the field equations. The electroweak theory of Weinberg and Salam was shown to be renormalizable (finite) and hence consistent by Gerardus 't Hooft and Martinus Veltman. The Glashow-Weinberg-Salam theory (GWS-Theory) is a triumph and, in certain applications, gives an accuracy on a par with quantum electrodynamics.

Also during the 1970s parallel developments in the study of phase transitions in condensed matter physics led Leo Kadanoff, Michael Fisher and Kenneth Wilson (extending work of Ernst Stueckelberg, Andre Peterman, Murray Gell-Mann, and Francis Low) to a set of ideas and methods known as the renormalization group. By providing a better physical understanding of the renormalization procedure invented in the 1940s, the renormalization group sparked what has been called the "grand synthesis" of theoretical physics, uniting the quantum field theoretical techniques used in particle physics and condensed matter physics into a single theoretical framework.

The study of quantum field theory is alive and flourishing, as are applications of this method to many physical problems. It remains one of the most vital areas of theoretical physics today, providing a common language to many branches of physics.

Principles of quantum field theory

Classical fields and quantum fields

Quantum mechanics, in its most general formulation, is a theory of abstract operators (observables) acting on an abstract state space (Hilbert space), where the observables represent physically observable quantities and the state space represents the possible states of the system under study. Furthermore, each observable corresponds, in a technical sense, to the classical idea of a degree of freedom. For instance, the fundamental observables associated with the motion of a single quantum mechanical particle are the position and momentum operators $\hat{x}$ and $\hat{p}$.

Ordinary quantum mechanics deals with systems such as this, which possess a small set of degrees of freedom.

(It is important to note, at this point, that this article does not use the word "particle" in the context of wave-particle duality. In quantum field theory, "particle" is a generic term for any discrete quantum mechanical entity, such as an electron or photon, which can behave like classical particles or classical waves under different experimental conditions.)

A **quantum field** is a quantum mechanical system containing a large, and possibly infinite, number of degrees of freedom. A classical field contains a set of degrees of freedom at each point of space; for instance, the classical

electromagnetic field defines two vectors — the electric field and the magnetic field — that can in principle take on distinct values for each position $\mathbf{r}$. When the field *as a whole* is considered as a quantum mechanical system, its observables form an infinite (in fact uncountable) set, because $\mathbf{r}$ is continuous.

Furthermore, the degrees of freedom in a quantum field are arranged in "repeated" sets. For example, the degrees of freedom in an electromagnetic field can be grouped according to the position $\mathbf{r}$, with exactly two vectors for each $\mathbf{r}$. Note that $\mathbf{r}$ is an ordinary number that "indexes" the observables; it is not to be confused with the position operator $\hat{x}$ encountered in ordinary quantum mechanics, which is an observable. (Thus, ordinary quantum mechanics is sometimes referred to as "zero-dimensional quantum field theory", because it contains only a single set of observables.)

It is also important to note that there is nothing special about $\mathbf{r}$ because, as it turns out, there is generally more than one way of indexing the degrees of freedom in the field.

In the following sections, we will show how these ideas can be used to construct a quantum mechanical theory with the desired properties. We will begin by discussing single-particle quantum mechanics and the associated theory of many-particle quantum mechanics. Then, by finding a way to index the degrees of freedom in the many-particle problem, we will construct a quantum field and study its implications.

Single-particle and many-particle quantum mechanics

In quantum mechanics, the time-dependent Schrödinger equation for a single particle is

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) + V(x) \psi(x, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t),$$

where m is the particle's mass, V is the applied potential, and $\psi(x, t)$ denotes the wavefunction.

We wish to consider how this problem generalizes to N particles. There are two motivations for studying the many-particle problem. The first is a straightforward need in condensed matter physics, where typically the number of particles is on the order of Avogadro's number (6.0221415×10^{23}). The second motivation for the many-particle problem arises from particle physics and the desire to incorporate the effects of special relativity. If one attempts to include the relativistic rest energy into the above equation (in quantum mechanics where position is an observable), the result is either the Klein-Gordon equation or the Dirac equation. However, these equations have many unsatisfactory qualities; for instance, they possess energy eigenvalues that extend to $-\infty$, so that there seems to be no easy definition of a ground state. It turns out that such inconsistencies arise from relativistic wavefunctions having a probabilistic interpretation in position space, as probability conservation is not a relativistically covariant concept. In quantum field theory, unlike in quantum mechanics, position is not an observable, and thus, one does not need the concept of a position-space probability density. For quantum fields whose interaction can be treated perturbatively, this is equivalent to neglecting the possibility of dynamically creating or destroying particles, which is a crucial aspect of relativistic quantum theory. Einstein's famous mass-energy relation allows for the possibility that sufficiently massive particles can decay into several lighter particles, and sufficiently energetic particles can combine to form massive particles. For example, an electron and a positron can annihilate each other to create photons. This suggests that a consistent relativistic quantum theory should be able to describe many-particle dynamics.

Furthermore, we will assume that the N particles are indistinguishable. As described in the article on identical particles, this implies that the state of the entire system must be either symmetric (bosons) or antisymmetric (fermions) when the coordinates of its constituent particles are exchanged. These multi-particle states are rather complicated to write. For example, the general quantum state of a system of N bosons is written as

$$|\phi_1 \cdots \phi_N\rangle = \sqrt{\frac{\prod_j N_j!}{N!}} \sum_{\mathbf{p} \in S_N} |\phi_{p(1)}\rangle \cdots |\phi_{p(N)}\rangle,$$

where $|\phi_i\rangle$ are the single-particle states, N_j is the number of particles occupying state j , and the sum is taken over all possible permutations $\mathbf{p}$ acting on N elements. In general, this is a sum of $N!(N \text{ factorial})$ distinct

terms, which quickly becomes unmanageable as N increases. The way to simplify this problem is to turn it into a quantum field theory.

Second quantization

In this section, we will describe a method for constructing a quantum field theory called **second quantization**. This basically involves choosing a way to index the quantum mechanical degrees of freedom in the space of multiple identical-particle states. It is based on the Hamiltonian formulation of quantum mechanics; several other approaches exist, such as the Feynman path integral,^[4] which uses a Lagrangian formulation. For an overview, see the article on quantization.

Second quantization of bosons

For simplicity, we will first discuss second quantization for bosons, which form perfectly symmetric quantum states. Let us denote the mutually orthogonal single-particle states by $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$, and so on. For example, the 3-particle state with one particle in state $|\phi_1\rangle$ and two in state $|\phi_2\rangle$ is

$$\frac{1}{\sqrt{3}} [|\phi_1\rangle|\phi_2\rangle|\phi_2\rangle + |\phi_2\rangle|\phi_1\rangle|\phi_2\rangle + |\phi_2\rangle|\phi_2\rangle|\phi_1\rangle].$$

The first step in second quantization is to express such quantum states in terms of **occupation numbers**, by listing the number of particles occupying each of the single-particle states $|\phi_1\rangle, |\phi_2\rangle$, etc. This is simply another way of labelling the states. For instance, the above 3-particle state is denoted as

$$|1, 2, 0, 0, \dots\rangle.$$

The next step is to expand the N -particle state space to include the state spaces for all possible values of N . This extended state space, known as a Fock space, is composed of the state space of a system with no particles (the so-called vacuum state), plus the state space of a 1-particle system, plus the state space of a 2-particle system, and so forth. It is easy to see that there is a one-to-one correspondence between the occupation number representation and valid boson states in the Fock space.

At this point, the quantum mechanical system has become a quantum field in the sense we described above. The field's elementary degrees of freedom are the occupation numbers, and each occupation number is indexed by a number $j \dots$, indicating which of the single-particle states $|\phi_1\rangle, |\phi_2\rangle, \dots |\phi_j\rangle \dots$ it refers to.

The properties of this quantum field can be explored by defining creation and annihilation operators, which add and subtract particles. They are analogous to "ladder operators" in the quantum harmonic oscillator problem, which added and subtracted energy quanta. However, these operators literally create and annihilate particles of a given quantum state. The bosonic annihilation operator a_2 and creation operator $a_2^\dagger$ have the following effects:

$$\begin{aligned} a_2 |N_1, N_2, N_3, \dots\rangle &= \sqrt{N_2} |N_1, (N_2 - 1), N_3, \dots\rangle, \\ a_2^\dagger |N_1, N_2, N_3, \dots\rangle &= \sqrt{N_2 + 1} |N_1, (N_2 + 1), N_3, \dots\rangle. \end{aligned}$$

It can be shown that these are operators in the usual quantum mechanical sense, i.e. linear operators acting on the Fock space. Furthermore, they are indeed Hermitian conjugates, which justifies the way we have written them. They can be shown to obey the commutation relation

$$[a_i, a_j] = 0 \quad , \quad [a_i^\dagger, a_j^\dagger] = 0 \quad , \quad [a_i, a_j^\dagger] = \delta_{ij},$$

where δ stands for the Kronecker delta. These are precisely the relations obeyed by the ladder operators for an infinite set of independent quantum harmonic oscillators, one for each single-particle state. Adding or removing bosons from each state is therefore analogous to exciting or de-exciting a quantum of energy in a harmonic oscillator.

The Hamiltonian of the quantum field (which, through the Schrödinger equation, determines its dynamics) can be written in terms of creation and annihilation operators. For instance, the Hamiltonian of a field of free

(non-interacting) bosons is

$$H = \sum_{\mathbf{k}} E_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}},$$

where $E_{\mathbf{k}}$ is the energy of the $\mathbf{k}$ -th single-particle energy eigenstate. Note that

$$a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} | \dots, N_{\mathbf{k}}, \dots \rangle = N_{\mathbf{k}} | \dots, N_{\mathbf{k}}, \dots \rangle.$$

Second quantization of fermions

It turns out that a different definition of creation and annihilation must be used for describing fermions. According to the Pauli exclusion principle, fermions cannot share quantum states, so their occupation numbers N_i can only take on the value 0 or 1. The fermionic annihilation operators c and creation operators $c^{\dagger}$ are defined by their actions on a Fock state thus

$$\begin{aligned} c_j |N_1, N_2, \dots, N_j = 0, \dots\rangle &= 0 \\ c_j |N_1, N_2, \dots, N_j = 1, \dots\rangle &= (-1)^{(N_1 + \dots + N_{j-1})} |N_1, N_2, \dots, N_j = 0, \dots\rangle \\ c_j^{\dagger} |N_1, N_2, \dots, N_j = 0, \dots\rangle &= (-1)^{(N_1 + \dots + N_{j-1})} |N_1, N_2, \dots, N_j = 1, \dots\rangle \\ c_j^{\dagger} |N_1, N_2, \dots, N_j = 1, \dots\rangle &= 0. \end{aligned}$$

These obey an anticommutation relation:

$$\{c_i, c_j\} = 0 \quad , \quad \{c_i^{\dagger}, c_j^{\dagger}\} = 0 \quad , \quad \{c_i, c_j^{\dagger}\} = \delta_{ij}.$$

One may notice from this that applying a fermionic creation operator twice gives zero, so it is impossible for the particles to share single-particle states, in accordance with the exclusion principle.

Field operators

We have previously mentioned that there can be more than one way of indexing the degrees of freedom in a quantum field. Second quantization indexes the field by enumerating the single-particle quantum states. However, as we have discussed, it is more natural to think about a "field", such as the electromagnetic field, as a set of degrees of freedom indexed by position.

To this end, we can define *field operators* that create or destroy a particle at a particular point in space. In particle physics, these operators turn out to be more convenient to work with, because they make it easier to formulate theories that satisfy the demands of relativity.

Single-particle states are usually enumerated in terms of their momenta (as in the particle in a box problem.) We can construct field operators by applying the Fourier transform to the creation and annihilation operators for these states. For example, the bosonic field annihilation operator $\phi(\mathbf{r})$ is

$$\phi(\mathbf{r}) \stackrel{\text{def}}{=} \sum_j e^{i\mathbf{k}_j \cdot \mathbf{r}} a_j.$$

The bosonic field operators obey the commutation relation

$$[\phi(\mathbf{r}), \phi(\mathbf{r}')] = 0 \quad , \quad [\phi^{\dagger}(\mathbf{r}), \phi^{\dagger}(\mathbf{r}')] = 0 \quad , \quad [\phi(\mathbf{r}), \phi^{\dagger}(\mathbf{r}')] = \delta^3(\mathbf{r} - \mathbf{r}')$$

where $\delta(x)$ stands for the Dirac delta function. As before, the fermionic relations are the same, with the commutators replaced by anticommutators.

It should be emphasized that the field operator is *not* the same thing as a single-particle wavefunction. The former is an operator acting on the Fock space, and the latter is a quantum-mechanical amplitude for finding a particle in some position. However, they are closely related, and are indeed commonly denoted with the same symbol. If we have a Hamiltonian with a space representation, say

$$H = -\frac{\hbar^2}{2m} \sum_i \nabla_i^2 + \sum_{i < j} U(|\mathbf{r}_i - \mathbf{r}_j|)$$

where the indices i and j run over all particles, then the field theory Hamiltonian (in the non-relativistic limit and for negligible self-interactions) is

$$H = -\frac{\hbar^2}{2m} \int d^3r \phi^\dagger(\mathbf{r}) \nabla^2 \phi(\mathbf{r}) + \int d^3r \int d^3r' \phi^\dagger(\mathbf{r}) \phi^\dagger(\mathbf{r}') U(|\mathbf{r} - \mathbf{r}'|) \phi(\mathbf{r}') \phi(\mathbf{r}).$$

This looks remarkably like an expression for the expectation value of the energy, with ϕ playing the role of the wavefunction. This relationship between the field operators and wavefunctions makes it very easy to formulate field theories starting from space-projected Hamiltonians.

Implications of quantum field theory

Unification of fields and particles

The "second quantization" procedure that we have outlined in the previous section takes a set of single-particle quantum states as a starting point. Sometimes, it is impossible to define such single-particle states, and one must proceed directly to quantum field theory. For example, a quantum theory of the electromagnetic field *must* be a quantum field theory, because it is impossible (for various reasons) to define a wavefunction for a single photon. In such situations, the quantum field theory can be constructed by examining the mechanical properties of the classical field and guessing the corresponding quantum theory. For free (non-interacting) quantum fields, the quantum field theories obtained in this way have the same properties as those obtained using second quantization, such as well-defined creation and annihilation operators obeying commutation or anticommutation relations.

Quantum field theory thus provides a unified framework for describing "field-like" objects (such as the electromagnetic field, whose excitations are photons) and "particle-like" objects (such as electrons, which are treated as excitations of an underlying electron field), so long as one can treat interactions as "perturbations" of free fields. There are still unsolved problems relating to the more general case of interacting fields that may or may not be adequately described by perturbation theory. For more on this topic, see Haag's theorem.

Physical meaning of particle indistinguishability

The second quantization procedure relies crucially on the particles being identical. We would not have been able to construct a quantum field theory from a distinguishable many-particle system, because there would have been no way of separating and indexing the degrees of freedom.

Many physicists prefer to take the converse interpretation, which is that *quantum field theory explains what identical particles are*. In ordinary quantum mechanics, there is not much theoretical motivation for using symmetric (bosonic) or antisymmetric (fermionic) states, and the need for such states is simply regarded as an empirical fact. From the point of view of quantum field theory, particles are identical if and only if they are excitations of the same underlying quantum field. Thus, the question "why are all electrons identical?" arises from mistakenly regarding individual electrons as fundamental objects, when in fact it is only the electron field that is fundamental.

Particle conservation and non-conservation

During second quantization, we started with a Hamiltonian and state space describing a fixed number of particles (N), and ended with a Hamiltonian and state space for an arbitrary number of particles. Of course, in many common situations N is an important and perfectly well-defined quantity, e.g. if we are describing a gas of atoms sealed in a box. From the point of view of quantum field theory, such situations are described by quantum states that are eigenstates of the number operator $\hat{N}$, which measures the total number of particles present. As with any quantum mechanical observable, $\hat{N}$ is conserved if it commutes with the Hamiltonian. In that case, the quantum state is trapped in the N -particle subspace of the total Fock space, and the situation could equally well be described by ordinary N -particle quantum mechanics. (Strictly speaking, this is only true in the noninteracting case or in the low energy density limit of renormalized quantum field theories)

For example, we can see that the free-boson Hamiltonian described above conserves particle number. Whenever the Hamiltonian operates on a state, each particle destroyed by an annihilation operator $a_{\mathbf{k}}$ is immediately put back by the creation operator $a_{\mathbf{k}}^\dagger$.

On the other hand, it is possible, and indeed common, to encounter quantum states that are *not* eigenstates of $\hat{N}$, which do not have well-defined particle numbers. Such states are difficult or impossible to handle using ordinary quantum mechanics, but they can be easily described in quantum field theory as quantum superpositions of states having different values of N . For example, suppose we have a bosonic field whose particles can be created or destroyed by interactions with a fermionic field. The Hamiltonian of the combined system would be given by the Hamiltonians of the free boson and free fermion fields, plus a "potential energy" term such as

$$H_I = \sum_{\mathbf{k}, \mathbf{q}} V_{\mathbf{q}} (a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger) c_{\mathbf{k}+\mathbf{q}}^\dagger c_{\mathbf{k}},$$

where $a_{\mathbf{k}}^\dagger$ and $a_{\mathbf{k}}$ denotes the bosonic creation and annihilation operators, $c_{\mathbf{k}}^\dagger$ and $c_{\mathbf{k}}$ denotes the fermionic creation and annihilation operators, and $V_{\mathbf{q}}$ is a parameter that describes the strength of the interaction. This "interaction term" describes processes in which a fermion in state $\mathbf{k}$ either absorbs or emits a boson, thereby being kicked into a different eigenstate $\mathbf{k} + \mathbf{q}$. (In fact, this type of Hamiltonian is used to describe interaction between conduction electrons and phonons in metals. The interaction between electrons and photons is treated in a similar way, but is a little more complicated because the role of spin must be taken into account.) One thing to notice here is that even if we start out with a fixed number of bosons, we will typically end up with a superposition of states with different numbers of bosons at later times. The number of fermions, however, is conserved in this case. In condensed matter physics, states with ill-defined particle numbers are particularly important for describing the various superfluids. Many of the defining characteristics of a superfluid arise from the notion that its quantum state is a superposition of states with different particle numbers. In addition, the concept of a coherent state (used to model the laser and the BCS ground state) refers to a state with an ill-defined particle number but a well-defined phase.

Axiomatic approaches

The preceding description of quantum field theory follows the spirit in which most physicists approach the subject. However, it is not mathematically rigorous. Over the past several decades, there have been many attempts to put quantum field theory on a firm mathematical footing by formulating a set of axioms for it. These attempts fall into two broad classes.

The first class of axioms, first proposed during the 1950s, include the Wightman, Osterwalder-Schrader, and Haag-Kastler systems. They attempted to formalize the physicists' notion of an "operator-valued field" within the context of functional analysis, and enjoyed limited success. It was possible to prove that any quantum field theory satisfying these axioms satisfied certain general theorems, such as the spin-statistics theorem and the CPT theorem. Unfortunately, it proved extraordinarily difficult to show that any realistic field theory, including the Standard Model, satisfied these axioms. Most of the theories that could be treated with these analytic axioms were physically trivial, being restricted to low-dimensions and lacking interesting dynamics. The construction of theories satisfying one of these sets of axioms falls in the field of constructive quantum field theory. Important work was done in this area in the 1970s by Segal, Glimm, Jaffe and others.

During the 1980s, a second set of axioms based on geometric ideas was proposed. This line of investigation, which restricts its attention to a particular class of quantum field theories known as topological quantum field theories, is associated most closely with Michael Atiyah and Graeme Segal, and was notably expanded upon by Edward Witten, Richard Borcherds, and Maxim Kontsevich. However, most of the physically relevant quantum field theories, such as the Standard Model, are not topological quantum field theories; the quantum field theory of the fractional quantum Hall effect is a notable exception. The main impact of axiomatic topological quantum field theory has been on mathematics, with important applications in representation theory, algebraic topology, and differential geometry.

Finding the proper axioms for quantum field theory is still an open and difficult problem in mathematics. One of the Millennium Prize Problems—proving the existence of a mass gap in Yang-Mills theory—is linked to this issue.

Phenomena associated with quantum field theory

In the previous part of the article, we described the most general properties of quantum field theories. Some of the quantum field theories studied in various fields of theoretical physics possess additional special properties, such as renormalizability, gauge symmetry, and supersymmetry. These are described in the following sections.

Renormalization

Early in the history of quantum field theory, it was found that many seemingly innocuous calculations, such as the perturbative shift in the energy of an electron due to the presence of the electromagnetic field, give infinite results. The reason is that the perturbation theory for the shift in an energy involves a sum over all other energy levels, and there are infinitely many levels at short distances that each give a finite contribution.

Many of these problems are related to failures in classical electrodynamics that were identified but unsolved in the 19th century, and they basically stem from the fact that many of the supposedly "intrinsic" properties of an electron are tied to the electromagnetic field that it carries around with it. The energy carried by a single electron— its self energy— is not simply the bare value, but also includes the energy contained in its electromagnetic field, its attendant cloud of photons. The energy in a field of a spherical source diverges in both classical and quantum mechanics, but as discovered by Weisskopf, in quantum mechanics the divergence is much milder, going only as the logarithm of the radius of the sphere.

The solution to the problem, presciently suggested by Stueckelberg, independently by Bethe after the crucial experiment by Lamb, implemented at one loop by Schwinger, and systematically extended to all loops by Feynman and Dyson, with converging work by Tomonaga in isolated postwar Japan, is called renormalization. The technique of renormalization recognizes that the problem is essentially purely mathematical, that extremely short distances are at fault. In order to define a theory on a continuum, first place a cutoff on the fields, by postulating that quanta cannot have energies above some extremely high value. This has the effect of replacing continuous space by a structure where very short wavelengths do not exist, as on a lattice. Lattices break rotational symmetry, and one of the crucial contributions made by Feynman, Pauli and Villars, and modernized by 't Hooft and Veltman, is a symmetry preserving cutoff for perturbation theory. There is no known symmetrical cutoff outside of perturbation theory, so for rigorous or numerical work people often use an actual lattice.

On a lattice, every quantity is finite but depends on the spacing. When taking the limit of zero spacing, we make sure that the physically observable quantities like the observed electron mass stay fixed, which means that the constants in the Lagrangian defining the theory depend on the spacing. Hopefully, by allowing the constants to vary with the lattice spacing, all the results at long distances become insensitive to the lattice, defining a continuum limit.

The renormalization procedure only works for a certain class of quantum field theories, called **renormalizable quantum field theories**. A theory is **perturbatively renormalizable** when the constants in the Lagrangian only diverge at worst as logarithms of the lattice spacing for very short spacings. The continuum limit is then well defined in perturbation theory, and even if it is not fully well defined non-perturbatively, the problems only show up at distance scales that are exponentially small in the inverse coupling for weak couplings. The Standard Model of particle physics is perturbatively renormalizable, and so are its component theories (quantum electrodynamics/electroweak theory and quantum chromodynamics). Of the three components, quantum electrodynamics is believed to not have a continuum limit, while the asymptotically free SU(2) and SU(3) weak hypercharge and strong color interactions are nonperturbatively well defined.

The renormalization group describes how renormalizable theories emerge as the long distance low-energy effective field theory for any given high-energy theory. Because of this, renormalizable theories are insensitive to the precise

nature of the underlying high-energy short-distance phenomena. This is a blessing because it allows physicists to formulate low energy theories without knowing the details of high energy phenomenon. It is also a curse, because once a renormalizable theory like the standard model is found to work, it gives very few clues to higher energy processes. The only way high energy processes can be seen in the standard model is when they allow otherwise forbidden events, or if they predict quantitative relations between the coupling constants.

Gauge freedom

A gauge theory is a theory that admits a symmetry with a local parameter. For example, in every quantum theory the global phase of the wave function is arbitrary and does not represent something physical. Consequently, the theory is invariant under a global change of phases (adding a constant to the phase of all wave functions, everywhere); this is a global symmetry. In quantum electrodynamics, the theory is also invariant under a *local* change of phase, that is - one may shift the phase of all wave functions so that the shift may be different at every point in space-time. This is a *local* symmetry. However, in order for a well-defined derivative operator to exist, one must introduce a new field, the gauge field, which also transforms in order for the local change of variables (the phase in our example) not to affect the derivative. In quantum electrodynamics this gauge field is the electromagnetic field. The change of local gauge of variables is termed gauge transformation.

In quantum field theory the excitations of fields represent particles. The particle associated with excitations of the gauge field is the gauge boson, which is the photon in the case of quantum electrodynamics.

The degrees of freedom in quantum field theory are local fluctuations of the fields. The existence of a gauge symmetry reduces the number of degrees of freedom, simply because some fluctuations of the fields can be transformed to zero by gauge transformations, so they are equivalent to having no fluctuations at all, and they therefore have no physical meaning. Such fluctuations are usually called "non-physical degrees of freedom" or *gauge artifacts*; usually some of them have a negative norm, making them inadequate for a consistent theory. Therefore, if a classical field theory has a gauge symmetry, then its quantized version (i.e. the corresponding quantum field theory) will have this symmetry as well. In other words, a gauge symmetry cannot have a quantum anomaly. If a gauge symmetry is anomalous (i.e. not kept in the quantum theory) then the theory is non-consistent: for example, in quantum electrodynamics, had there been a gauge anomaly, this would require the appearance of photons with longitudinal polarization and polarization in the time direction, the latter having a negative norm, rendering the theory inconsistent; another possibility would be for these photons to appear only in intermediate processes but not in the final products of any interaction, making the theory non unitary and again inconsistent (see optical theorem).

In general, the gauge transformations of a theory consist of several different transformations, which may not be commutative. These transformations are together described by a mathematical object known as a gauge group. Infinitesimal gauge transformations are the gauge group generators. Therefore the number of gauge bosons is the group dimension (i.e. number of generators forming a basis).

All the fundamental interactions in nature are described by gauge theories. These are:

- Quantum electrodynamics, whose gauge transformation is a local change of phase, so that the gauge group is $U(1)$. The gauge boson is the photon.
- Quantum chromodynamics, whose gauge group is $SU(3)$. The gauge bosons are eight gluons.
- The electroweak Theory, whose gauge group is $U(1) \otimes SU(2)$ (a direct product of $U(1)$ and $SU(2)$).
- Gravity, whose classical theory is general relativity, admits the equivalence principle, which is a form of gauge symmetry. However, it is explicitly non-renormalizable.

Multivalued Gauge Transformations

The gauge transformations which leave the theory invariant involve by definition only single-valued gauge functions $\Lambda(x_i)$ which satisfy the Schwarz integrability criterion

$$\partial_{x_i x_j} \Lambda = \partial_{x_j x_i} \Lambda.$$

An interesting extension of gauge transformations arises if the gauge functions $\Lambda(x_i)$ are allowed to be multivalued functions which violate the integrability criterion. These are capable of changing the physical field strengths and are therefore no proper symmetry transformations. Nevertheless, the transformed field equations describe correctly the physical laws in the presence of the newly generated field strengths. See the textbook by H. Kleinert cited below for the applications to phenomena in physics.

Supersymmetry

Supersymmetry assumes that every fundamental fermion has a superpartner that is a boson and vice versa. It was introduced in order to solve the so-called Hierarchy Problem, that is, to explain why particles not protected by any symmetry (like the Higgs boson) do not receive radiative corrections to its mass driving it to the larger scales (GUT, Planck...). It was soon realized that supersymmetry has other interesting properties: its gauged version is an extension of general relativity (Supergravity), and it is a key ingredient for the consistency of string theory.

The way supersymmetry protects the hierarchies is the following: since for every particle there is a superpartner with the same mass, any loop in a radiative correction is cancelled by the loop corresponding to its superpartner, rendering the theory UV finite.

Since no superpartners have yet been observed, if supersymmetry exists it must be broken (through a so-called soft term, which breaks supersymmetry without ruining its helpful features). The simplest models of this breaking require that the energy of the superpartners not be too high; in these cases, supersymmetry is expected to be observed by experiments at the Large Hadron Collider.

See also

- List of quantum field theories
- Constructive quantum field theory
- Feynman path integral
- Quantum chromodynamics
- Quantum electrodynamics
- Quantum flavordynamics
- Quantum geometrodynamics
- Quantum hydrodynamics
- Quantum magnetodynamics
- Quantum triviality
- Schwinger-Dyson equation
- Einstein-Maxwell-Dirac equations
- Relation between Schrödinger's equation and the path integral formulation of quantum mechanics
- Basic concepts of quantum mechanics
- Relationship between string theory and quantum field theory
- Abraham-Lorentz force
- Form factor
- Photon polarization
- Theoretical and experimental justification for the Schrödinger equation
- Invariance mechanics
- Green-Kubo relations
- Green's function (many-body theory)
- Common integrals in quantum field theory
- Wheeler-Feynman absorber theory
- Wigner's theorem
- Wigner's classification
- Static forces and virtual-particle exchange

Notes

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- [2] (http://physics.princeton.edu/~mcdonald/examples/QM/thorn_ajp_72_1210_04.pdf)
- [3] Dirac, P.A.M. (1927). *The Quantum Theory of the Emission and Absorption of Radiation*, Proceedings of the Royal Society of London, Series A, Vol. 114, p. 243.
- [4] Abraham Pais, *Inward Bound: Of Matter and Forces in the Physical World* ISBN 0-19-851997-4. Pais recounts how his astonishment at the rapidity with which Feynman could calculate using his method. Feynman's method is now part of the standard methods for physicists.

Further reading

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External links

- Stanford Encyclopedia of Philosophy: "Quantum Field Theory, (<http://plato.stanford.edu/entries/quantum-field-theory/>)" by Meinard Kuhlmann.
- Siegel, Warren, 2005. *Fields*. (<http://insti.physics.sunysb.edu/~siegel/errata.html>) A free text, also available from arXiv:hep-th/9912205.
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- Free condensed matter books and notes (<http://www.freebookcentre.net/Physics/Condensed-Matter-Books.html>).
- Quantum field theory texts (<http://motls.blogspot.com/2006/01/qft-didactics.html>), a list with links to amazon.com.
- Quantum Field Theory (<http://www.nat.vu.nl/~mulders/QFT-0.pdf>) by P. J. Mulders
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- Quantum Field Theory Video Lectures (<http://www.physics.harvard.edu/about/Phys253.html>) by Sidney R. Coleman
- Quantum Field Theory Lecture Notes (http://www.physics.gla.ac.uk/~dmiller/lectures/RQF_1-6_2010.pdf) by D.J. Miller
- Quantum Field Theory Lecture Notes II (http://www.physics.gla.ac.uk/~dmiller/lectures/RQF_7-9_2010.pdf) by D.J. Miller

Scalar field theory

In theoretical physics, **scalar field theory** can refer to a classical or quantum theory of scalar fields. A field which is invariant under any Lorentz transformation is called a "scalar", in contrast to a vector or tensor field. The quanta of the quantized scalar field are spin-zero particles, and as such are bosons.

No fundamental scalar fields have been observed in nature, though the Higgs boson may yet prove the first example. However, scalar fields appear in the effective field theory descriptions of many physical phenomena. An example is the pion, which is actually a "pseudoscalar", which means it is not invariant under parity transformations which invert the spatial directions, distinguishing it from a true scalar, which is parity-invariant. Because of the relative simplicity of the mathematics involved, scalar fields are often the first field introduced to a student of classical or quantum field theory.

In this article, the repeated index notation indicates the Einstein summation convention for summation over repeated indices. The theories described are defined in flat, D-dimensional Minkowski space, with (D-1) spatial dimension and one time dimension and are, by construction, relativistically covariant. The Minkowski space metric, $\eta_{\mu\nu}$, has a particularly simple form: it is diagonal, and here we use the + − − − sign convention.

Classical scalar field theory

Linear (free) theory

The most basic scalar field theory is the linear theory. The action for the free relativistic scalar field theory is

$$\begin{aligned}\mathcal{S} &= \int d^{D-1}x dt \mathcal{L} = \int d^{D-1}x dt \left[\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right] \\ &= \int d^{D-1}x dt \left[\frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} \delta^{ij} \partial_i \phi \partial_j \phi - \frac{1}{2} m^2 \phi^2 \right],\end{aligned}$$

where $\mathcal{L}$ is known as a Lagrangian density. This is an example of a quadratic action, since each of the terms is quadratic in the field, ϕ . The term proportional to m^2 is sometimes known as a mass term, due to its interpretation in the quantized version of this theory in terms of particle mass.

The equation of motion for this theory is obtained by extremizing the action above. It takes the following form, linear in ϕ :

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi + m^2 \phi = \partial_t^2 \phi - \nabla^2 \phi + m^2 \phi = 0$$

Note that this is the same as the Klein–Gordon equation, but that here the interpretation is as a classical field equation, rather than as a quantum mechanical wave equation.

Nonlinear (interacting) theory

The most common generalization of the linear theory above is to add a scalar potential $V(\phi)$ to the equations of motion, where typically, V is a polynomial in ϕ of order 3 or more (often a monomial). Such a theory is sometimes said to be interacting, because the Euler-Lagrange equation is now nonlinear, implying a self-interaction. The action for the most general such theory is

$$\begin{aligned}\mathcal{S} &= \int d^{D-1}x dt \mathcal{L} = \int d^{D-1}x dt \left[\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \\ &= \int d^{D-1}x dt \left[\frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} \delta^{ij} \partial_i \phi \partial_j \phi - \frac{1}{2} m^2 \phi^2 - \sum_{n=3}^{\infty} \frac{1}{n!} g_n \phi^n \right]\end{aligned}$$

The $n!$ factors in the expansion are introduced because they are useful in the Feynman diagram expansion of the quantum theory, as described below. The corresponding Euler-Lagrange equation of motion is

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi + V'(\phi) = \partial_t^2 \phi - \nabla^2 \phi + V'(\phi) = 0.$$

Dimensional analysis and scaling

Physical quantities in these scalar field theories may have dimensions of length, time or mass, or some combination of the three. However, in a relativistic theory, any quantity t , with dimensions of time, can be 'converted' into a length, $l = ct$, by using the velocity of light, c .

Similarly, any length l is equivalent to an inverse mass, $l = \frac{\hbar}{mc}$, using Planck's constant, $\hbar$. Heuristically, one

can think of a time as a length, or either time or length as an inverse mass. In short, one can think of the dimensions of any physical quantity as defined in terms of just one independent dimension, rather than in terms of all three. This is most often termed the mass dimension of the quantity.

One objection is that this theory is classical, and therefore it is not obvious that Planck's constant should be a part of the theory at all. In a sense this is a valid objection, and if desired one can indeed recast the theory without mass dimensions at all. However, this would be at the expense of making the connection with the quantum scalar field slightly more obscure. Given that one has dimensions of mass, Planck's constant is thought of here as an essentially arbitrary fixed quantity with dimensions appropriate to convert between mass and inverse length. This is consistent

with the Feynman path integral approach to quantization, where the only reason for Planck's constant to appear stems from the same type of dimensional argument, since the action must be divided by some parameter with these dimensions to render the phase dimensionless.

Scaling Dimension

The classical scaling dimension, or mass dimension, Δ , of ϕ describes the transformation of the field under a rescaling of coordinates:

$$x \rightarrow \lambda x$$

$$\phi \rightarrow \lambda^{-\Delta} \phi$$

The units of action are the same as the units of $\hbar$, and so the action itself has zero mass dimension. This fixes the scaling dimension of ϕ to be

$$\Delta = \frac{D-2}{2}.$$

Scale invariance

There is a specific sense in which some scalar field theories are scale-invariant. While the actions above are all constructed to have zero mass dimension, not all actions are invariant under the scaling transformation

$$x \rightarrow \lambda x$$

$$\phi \rightarrow \lambda^{-\Delta} \phi$$

The reason that not all actions are invariant is that one usually thinks of the parameters m and g_n as fixed quantities, which are not rescaled under the transformation above. The condition for a scalar field theory to be scale invariant is then quite obvious: all of the parameters appearing in the action should be dimensionless quantities. In other words, a scale invariant theory is one without any fixed length scale (or equivalently, mass scale) in the theory.

For a scalar field theory with D spacetime dimensions, the only dimensionless parameter g_n satisfies $n = \frac{2D}{D-2}$

. For example, in $D=4$ only g_4 is classically dimensionless, and so the only classically scale-invariant scalar field theory in $D=4$ is the massless ϕ^4 theory. Classical scale invariance normally does not imply quantum scale invariance. See the discussion of the beta function below.

Conformal invariance

A transformation

$$x \rightarrow \tilde{x}(x)$$

is said to be conformal if the transformation satisfies

$$\frac{\partial \tilde{x}^\mu}{\partial x^\rho} \frac{\partial \tilde{x}^\nu}{\partial x^\sigma} \eta_{\mu\nu} = \lambda^2(x) \eta_{\rho\sigma}$$

for some function $\lambda^2(x)$. The conformal group contains as subgroups the isometries of the metric $\eta_{\mu\nu}$ (the Poincare group) and also the scaling transformations (or dilatations) considered above. In fact, the scale-invariant theories in the previous section are also conformally-invariant.

ϕ^4 theory

Massive ϕ^4 theory illustrates a number of interesting phenomena in scalar field theory.

The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}\delta^{ij}\partial_i\phi\partial_j\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4.$$

Spontaneous symmetry breaking

This Lagrangian has a Z_2 symmetry under the transformation $\phi \rightarrow -\phi$

This is an example of an internal symmetry, in contrast to a space-time symmetry.

If m^2 is positive, the potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4$ has a single minimum, at the origin. The solution $\phi = 0$ is clearly invariant under the Z_2 symmetry. Conversely, if m^2 is negative, then one can readily see that the potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4$ has two minima. This is known as a *double well potential*, and the lowest energy states (known as the vacua, in quantum field theoretical language) in such a theory are **not** invariant under the Z_2 symmetry of the action (in fact it maps each of the two vacua into the other). In this case, the Z_2 symmetry is said to be spontaneously broken.

Kink solutions

The ϕ^4 theory with a negative m^2 also has a kink solution, which is a canonical example of a soliton. Such a solution is of the form

$$\phi(\vec{x}, t) = \pm \frac{m}{2\sqrt{g}} \tanh\left(\frac{m(x - x_0)}{\sqrt{2}}\right)$$

where x is one of the spatial variables (ϕ is taken to be independent of t , and the remaining spatial variables). The solution interpolates between the two different vacua of the double well potential. It is not possible to deform the kink into a constant solution without passing through a solution of infinite energy, and for this reason the kink is said to be stable. For $D > 2$, i.e. theories with more than one spatial dimension, this solution is called a domain wall.

Another well-known example of a scalar field theory with kink solutions is the sine-Gordon theory.

Complex scalar field theory

In a complex scalar field theory, the scalar field takes values in the complex numbers, rather than the real numbers. The action considered normally takes the form

$$\mathcal{S} = \int d^{D-1}x dt \mathcal{L} = \int d^{D-1}x dt [\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - V(|\phi|^2)]$$

This has a $U(1)$ symmetry, whose action on the space of fields rotates $\phi \rightarrow e^{i\alpha}\phi$, for some real phase angle α .

As for the real scalar field, spontaneous symmetry breaking is found if m^2 is negative. This gives rise to a Mexican hat potential which is analogous to the double-well potential in real scalar field theory, but now the choice of vacuum breaks a continuous $U(1)$ symmetry instead of a discrete one. This leads to a Goldstone boson.

$O(N)$ theory

One can express the complex scalar field theory in terms of two real fields, $\phi^1 = \text{Re}\phi$ and $\phi^2 = \text{Im}\phi$ which transform in the vector representation of the $U(1) = O(2)$ internal symmetry. Although such fields transform as a vector under the internal symmetry, they are still Lorentz scalars. This can be generalised to a theory of N scalar fields transforming in the vector representation of the $O(N)$ symmetry. The Lagrangian for an $O(N)$ -invariant scalar field theory is typically of the form

$$\mathcal{L} = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \cdot \partial_\nu \phi - V(\phi \cdot \phi)$$

using an appropriate $O(N)$ -invariant inner product.

Quantum scalar field theory

In quantum field theory, the fields, and all observables constructed from them, are replaced by quantum operators on a Hilbert space. This Hilbert space is built on a vacuum state, and dynamics are governed by a Hamiltonian, a positive operator which annihilates the vacuum. A construction of a quantum scalar field theory may be found in the canonical quantization article, which uses canonical commutation relations among the fields as a basis for the construction. In brief, the basic variables are the field ϕ and its canonical momentum π . Both fields are Hermitian. At spatial points $\vec{x}, \vec{y}$ at equal times, the canonical commutation relations are given by

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0, \quad [\phi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x} - \vec{y}),$$

and the free Hamiltonian is

$$H = \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 \right].$$

A spatial Fourier transform leads to a momentum space fields

$$\tilde{\phi}(\vec{k}) = \int d^3x e^{-i\vec{k} \cdot \vec{x}} \phi(\vec{x}), \quad \tilde{\pi}(\vec{k}) = \int d^3x e^{-i\vec{k} \cdot \vec{x}} \pi(\vec{x})$$

which are used to define annihilation and creation operators

$$a(\vec{k}) = (E\tilde{\phi}(\vec{k}) + i\tilde{\pi}(\vec{k})), \quad a^\dagger(\vec{k}) = (E\tilde{\phi}(\vec{k}) - i\tilde{\pi}(\vec{k})),$$

where $E = \sqrt{k^2 + m^2}$. These operators satisfy the commutation relations

$$[a(\vec{k}_1), a(\vec{k}_2)] = [a^\dagger(\vec{k}_1), a^\dagger(\vec{k}_2)] = 0, \quad [a(\vec{k}_1), a^\dagger(\vec{k}_2)] = (2\pi)^3 2E \delta(\vec{k}_1 - \vec{k}_2).$$

The state $|0\rangle$ annihilated by all of the operators a is identified as the *bare vacuum*, and a particle with momentum $\vec{k}$ is created by applying $a^\dagger(\vec{k})$ to the vacuum. Applying all possible combinations of creation operators to the vacuum constructs the Hilbert space. This construction is called Fock space. The vacuum is annihilated by the Hamiltonian

$$H = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} a^\dagger(\vec{k}) a(\vec{k}),$$

where the zero-point energy has been removed by Wick-ordering. (See canonical quantization.)

Interactions can be included by adding an interaction Hamiltonian. For a ϕ^4 theory, this corresponds to adding a Wick-ordered term $g:\phi^4:/4!$ to the Hamiltonian, and integrating over x . Scattering amplitudes may be calculated from this Hamiltonian in the interaction picture. These are constructed in perturbation theory by means of the Dyson series, which gives the time-ordered products, or n -particle Green's functions $\langle 0 | T \{ \phi(x_1) \cdots \phi(x_n) \} | 0 \rangle$ as described in the Dyson series article. The Green's functions may also be obtained from a generating function that is constructed as a solution to the Schwinger-Dyson equation.

Feynman Path Integral

The Feynman diagram expansion may be obtained also from the Feynman path integral formulation.^[1] The time ordered vacuum expectation values of polynomials in ϕ , known as the n -particle Green's functions, are constructed by integrating over all possible fields, normalized by the vacuum expectation value with no external fields,

$$\langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle = \frac{\int \mathcal{D}\phi \phi(x_1)\cdots\phi(x_n) e^{i\int d^4x \left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4\right)}}{\int \mathcal{D}\phi e^{i\int d^4x \left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4\right)}}.$$

All of these Green's functions may be obtained by expanding the exponential in $J(x)\phi(x)$ in the generating function

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x \left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4 + J\phi\right)} = Z[0] \sum_{n=0}^{\infty} \frac{1}{n!} \langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle.$$

A Wick rotation may be applied to make time imaginary. Changing the signature to (++++) then turns the Feynman integral into a statistical mechanics partition function in Euclidean space,

$$Z[J] = \int \mathcal{D}\phi e^{-\int d^4x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 + J\phi\right)}.$$

Normally, this is applied to the scattering of particles with fixed momenta, in which case, a Fourier transform is useful, giving instead

$$\tilde{Z}[\tilde{J}] = \int \mathcal{D}\tilde{\phi} e^{-\int d^4p \left(\frac{1}{2}(p^2+m^2)\tilde{\phi}^2 + \frac{\lambda}{4!}\tilde{\phi}^4 - \tilde{J}\tilde{\phi}\right)}.$$

The standard trick to evaluate this functional integral is to write it as a product of exponential factors, schematically,

$$\tilde{Z}[\tilde{J}] \sim \int \mathcal{D}\tilde{\phi} \prod_p \left[e^{-(p^2+m^2)\tilde{\phi}^2/2} e^{-g\tilde{\phi}^4/4!} e^{\tilde{J}\tilde{\phi}} \right].$$

The second two exponential factors can be expanded as power series, and the combinatorics of this expansion can be represented graphically. The integral with $\lambda = 0$ can be treated as a product of infinitely many elementary Gaussian integrals, and the result may be expressed as a sum of Feynman diagrams, calculated using the following Feynman rules:

- Each field $\tilde{\phi}(p)$ in the n -point Euclidean Green's function is represented by an external line (half-edge) in the graph, and associated with momentum p .
- Each vertex is represented by a factor $-g$.
- At a given order g^k , all diagrams with n external lines and k vertices are constructed such that the momenta flowing into each vertex is zero. Each internal line is represented by a propagator $1/(q^2 + m^2)$, where q is the momentum flowing through that line.
- Any unconstrained momenta are integrated over all values.
- The result is divided by a symmetry factor, which is the number of ways the lines and vertices of the graph can be rearranged without changing its connectivity.
- Do not include graphs containing "vacuum bubbles", connected subgraphs with no external lines.

The last rule takes into account the effect of dividing by $\tilde{Z}[0]$. The Minkowski-space Feynman rules are similar, except that each vertex is represented by $-ig$, while each internal line is represented by a propagator $i/(q^2 - m^2 + i\epsilon)$, where the ' ϵ term represents the small Wick rotation needed to make the Minkowski-space Gaussian integral converge'.

Renormalization

The integrals over unconstrained momenta, called "loop integrals", in the Feynman graphs typically diverge. This is normally handled by renormalization, which is a procedure of adding divergent counter-terms to the Lagrangian in such a way that the diagrams constructed from the original Lagrangian and counter-terms is finite.^[2] A renormalization scale must be introduced in the process, and the coupling constant and mass become dependent upon it.

The dependence of a coupling constant g on the scale λ is encoded by a beta function, $\beta(g)$, defined by the relation

$$\beta(g) = \lambda \frac{\partial g}{\partial \lambda}$$

This dependence on the energy scale is known as the running of the coupling parameter, and theory of this kind of scale-dependence in quantum field theory is described by the renormalization group.

Beta-functions are usually computed in an approximation scheme, most commonly perturbation theory, where one assumes that the coupling constant is small. One can then make an expansion in powers of the coupling parameters and truncate the higher-order terms (also known as higher loop contributions, due to the number of loops in the corresponding Feynman graphs).

The beta-function at one loop (the first perturbative contribution) for the ϕ^4 theory is

$$\beta(g) = \frac{3}{16\pi^2} g^2 + O(g^3)$$

The fact that the sign in front of the lowest-order term is positive suggests that the coupling constant increases with energy. If this behavior persists at large couplings, this would indicate the presence of a Landau pole at finite energy, or quantum triviality. The question can only be answered non-perturbatively, since it involves strong coupling.

A quantum field theory is trivial when the running coupling, computed through its beta function, goes to zero when the cutoff is removed. Consequently, the propagator becomes that of a free particle and the field is no longer interacting. Alternatively, the field theory may be interpreted as an effective theory, in which the cutoff is not removed, giving finite interactions but leading to a Landau pole at some energy scale. For a ϕ^4 interaction, Michael Aizenman proved that the theory is indeed trivial for space-time dimension $D \geq 5$.^[3] For $D = 4$ the triviality has yet to be proven rigorously, but lattice computations have confirmed this. (See Landau pole for details and references.) This fact is relevant as the Higgs field, for which triviality bounds are used to set limits on the Higgs mass, based on the new physics must enter at a higher scale (perhaps the Planck scale) to prevent the Landau pole from being reached.

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- Zinn-Justin, Jean ; *Quantum Field Theory and Critical Phenomena*, Oxford University Press (2002)

External links

- Pedagogic Aides to Quantum Field Theory (<http://www.quantumfieldtheory.info>) Click on the link for Chap. 3 to find an extensive, simplified introduction to scalars in relativistic quantum mechanics and quantum field theory.
- 't Hooft, G., "The Conceptual Basis of Quantum Field Theory" (*online version* (<http://www.phys.uu.nl/~thooft/lectures/basisqft.pdf>)).

Yang-Mills theory

Yang–Mills theory is a gauge theory based on the $SU(N)$ group. Wolfgang Pauli formulated in 1953 the first consistent generalization of the five-dimensional theory of Kaluza, Klein, Fock and others to a higher dimensional internal space.^[1] Because Pauli saw no way to give masses to the gauge bosons, he refrained from publishing his results formally.^[1]

Although Pauli did not publish this theory, he gave talks widely attended by physicists of the time.^[2] In early 1954, Yang and Mills ^[3] developed a modern formulation in an effort to extend the original concept of gauge theory for abelian groups, e.g. quantum electrodynamics, to nonabelian groups to provide an explanation for strong interactions. This initial idea was not a success, since the quanta of the Yang–Mills field must be massless in order to maintain gauge invariance. The massless particles should have long range effects, but these effects are not seen in experiments. The idea was set aside until 1960, when the concept of particles acquiring mass through symmetry breaking in massless theories was put forward, initially by Jeffrey Goldstone, Yoichiro Nambu, and Giovanni Jona-Lasinio.

This prompted a significant restart of Yang–Mills theory studies that proved successful in the formulation of both electroweak unification and quantum chromodynamics (QCD). The electroweak interaction is described by $SU(2) \times U(1)$ group while QCD is an $SU(3)$ gauge theory. The electroweak theory is obtained by combining $SU(2)$ with $U(1)$, where quantum electrodynamics (QED) is described by a $U(1)$ group, and is replaced in the unified electroweak theory by a $U(1)$ group representing a weak hypercharge rather than electric charge. The massless bosons from the $SU(2) \times U(1)$ theory mix after spontaneous symmetry breaking to produce the 3 massive weak bosons, and the photon field. The Standard Model combines the strong interaction, with the unified electroweak interaction (unifying the weak and electromagnetic interaction) through the symmetry group $SU(2) \times U(1) \times SU(3)$. In the current epoch the strong interaction is not unified with the electroweak interaction, but from the observed running of the coupling constants it is believed they all converge to a single value at very high energies.

Phenomenology at lower energies in quantum chromodynamics is not completely understood due to the difficulties of managing such a theory with a strong coupling. This is the reason confinement has not been theoretically proven, though it is a consistent experimental observation. Proof that QCD confines at low energy is a mathematical problem of great relevance, and an award has been proposed by the Clay Mathematics Institute for whoever is able to show that the Yang–Mills theory has a mass gap.

Mathematical overview

Yang–Mills theories are a special example of gauge theory with symmetry non-abelian group given by the Lagrangian

$$\mathcal{L}_{\text{gf}} = -\frac{1}{4}\text{Tr}(F^2) = -\frac{1}{4}F^{\mu\nu a}F_{\mu\nu}^a$$

with the generators of the Lie algebra corresponding to the F -quantities (the curvature or field-strength form) satisfying

$$[T^a, T^b] = if^{abc}T^c$$

and the covariant derivative defined as

$$D_\mu = I\partial_\mu - igT^a A_\mu^a$$

where I is the identity for the group generators, A_μ^a is the vector potential, and g is the coupling constant. In four dimensions, the coupling constant g is a pure number and for a $SU(N)$ group one has $a, b, c = 1 \dots N^2 - 1$. The relation

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$$

can be derived by the commutator

$$[D_\mu, D_\nu] = -igT^a F_{\mu\nu}^a.$$

The field has the property of being self-interacting and equations of motion that one obtains are said to be semilinear, as nonlinearities are both with and without derivatives. This means that one can manage this theory only by perturbation theory, with small nonlinearities.

Note that the transition between "upper" ("contravariant") and "lower" ("covariant") vector or tensor components is trivial for a indices (e.g. $f^{abc} = f_{abc}$), whereas for μ and ν it is nontrivial, corresponding e.g. to the usual Lorentz signature, $\eta_{\mu\nu} = \text{diag}(+ - - -)$.

From the given Lagrangian one can derive the equations of motion given by

$$\partial^\mu F_{\mu\nu}^a + gf^{abc}A^{\mu b}F_{\mu\nu}^c = 0.$$

Putting $F_{\mu\nu} = T^a F_{\mu\nu}^a$, these can be rewritten as

$$(D^\mu F_{\mu\nu})^a = 0.$$

A Bianchi identity holds

$$(D_\mu F_{\nu\kappa})^a + (D_\kappa F_{\mu\nu})^a + (D_\nu F_{\kappa\mu})^a = 0.$$

A source J_μ^a enters into the equations of motion as

$$\partial^\mu F_{\mu\nu}^a + gf^{abc}A^{\mu b}F_{\mu\nu}^c = -J_\nu^a.$$

Note that the currents must properly change under gauge group transformations.

Quantization of Yang–Mills theory

The most appropriate method to quantize the Yang–Mills theory is by functional methods, i.e. path integrals. One introduces a generating functional for n -point functions as

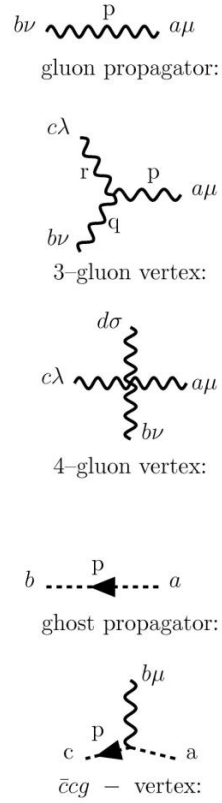
$$Z[j] = \int [dA] e^{-\frac{i}{4} \int d^4x \text{Tr}(F^{\mu\nu} F_{\mu\nu}) + i \int d^4x j_\mu^a(x) A^{\mu a}(x)}$$

but this integral has no meaning as is because the potential vector can be arbitrarily chosen due to the gauge freedom. This problem was already known for quantum electrodynamics but here becomes more severe due to non-abelian properties of the gauge group. A way out has been given by Ludvig Faddeev and Victor Popov with the introduction of a **ghost field** (see Faddeev–Popov ghost) that has the property of being unphysical since, although it agrees with Fermi–Dirac statistics, it is a complex scalar field, violating in this way the spin-statistics theorem. So, we can write

the generating functional as

$$Z[j, \bar{\varepsilon}, \varepsilon] = \int [dA][d\bar{c}][dc] e^{\left[-\frac{i}{4} \int d^4x \text{Tr}(F^{\mu\nu} F_{\mu\nu}) - i \int d^4x (\bar{c}^a \partial_\mu \partial^\mu c^a + g \bar{c}^a f^{abc} \partial_\mu A^{b\mu} c^c) - i \int d^4x \frac{1}{2\alpha} (\partial \cdot A)^2\right]} \times \\ e^{i \int d^4x j_\mu^a(x) A^{a\mu}(x) + i \int d^4x [\bar{c}^a(x) \varepsilon^a(x) + \bar{\varepsilon}^a(x) c^a(x)]}$$

that is the expression commonly used to derive Feynman's rules (see Feynman diagram). Here we have c^a for the ghost field while α fixes the gauge's choice for the quantization. Feynman's rules obtained from this functional are the following



gluon propagator: $D_{\mu\nu}^{ab}(p) = \frac{-i\delta^{ab}}{p^2 + i0} \left[\eta_{\mu\nu} - \frac{(1-\xi)p_\mu p_\nu}{p^2 + i0} \right]$

3-gluon vertex: $\Gamma_{\mu\nu\lambda}^{abc}(p, q, r) = -gf^{abc}[(p-q)_\lambda \eta_{\mu\nu} + (q-r)_\mu \eta_{\nu\lambda} + (r-p)_\nu \eta_{\mu\lambda}]$

4-gluon vertex: $\Gamma_{\mu\nu\lambda\sigma}^{abcd} = -ig^2 f^{abe} f^{cde} (\eta_{\mu\lambda} \eta_{\nu\sigma} - \eta_{\mu\sigma} \eta_{\nu\lambda}) - ig^2 f^{ace} f^{bde} (\eta_{\mu\nu} \eta_{\sigma\lambda} - \eta_{\mu\sigma} \eta_{\nu\lambda}) - ig^2 f^{ade} f^{bce} (\eta_{\mu\nu} \eta_{\sigma\lambda} - \eta_{\mu\lambda} \eta_{\nu\sigma})$

ghost propagator: $C^{ab}(p) = \frac{i\delta^{ab}}{p^2 + i0}$

$\bar{c}cg$ - vertex: $\Gamma^{abc}(p) = gf^{abc} p_\mu$

These rules for Feynman diagrams are easily obtained when we realize that the generating functional given above can be rewritten as

$$Z[j, \bar{\varepsilon}, \varepsilon] = e^{-ig \int d^4x \frac{\delta}{i\delta \bar{\varepsilon}^a(x)} f^{abc} \partial_\mu \frac{i\delta}{\delta j_\mu^b(x)} \frac{i\delta}{\delta \varepsilon^c(x)}} e^{-ig \int d^4x f^{abc} \partial_\mu \frac{i\delta}{\delta j_\mu^b(x)} \frac{i\delta}{\delta j_\mu^c(x)} \frac{i\delta}{\delta j_\mu^a(x)}} \times \\ e^{-i \frac{g^2}{4} \int d^4x f^{abc} f^{ars} \frac{i\delta}{\delta j_\mu^b(x)} \frac{i\delta}{\delta j_\nu^c(x)} \frac{i\delta}{\delta j_\mu^r(x)} \frac{i\delta}{\delta j_\nu^s(x)}} Z_0[j, \bar{\varepsilon}, \varepsilon]$$

being

$$Z_0[j, \bar{\varepsilon}, \varepsilon] = e^{-\int d^4x d^4y \bar{\varepsilon}^a(x) C^{ab}(x-y) \varepsilon^b(y)} e^{\frac{1}{2} \int d^4x d^4y j_\mu^a(x) D^{ab\mu\nu}(x-y) j_\nu^b(y)}$$

the generating functional of the free theory. Expanding in g and computing the functional derivatives, we are able to obtain all the n -point functions with perturbation theory. Using LSZ reduction formula we get from the n -point functions the corresponding amplitudes for the given processes and cross sections and decay rates are promptly obtained. The theory is renormalizable and corrections are finite at any order of perturbation theory.

For quantum electrodynamics, being in this case abelian the gauge group (see abelian group), the ghost field decouples. This can be easily realized when we look at the coupling between the gauge field and the ghost field that is $\bar{c}^a f^{abc} \partial_\mu A^{b\mu} c^c$. For the Abelian case all the structure constants f^{abc} are zero and so there is no coupling. In the non-Abelian case, the ghost field appears as a useful way to rewrite the quantum field theory without physical consequences on the observables of the theory as cross sections or decay rates.

One of the most important results obtained for Yang–Mills theory is asymptotic freedom. This result can be obtained assuming the coupling constant g as small (so small nonlinearities), as indeed happens to high energies, and applying perturbation theory. The relevance of this result is due to the fact that a Yang–Mills theory describes strong interactions and asymptotic freedom permits to treat properly experimental results coming from deep inelastic scattering.

In order to obtain the behavior at high energies of the Yang–Mills theory, and so to prove asymptotic freedom, one does perturbation theory assuming a small coupling. This is verified a posteriori in the ultraviolet limit. In the opposite limit, infrared limit, the situation is quite the opposite being the coupling too large for perturbation theory to be reliable. Indeed, most of the difficulties that current research meets is just managing the theory at low energies that is the interesting one being inherent to the description of hadronic matter and, more generally, to all the observed bound states of gluons and quarks and their confinement (see hadrons). Then, the most used method to study the theory in this limit is to try to solve it on computers (see lattice gauge theory). In this case, large computational resources are needed to be sure the right limit of infinite volume (smaller lattice spacing) is hit. This is the limit the results have to be compared with. Smaller spacing and larger coupling are not independent each others and to accomplish both larger computational resources are demanded. As for today, the situation appears somewhat satisfactory for the hadronic spectrum and the computation of the gluon and ghost propagators but the glueball and hybrids spectra are yet a questioned matter also in view of the experimental observation of such exotic states. Indeed, the σ resonance^{[4] [5]} is not seen in any of such lattice computations and contrasting interpretations have been put forward. This is currently a hotly debated issue.

Beta function

One of the key properties of a quantum field theory is the behavior at all the energy range of the running coupling. Such a behavior can be obtained from a theory once its beta function is known. Our ability of extracting results from a quantum field theory relies on perturbation theory. Once the beta function is known, the behavior at all energy scales of the running coupling is obtained through the equation

$$\mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s).$$

being $\alpha_s = g^2/4\pi$. Yang–Mills theory has the property of being asymptotically free in the large energy limit (ultraviolet limit). This means that, in this limit, beta function has a minus sign driving the behavior of the running coupling toward even smaller values as the energy increases. Perturbation theory permits to evaluate beta function in this limit producing the following result for SU(N)

$$\beta(\alpha_s) = -\frac{11N}{12\pi}\alpha_s^2 - \frac{17N^2}{24\pi^2}\alpha_s^3 + O(\alpha_s^4).$$

In the opposite limit of low energies (infrared limit), beta function is not known. It is note the exact one for a supersymmetric Yang–Mills theory. This has been obtained by Novikov, Shifman, Vainshtein and Zakharov^[6] and can be written as

$$\beta(\alpha_s) = -\frac{\alpha_s^2}{4\pi} \frac{3N}{1 - \frac{N\alpha_s}{2\pi}}.$$

With this starting point, Thomas Rytov and Francesco Sannino were able to postulate a non-supersymmetric version of it writing down^[7]

$$\beta(\alpha_s) = -\alpha_s^2 \frac{11N}{12\pi} \frac{1}{1 - \frac{17N}{11} \frac{\alpha_s}{2\pi}}.$$

As can be seen from the beta function of the supersymmetric theory, the limit of a large coupling (infrared limit) implies

$$\beta(\alpha_s) \approx \frac{3}{2}\alpha_s.$$

and so the running coupling in the deep infrared limit goes to zero making this theory trivial. This implies that the coupling reaches a maximum at some value of the energy turning again to zero as the energy is lowered. Then, if Rytov and Sannino hypothesis is correct, the same should be true for ordinary Yang–Mills theory. This would be in agreement with recent lattice computations^[8].

Open problems

The Yang–Mills theories were generally acknowledged in the physics community after Gerard 't Hooft, in 1972, could prove their renormalizability. This applies even if the gauge bosons described by this theory are massive, as in the electroweak theory. However, the mass is only an "acquired" one, namely, as suggested, by the famous Higgs mechanism.

Concerning the mathematics, it should be noted that presently, i.e. in 2009, the Yang–Mills theory is a very active field of research, yielding e.g. a classification of differentiable structures of four-dimensional manifolds by Simon Donaldson. Furthermore, the field of Yang–Mills theories was included in the Clay Mathematics Institute's list of "Millennium Prize Problems". Here the prize-problem consists, especially, in a proof of the conjecture that the lowest excitations of a pure Yang–Mills theory (i.e. without matter fields) have a finite mass-gap with regard to the vacuum state. Another open problem, connected with this conjecture, is a proof of the confinement property in the presence of additional Fermion particles.

In physics the survey of Yang–Mills theories does not usually start from perturbation analysis or analytical methods, but more recently from systematic application of numerical methods to lattice gauge theories.

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External links

- Yang-Mills theory on DispersiveWiki (http://tosio.math.toronto.edu/wiki/index.php/Yang-Mills_equations)
- The Clay Mathematics Institute (<http://www.claymath.org>)
- The Millennium Prize Problems (<http://www.claymath.org/prizeproblems>)

Yangian

Yangian is an important structure in modern representation theory, a type of a quantum group with origins in physics. Yangians first appeared in the work of Ludvig Faddeev and his school concerning the quantum inverse scattering method in the late 1970s and early 1980s. Initially they were considered a convenient tool to generate the solutions of the quantum Yang–Baxter equation. The name *Yangian* was introduced by Vladimir Drinfeld in 1985 in honor of C.N. Yang.

Description

For any finite-dimensional semisimple Lie algebra a , Drinfeld defined an infinite-dimensional Hopf algebra $Y(a)$, called the *Yangian* of a . This Hopf algebra is a deformation of the universal enveloping algebra $U(a[z])$ of the Lie algebra of polynomial loops of a given by explicit generators and relations. The relations can be encoded by identities involving a rational R -matrix. Replacing it with a trigonometric R -matrix, one arrives at affine quantum groups, defined in the same paper of Drinfeld.

In the case of the general linear Lie algebra gl_N , the Yangian admits a simpler description in terms of a single *ternary* (or *RTT*) *relation* on the matrix generators due to Faddeev and coauthors. The Yangian $Y(gl_N)$ is defined to be the algebra generated by elements $t_{ij}^{(p)}$ with $1 \leq i, j \leq N$ and $p \geq 0$, subject to the relations

$$[t_{ij}^{(p+1)}, t_{kl}^{(q)}] - [t_{ij}^{(p)}, t_{kl}^{(q+1)}] = -(t_{kj}^{(p)} t_{il}^{(q)} - t_{kl}^{(q)} t_{ij}^{(p)}).$$

Defining $t_{ij}^{(-1)} = \delta_{ij}$, setting

$$T(z) = \sum_{p \geq -1} t_{ij}^{(p)} z^{-p+1}$$

and introducing the R -matrix $R(z) = I + z^{-1} P$ on $\mathbf{C}^N \otimes \mathbf{C}^N$, where P is the operator permuting the tensor factors, the above relations can be written more simply as the ternary relation:

$$R_{23}(z-w)T_{12}(z)T_{13}(w) = T_{13}(w)T_{12}(z)R_{23}(z-w).$$

The Yangian becomes a Hopf algebra with comultiplication Δ , counit ϵ and antipode s given by

$$(\Delta \otimes \text{id})T(z) = T_{12}(z)T_{13}(z), \quad (\epsilon \otimes \text{id})T(z) = I, \quad (s \otimes \text{id})T(z) = T(z)^{-1}.$$

At special values of the spectral parameter $(z - w)$, the R -matrix degenerates to a rank one projection. This can be used to define the **quantum determinant** of $T(z)$, which generates the center of the Yangian.

The **twisted Yangian** $Y^-(gl_{2N})$, introduced by G. I. Olshansky, is the sub-Hopf algebra generated by the coefficients of

$$S(z) = T(z)\sigma T(-z),$$

where σ is the involution of gl_{2N} given by

$$\sigma(E_{ij}) = (-1)^{i+j} E_{2N-j+1, 2N-i+1}.$$

Applications to classical representation theory

G.I. Olshansky and I. Cherednik discovered that the Yangian of gl_N is closely related with the branching properties of irreducible finite-dimensional representations of general linear algebras. In particular, the classical Gelfand–Tsetlin construction of a basis in the space of such a representation has a natural interpretation in the language of Yangians, studied by M. Nazarov and V. Tarasov. Olshansky, Nazarov and Molev later discovered a generalization of this theory to other classical Lie algebras, based on the twisted Yangian.

Applications to physics

Yangian appears as a symmetry group in different models in physics. The most famous one is super-symmetric Yang–Mills field in four dimensions. Yangian also appears as a symmetry group of one dimensional exactly solvable models such as spin chains, Hubbard model ^[1] and in models of one dimensional relativistic quantum field theory.

Representation theory of Yangians

Irreducible finite-dimensional representations of Yangians were parametrized by Drinfeld in a way similar to the highest weight theory in the representation theory of semisimple Lie algebras. The role of the highest weight is played by a finite set of *Drinfeld polynomials*. Drinfeld also discovered a generalization of the classical Schur–Weyl duality between representations of general linear and symmetric groups that involves the Yangian of sl_N and the degenerate affine Hecke algebra (graded Hecke algebra of type A, in George Lusztig's terminology).

Representations of Yangians have been extensively studied, but the theory is still under active development.

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Quantum spacetime

In mathematical physics **quantum spacetime** is the proposal that the actual spacetime that we live in is more accurately described not by usual local coordinates x, y, z, t but operator or algebra variables where the order of at least some of the products matters. The idea is borrowed from the canonical commutation relations in quantum mechanics where position and momentum variables x, p are mutually noncommutative, but is postulated now for relations between one or more of the spacetime variables themselves. Just as with Heisenberg's uncertainty principle, a quantum spacetime necessarily comes with uncertainty relations in the sense that not all x, y, z, t can simultaneously be ascribed actual numerical values.

There are fundamental physical reasons to believe that spacetime is better modeled in this way. Because of wave particle duality the energy needed to probe smaller and smaller distances would be greater and greater, until the test particles doing the probing would form black holes and destroy the very geometry trying to be measured. This means that our usual picture of continuum spacetime **must** itself break down as we approach such Planck scale distances. Quantum spacetime is a particular attempt to address this using ideas from quantum mechanics and is plausibly expected on the grounds that the corrections to geometry are being induced by quantum gravity.

The mathematics allowing such a physical possibility has been developed under the general headings of noncommutative geometry and quantum geometry. In practice the more well-known Connes approach to noncommutative geometry has not been applied so much here and the currently best-known models of quantum spacetime fall within a more pedestrian quantum groups approach to noncommutative geometry, based on symmetry. It is important to insist that any noncommutative algebra with four generators qualifies properly to be called quantum spacetime. One should require at least the following:

- There should be a plausible expectation that such an algebra might actually arise in an effective description of quantum gravity effects in some regime of that theory. In this context there should be a parameter λ , say, controlling the extent of deviation from ordinary spacetime and ultimately identifiable with physical constants such as the Planck length. One should obtain ordinary Lorentzian spacetime as $\lambda \rightarrow 0$.
- Local Lorentz group and Poincare group symmetries should be retained in some sufficient but possibly generalised form. These symmetries of ordinary spacetime are needed for the formulation of Special Relativity and their generalisation often takes the form of a quantum group acting on the quantum spacetime algebra.
- There should be a notion of quantum differential calculus on the quantum spacetime algebra, compatible with the (quantum) symmetry and preferably reducing to classical high school differential calculus as $\lambda \rightarrow 0$.

These and some partial understanding of the rest of the story are the minimum needed to have wave equations for particles and fields and hence first predictions for the deviations from classical spacetime physics. This in turn is needed if the theory is ever to be tested.

Several models were found in the 1990s meeting the above criteria to lesser or greater extents. The most important of these has currently testable predictions making the search for quantum spacetime not only a theoretical fancy but a potentially an actual new discovery about our physical world.

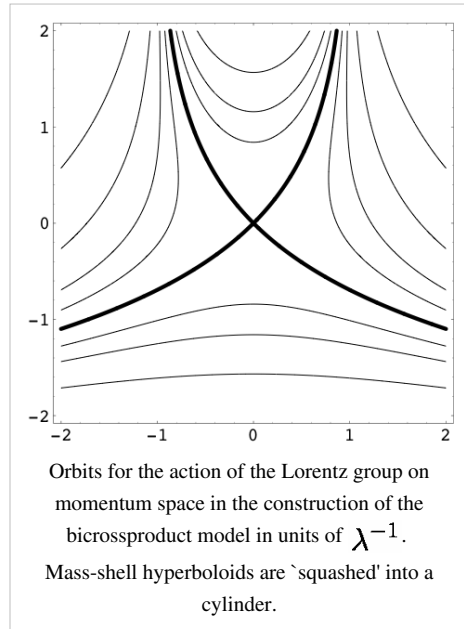
Bicrossproduct model spacetime

Was introduced in 1994 by Shahn Majid and Henri Ruegg^[1] and has relations

$$[x_i, x_j] = 0, \quad [x_i, t] = i\lambda x_i$$

for spatial variables x_i and the one time variable t . Here λ has dimensions of length and is therefore expected to be something like the Planck length. The Poincaré group here is correspondingly deformed, now to a certain bicrossproduct quantum group with the following characteristic features.

The momentum generators p_i commute among themselves but addition of momenta, reflected in the quantum group structure, is deformed (momentum space becomes a non-abelian group). Meanwhile, the Lorentz group generators enjoy their usual relations among themselves but act non-linearly on the momentum space. The orbits for this action are depicted in the figure as a cross-section of p_0 against one of the p_i . The on-shell region describing particles in the upper centre of the image would normally be hyperboloids but these are now 'squashed up' into the cylinder



$$\sqrt{p_1^2 + p_2^2 + p_3^2} < \lambda^{-1}$$

in simplified units. The upshot is that as you try to Lorentz-boost the momentum of a particle you will never exceed the Planck momentum. The existence of a highest momentum scale or lowest distance scale fits the physical picture. The existence of such squashing behaviour comes from the non-linearity of the action and is an endemic feature of bicrossproduct quantum groups known since their introduction in 1988^[2]. Some physicists have been so impressed by this feature of the bicrossproduct model that they have dubbed it doubly special relativity but such nomenclature remains controversial and disputed.

Another consequence of the squashing is that the propagation of particles is deformed, even of light, leading to a variable speed of light prediction. Key to this prediction was a reason to believe that the particular p_0, p_i are plausibly the physical energy and spatial momentum (as opposed to some other function of them), provided in 1999 by Giovanni Amelino-Camelia and Majid^[3] through a study of plane waves for a quantum differential calculus in the model. They take the form

$$e^{i \sum_i p_i x_i} e^{i p_0 t}$$

in other words a form which is sufficiently close to classical that one might plausibly believe the interpretation. At the moment such wave analysis represents the best hope to obtain physically testable predictions from the model.

Prior to this work there were a number of unsupported claims to make predictions from the model based solely on the form of the Poincaré quantum group. There were also claims based on an earlier κ -Poincaré quantum group introduced by Jurek Lukierski and co-workers^[4] which should be viewed as an important precursor to the bicrossproduct one, albeit without the actual quantum spacetime and with different proposed generators for which the above picture does not apply. The bicrossproduct model spacetime has also been called κ -deformed spacetime with $\kappa = \lambda^{-1}$.

q -Deformed spacetime

Was introduced independently by a team^[5] working under Julius Wess in 1990 and by Majid and coworkers in a series of papers on braided matrices starting a year later^[6]. The point of view in the second approach is that usual Minkowski spacetime has a nice description via Pauli matrices as the space of 2×2 hermitian matrices. In quantum group theory and using braided monoidal category methods one has a natural q -version of this defined here for real values of q as a 'braided hermitian matrix' of generators and relations

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^\dagger, \quad \beta\alpha = q^2\alpha\beta, \quad [\alpha, \delta] = 0, \quad [\beta, \gamma] = (1-q^{-2})\alpha(\delta-\alpha), \quad [\delta, \beta] = (1-q^{-2})\alpha\beta$$

These relations say that the generators commute as $q \rightarrow 1$ thereby recovering usual Minkowski space. One can work with more familiar variables x, y, z, t as linear combinations of these. In particular, time

$$t = \text{Trace}_q \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = q\delta + q^{-1}\alpha$$

is given by a natural braided trace of the matrix and commutes with the other generators (so this model has a very different flavour from the bicrossproduct one). The braided-matrix picture also leads naturally to a quantity

$$\det_q \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \alpha\delta - q^2\gamma\beta$$

which as $q \rightarrow 1$ returns us the usual Minkowski distance (this translates to a metric in the quantum differential geometry). The parameter $q = e^\lambda$ or $q = e^{i\lambda}$ is dimensionless and λ is thought to be a ratio of the Planck scale and the cosmological length. That is, there are indications that that this model relates to quantum gravity **with** non-zero cosmological constant, the choice of q depending on whether this is positive or negative. We have described the mathematically better understood but perhaps less physically justified positive case here.

A full understanding of this model requires (and was concurrent with the development of) a full theory of 'braided linear algebra' for such spaces. The momentum space for the theory is another copy of the same algebra and there is a certain 'braided addition' of momentum on it expressed as the structure of a braided Hopf algebra or quantum group *in* a certain braided monoidal category). This theory by 1993 had provided the corresponding q -deformed Poincaré group as generated by such translations and q -Lorentz transformations, completing the interpretation as a quantum spacetime^[7].

In the process it was discovered that the Poincaré group not only had to be deformed but had to be extended to include dilations of the quantum spacetime. For such a theory to be exact we would need all particles in the theory to be massless, which is consistent with experiment as masses of elementary particles are indeed vanishingly small compared to the Planck mass. If current thinking in cosmology is correct then this model is more appropriate, but it is significantly more complicated and for this reason its physical predictions have yet to be worked out.

Fuzzy or spin model spacetime

Refers in modern usage to the angular momentum algebra

$$[x_1, x_2] = 2i\lambda x_3, \quad [x_2, x_3] = 2i\lambda x_1, \quad [x_3, x_1] = 2i\lambda x_2$$

familiar from quantum mechanics but regarded now as coordinates of a quantum space or spacetime. This is primarily of interest as a toy model of quantum gravity where we work only in 3 spacetime dimensions (not the correct 4) and here presented with a Euclidean not Lorentzian signature. It was first proposed^[8] in this context by Gerardus 't Hooft while its full development including a quantum differential calculus and an action of a certain 'quantum double' quantum group as deformed Euclidean group of motions was obtained by Majid and E. Batista^[9]

A striking feature of the noncommutative geometry here is that the smallest covariant quantum differential calculus has one dimension higher than expected, namely 4, suggesting that the above can also be viewed as the spatial part of a 4-dimensional quantum spacetime. The model should not be confused with fuzzy spheres which are

finite-dimensional matrix algebras which one can think of as spheres in the spin model spacetime of fixed radius.

Heisenberg model spacetimes

The first concrete model of quantum spacetime is often attributed to Hartland Snyder in 1947, however his paper^[10] actually proposes that

$$[x_\mu, x_\nu] = iM_{\mu\nu}$$

where $M_{\mu\nu}$ generate and are interpreted as the Lorentz group. This is not an actual quantum spacetime in the sense above because the spacetime coordinates x_μ do not form a self-contained algebra among themselves, rather Snyder was proposing a radical unification of spacetime with the Lorentz and Poincaré groups.

The idea was revived in a modern context by Sergio Doplicher, Claus Fredenhagen and John Roberts in 1995^[11] by letting $M_{\mu\nu}$ simply be viewed as some function of x_μ as defined by the above relation, and any relations involving it viewed as higher order relations among the x_μ . The Lorentz symmetry is arranged so as to transform the indices as usual and without being deformed.

An even simpler variant of this model is to let M here be a numerical antisymmetric tensor, in which context it is usually denoted θ , so the relations are $[x_\mu, x_\nu] = i\theta_{\mu\nu}$. In even dimensions D any nondegenerate such theta can be transformed to a normal form in which this really is just the Heisenberg algebra but the difference that the variables are being proposed as those of spacetime. This proposal was for a time quite popular because of its familiar form of relations and because it has been argued^[12] that it emerges from the theory of open strings landing on D-branes, see noncommutative quantum field theory and Moyal plane. However, it should be realised that this D-brane lives in some of the higher spacetime dimensions in the theory and hence it is not our physical spacetime that string theory suggests to be effectively quantum in this way. You also have to subscribe to D-branes as an approach to quantum gravity in the first place. Even when posited as quantum spacetime it is hard to obtain physical predictions and one reason for this is that if θ is a tensor then by dimensional analysis it should have dimensions of length², and if this length is speculated to be the Planck length then the effects would be even harder to ever detect than for other models.

Noncommutative extensions to spacetime

Although not quantum spacetime in the sense above, another use of noncommutative geometry is to tack on 'noncommutative extra dimensions' at each point of ordinary spacetime. Instead of invisible curled up extra dimensions as in string theory, Alain Connes and coworkers have argued that the coordinate algebra of this extra part should be replaced by a finite-dimensional noncommutative algebra. For a certain reasonable choice of this algebra, its representation and extended Dirac operator, one is able to recover the Standard Model of elementary particles. In this point of view the different kinds of matter particles are manifestations of geometry in these extra noncommutative directions. Connes first works here date from 1989^[13] but has been developed considerably since then. Such an approach can theoretically be combined with quantum spacetime as above.

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External links

- Plus Magazine article on quantum geometry (<http://plus.maths.org/issue43/features/noncom/index-gifd.html>) by Marianne Freiberger
- S. Majid, ed. (2008), *On Space and Time*, Cambridge University Press (<http://www.ewidgetsonline.com/dxreader/Reader.aspx?token=KSvQC0T+8xw7PY+iwDj29A==&rand=1970407385&buyNowLink=http://www.cambridge.org/us/catalogue/AddToBasket.asp?isbn=9780521889261>)

Quantum gauge theory

In order to quantize a gauge theory, like for example Yang-Mills theory, Chern-Simons or BF model, one method is to perform a gauge fixing. This is done in the BRST and Batalin-Vilkovisky formulation. Another is to factor out the symmetry by dispensing with vector potentials altogether (they're not physically observable anyway) and work directly with Wilson loops, Wilson lines contracted with other charged fields at its endpoints and spin networks.

Older approaches to quantization for Abelian models use the Gupta-Bleuler formalism with a "semi-Hilbert space" with an indefinite sesquilinear form. However, it is much more elegant to just work with the quotient space of vector field configurations by gauge transformations.

An alternative approach using lattice approximations is covered in (Wick rotated) lattice gauge theory.

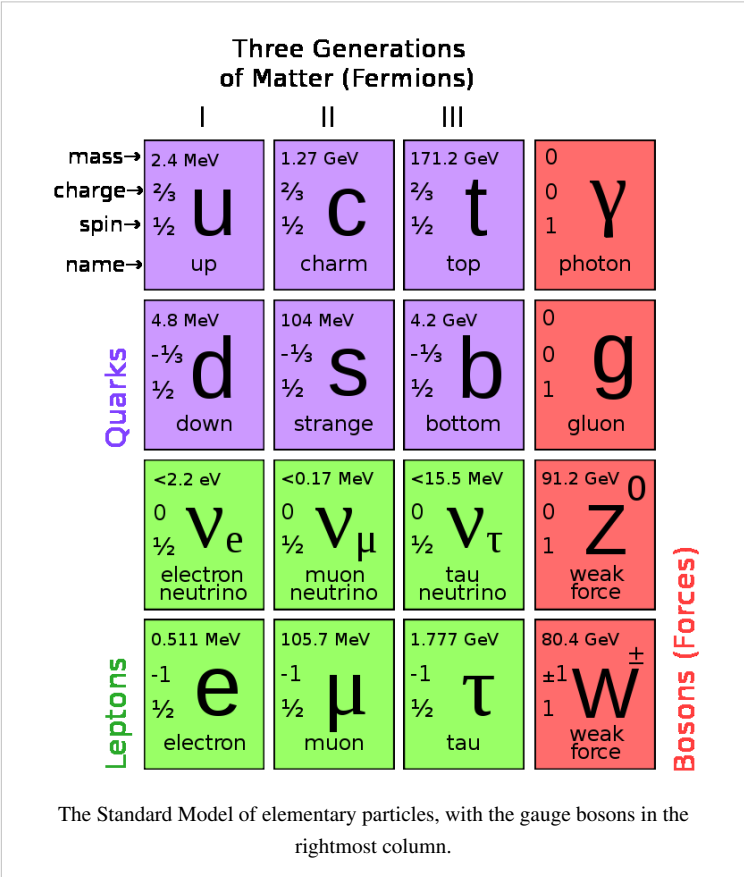
To establish the existence of the Yang-Mills theory and a mass gap is one of the seven Millennium Prize Problems of the Clay Mathematics Institute.

Standard Model

The **standard model** of particle physics is a theory concerning the electromagnetic, weak, and strong nuclear interactions, which mediate the dynamics of the known subatomic particles. Developed throughout the early and middle 20th century, the current formulation was finalized in the mid 1970s upon experimental confirmation of the existence of quarks. Since then, discoveries of the bottom quark (1977), the top quark (1995) and the tau neutrino (2000) have given credence to the standard model. Because of its success in explaining a wide variety of experimental results, the standard model is sometimes regarded as a theory of almost everything.

Still, the standard model falls short of being a complete theory of fundamental interactions because it does not incorporate the physics of general relativity, such as gravitation and dark energy. The theory does not contain any viable dark matter particle that possesses all of the required properties deduced from observational cosmology. It also does not correctly account for neutrino oscillations (and their non-zero masses). Although the standard model is theoretically self-consistent, it has several unnatural properties giving rise to puzzles like the strong CP problem and the hierarchy problem.

Nevertheless, the standard model is important to theoretical and experimental particle physicists alike. For theoreticians, the standard model is a paradigm example of a quantum field theory, which exhibits a wide range of physics including spontaneous symmetry breaking, anomalies, non-perturbative behavior, etc. It is used as a basis for



building more exotic models which incorporate hypothetical particles, extra dimensions and elaborate symmetries (such as supersymmetry) in an attempt to explain experimental results at variance with the standard model such as the existence of dark matter and neutrino oscillations. In turn, the experimenters have incorporated the standard model into simulators to help search for new physics beyond the standard model from relatively uninteresting background.

Recently, the standard model has found applications in other fields besides particle physics such as astrophysics and cosmology, in addition to nuclear physics.

Historical background

The first step towards the Standard Model was Sheldon Glashow's discovery, in 1960, of a way to combine the electromagnetic and weak interactions.^[1] In 1967, Steven Weinberg^[2] and Abdus Salam^[3] incorporated the Higgs mechanism^{[4] [5] [6]} into Glashow's electroweak theory, giving it its modern form.

The Higgs mechanism is believed to give rise to the masses of all the elementary particles in the Standard Model. This includes the masses of the W and Z bosons, and the masses of the fermions - i.e. the quarks and leptons.

After the neutral weak currents caused by Z boson exchange were discovered at CERN in 1973,^{[7] [8] [9] [10]} the electroweak theory became widely accepted and Glashow, Salam, and Weinberg shared the 1979 Nobel Prize in Physics for discovering it. The W and Z bosons were discovered experimentally in 1981, and their masses were found to be as the Standard Model predicted.

The theory of the strong interaction, to which many contributed, acquired its modern form around 1973–74, when experiments confirmed that the hadrons were composed of fractionally charged quarks.

Overview

At present, matter and energy are best understood in terms of the kinematics and interactions of elementary particles. To date, physics has reduced the laws governing the behavior and interaction of all known forms of matter and energy to a small set of fundamental laws and theories. A major goal of physics is to find the "common ground" that would unite all of these theories into one integrated theory of everything, of which all the other known laws would be special cases, and from which the behavior of all matter and energy could be derived (at least in principle).^[11]

The Standard Model groups two major extant theories—quantum electroweak and quantum chromodynamics—into an internally consistent theory that describes the interactions between all known particles in terms of quantum field theory. For a technical description of the fields and their interactions, see Standard Model (mathematical formulation).

Particle content

Fermions

Organization of Fermions

	Charge	First generation		Second generation		Third generation	
Quarks	$+\frac{2}{3}$	Up	u	Charm	c	Top	t
	$-\frac{1}{3}$	Down	d	Strange	s	Bottom	b
Leptons	-1	Electron	e^-	Muon	μ^-	Tau	τ^-
	0	Electron neutrino	ν_e	Muon neutrino	ν_μ	Tau neutrino	ν_τ

The Standard Model includes 12 elementary particles of spin- $\frac{1}{2}$ known as fermions. According to the spin-statistics theorem, fermions respect the Pauli exclusion principle. Each fermion has a corresponding antiparticle.

The fermions of the Standard Model are classified according to how they interact (or equivalently, by what charges they carry). There are six quarks (up, down, charm, strange, top, bottom), and six leptons (electron, electron neutrino, muon, muon neutrino, tau, tau neutrino). Pairs from each classification are grouped together to form a generation, with corresponding particles exhibiting similar physical behavior (see table).

The defining property of the quarks is that they carry color charge, and hence, interact via the strong interaction. A phenomenon called color confinement results in quarks being perpetually (or at least since very soon after the start of the Big Bang) bound to one another, forming color-neutral composite particles (hadrons) containing either a quark and an antiquark (mesons) or three quarks (baryons). The familiar proton and the neutron are the two baryons having the smallest mass. Quarks also carry electric charge and weak isospin. Hence they interact with other fermions both electromagnetically and via the weak nuclear interaction.

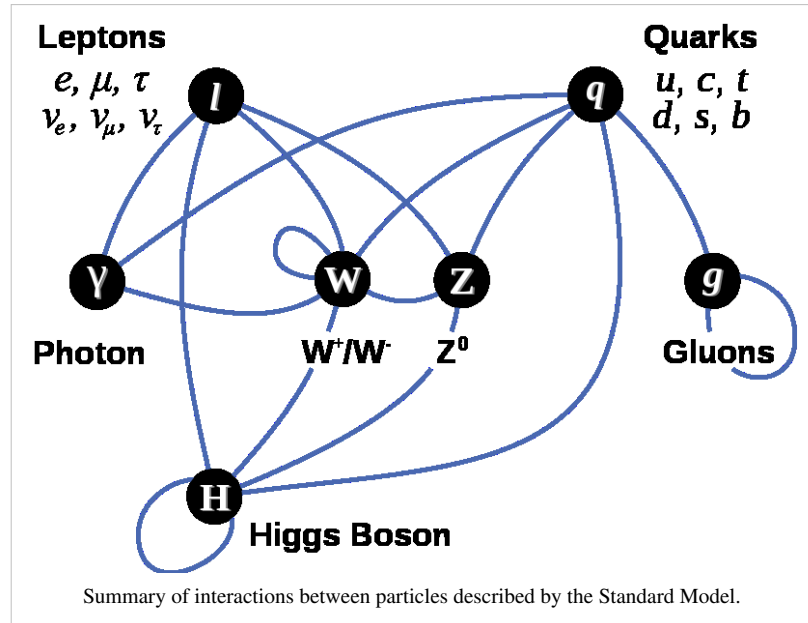
The remaining six fermions do not carry color charge and are called leptons. The three neutrinos do not carry electric charge either, so their motion is directly influenced only by the weak nuclear force, which makes them notoriously difficult to detect. However, by virtue of carrying an electric charge, the electron, muon, and tau all interact electromagnetically.

Each member of a generation has greater mass than the corresponding particles of lower generations. The first generation charged particles do not decay; hence all ordinary (baryonic) matter is made of such particles. Specifically, all atoms consist of electrons orbiting atomic nuclei ultimately constituted of up and down quarks. Second and third generations charged particles, on the other hand, decay with very short half lives, and are observed only in very high-energy environments. Neutrinos of all generations also do not decay, and pervade the universe, but rarely interact with baryonic matter.

Gauge bosons

In the Standard Model, gauge bosons are force carriers that mediate the strong, weak, and electromagnetic fundamental interactions.

Interactions in physics are the ways that particles influence other particles. At a macroscopic level, electromagnetism allows particles to interact with one another via electric and magnetic fields, and gravitation allows particles with mass to attract one another in accordance with Einstein's general relativity. The standard model explains such forces as resulting from matter particles exchanging other particles, known as



force mediating particles (Strictly speaking, this is only so if interpreting literally what is actually an *approximation method* known as perturbation theory, as opposed to the exact theory). When a force mediating particle is exchanged, at a macroscopic level the effect is equivalent to a force influencing both of them, and the particle is therefore said to have *mediated* (i.e., been the agent of) that force. The Feynman diagram calculations, which are a graphical form of the perturbation theory approximation, invoke "force mediating particles" and when applied to analyze high-energy scattering experiments are in reasonable agreement with the data. Perturbation theory (and with it the concept of "force mediating particle") in other situations fails. These include low-energy QCD, bound states, and solitons.

The gauge bosons of the Standard Model also all have spin (as do matter particles), but in their case, the value of the spin is 1, making them bosons. As a result, they do not follow the Pauli exclusion principle. The different types of gauge bosons are described below.

- Photons mediate the electromagnetic force between electrically charged particles. The photon is massless and is well-described by the theory of quantum electrodynamics.
- The W^+ , W^- , and Z gauge bosons mediate the weak interactions between particles of different flavors (all quarks and leptons). They are massive, with the Z being more massive than the $W^\pm$. The weak interactions involving the $W^\pm$ act on exclusively *left-handed* particles and *right-handed* antiparticles. Furthermore, the $W^\pm$ carry an electric charge of +1 and -1 and couple to the electromagnetic interactions. The electrically neutral Z boson interacts with both left-handed particles and antiparticles. These three gauge bosons along with the photons are grouped together which collectively mediate the electroweak interactions.
- The eight gluons mediate the strong interactions between color charged particles (the quarks). Gluons are massless. The eightfold multiplicity of gluons is labeled by a combination of color and an anticolor charge (e.g., red-antigreen).^[12] Because the gluon has an effective color charge, they can interact among themselves. The gluons and their interactions are described by the theory of quantum chromodynamics.

The interactions between all the particles described by the Standard Model are summarized by the diagram at the top of this section.

Higgs boson

The Higgs particle is a hypothetical massive scalar elementary particle theorized by Robert Brout, François Englert, Peter Higgs, Gerald Guralnik, C. R. Hagen, and Tom Kibble in 1964 (see 1964 PRL symmetry breaking papers) and is a key building block in the Standard Model.^{[13] [14] [15] [16]} It has no intrinsic spin, and for that reason is classified as a boson (like the gauge bosons, which have integer spin). Because an exceptionally large amount of energy and beam luminosity are theoretically required to observe a Higgs boson in high energy colliders, it is the only fundamental particle predicted by the Standard Model that has yet to be observed.

The Higgs boson plays a unique role in the Standard Model, by explaining why the other elementary particles, the photon and gluon excepted, are massive. In particular, the Higgs boson would explain why the photon has no mass, while the W and Z bosons are very heavy. Elementary particle masses, and the differences between electromagnetism (mediated by the photon) and the weak force (mediated by the W and Z bosons), are critical to many aspects of the structure of microscopic (and hence macroscopic) matter. In electroweak theory, the Higgs boson generates the masses of the leptons (electron, muon, and tau) and quarks.

As yet, no experiment has directly detected the existence of the Higgs boson. It is hoped that the Large Hadron Collider at CERN will confirm the existence of this particle. It is also possible that the Higgs boson may already have been produced but overlooked.^[17]

Field content

The standard model has the following fields:

Spin 1

1. A U(1) gauge field $B_{\mu\nu}$ with coupling g' (weak U(1), or weak hypercharge)
2. An SU(2) gauge field $W_{\mu\nu}$ with coupling g (weak SU(2), or weak isospin)
3. An SU(3) gauge field $G_{\mu\nu}$ with coupling g_s (strong SU(3), or color charge)

Spin $\frac{1}{2}$

The spin $\frac{1}{2}$ particles are in representations of the gauge groups. For the U(1) group, we list the value of the weak hypercharge instead. The left-handed fermionic fields are:

1. An SU(3) triplet, SU(2) doublet, with U(1) weak hypercharge $\frac{1}{3}$ (left-handed quarks)
2. An SU(3) triplet, SU(2) singlet, with U(1) weak hypercharge $\frac{2}{3}$ (left-handed down-type antiquark)
3. An SU(3) singlet, SU(2) doublet with U(1) weak hypercharge -1 (left-handed lepton)
4. An SU(3) triplet, SU(2) singlet, with U(1) weak hypercharge $-\frac{4}{3}$ (left-handed up-type antiquark)
5. An SU(3) singlet, SU(2) singlet with U(1) weak hypercharge 2 (left-handed antilepton)

By CPT symmetry, there is a set of right-handed fermions with the opposite quantum numbers.

This describes one *generation* of leptons and quarks, and there are three generations, so there are three copies of each field. Note that there are twice as many left-handed lepton field components as left-handed antilepton field components in each generation, but an equal number of left-handed quark and antiquark fields.

Spin 0

1. An SU(2) doublet H with U(1) hyper-charge -1 (Higgs field)

Note that $|H|^2$, summed over the two SU(2) components, is invariant under both SU(2) and under U(1), and so it can appear as a renormalizable term in the Lagrangian, as can its square.

This field acquires a vacuum expectation value, leaving a combination of the weak isospin, I_3 , and weak hypercharge unbroken. This is the electromagnetic gauge group, and the photon remains massless. The standard formula for the electric charge (which defines the normalization of the weak hypercharge, Y , which would otherwise be somewhat arbitrary) is:^[18]

$$Q = I_3 + \frac{Y}{2}.$$

Lagrangian

The Lagrangian for the spin 1 and spin $1/2$ fields is the most general renormalizable gauge field Lagrangian with no fine tunings:

- Spin 1:

$$\int -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}\text{tr}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}\text{tr}G_{\mu\nu}G^{\mu\nu}$$

where the traces are over the SU(2) and SU(3) indices hidden in W and G respectively. The two-index objects are the field strengths derived from W and G the vector fields. There are also two extra hidden parameters: the theta angles for SU(2) and SU(3).

The spin- $1/2$ particles can have no mass terms because there is no right/left helicity pair with the same SU(2) and SU(3) representation and the same weak hypercharge. This means that if the gauge charges were conserved in the vacuum, none of the spin $1/2$ particles could ever swap helicity, and they would all be massless.

For a neutral fermion, for example a hypothetical right-handed lepton N (or N^α in relativistic two-spinor notation), with no SU(3), SU(2) representation and zero charge, it is possible to add the term:

$$\int MN^\alpha N^\beta \epsilon_{\alpha\beta} + \bar{N}_{\dot{\alpha}} \bar{N}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}}.$$

This term gives the neutral fermion a Majorana mass. Since the generic value for M will be of order 1, such a particle would generically be unacceptably heavy. The interactions are completely determined by the theory – the leptons introduce no extra parameters.

Higgs mechanism

The Lagrangian for the Higgs includes the most general renormalizable self interaction:

$$S_{\text{Higgs}} = \int d^4x [(D_\mu H)^*(D^\mu H) + \lambda(|H|^2 - v^2)^2].$$

The parameter v^2 has dimensions of mass squared, and it gives the location where the classical Lagrangian is at a minimum. In order for the Higgs mechanism to work, v^2 must be a positive number. v has units of mass, and it is the only parameter in the standard model which is not dimensionless. It is also much smaller than the Planck scale; it is approximately equal to the Higgs mass, and sets the scale for the mass of everything else. This is the only real fine-tuning to a small nonzero value in the standard model, and it is called the Hierarchy problem.

It is traditional to choose the SU(2) gauge so that the Higgs doublet in the vacuum has expectation value $(v,0)$.

Masses and CKM matrix

The rest of the interactions are the most general spin-0 spin- $\frac{1}{2}$ Yukawa interactions, and there are many of these. These constitute most of the free parameters in the model. The Yukawa couplings generate the masses and mixings once the Higgs gets its vacuum expectation value.

The terms $\bar{L}^* H R$ generate a mass term for each of the three generations of leptons. There are 9 of these terms, but by relabeling L and R, the matrix can be diagonalized. Since only the upper component of H is nonzero, the upper SU(2) component of L mixes with R to make the electron, the muon, and the tau, leaving over a lower massless component, the neutrino. {Neutrino oscillation show neutrinos have mass. <http://operaweb.lngs.infn.it/spip.php?rubrique14> 31May2010 Press Release.}

The terms $\bar{Q} H U$ generate up masses, while $\bar{Q} H D$ generate down masses. But since there is more than one right-handed singlet in each generation, it is not possible to diagonalize both with a good basis for the fields, and there is an extra CKM matrix.

Theoretical aspects

Construction of the Standard Model Lagrangian

Parameters of the Standard Model

Symbol	Description	Renormalization scheme (point)	Value
m_e	Electron mass		511 keV
m_μ	Muon mass		105.7 MeV
m_τ	Tau mass		1.78 GeV
m_u	Up quark mass	$\mu_{\text{MS}} = 2 \text{ GeV}$	1.9 MeV
m_d	Down quark mass	$\mu_{\text{MS}} = 2 \text{ GeV}$	4.4 MeV
m_s	Strange quark mass	$\mu_{\text{MS}} = 2 \text{ GeV}$	87 MeV
m_c	Charm quark mass	$\mu_{\text{MS}} = m_c$	1.32 GeV
m_b	Bottom quark mass	$\mu_{\text{MS}} = m_b$	4.24 GeV
m_t	Top quark mass	On-shell scheme	172.7 GeV
θ_{12}	CKM 12-mixing angle		13.1°
θ_{23}	CKM 23-mixing angle		2.4°
θ_{13}	CKM 13-mixing angle		0.2°
δ	CKM CP-violating Phase		0.995
g_1	U(1) gauge coupling	$\mu_{\text{MS}} = m_Z$	0.357
g_2	SU(2) gauge coupling	$\mu_{\text{MS}} = m_Z$	0.652
g_3	SU(3) gauge coupling	$\mu_{\text{MS}} = m_Z$	1.221
θ_{QCD}	QCD vacuum angle		~0
μ	Higgs quadratic coupling		Unknown
λ	Higgs self-coupling strength		Unknown

Technically, quantum field theory provides the mathematical framework for the standard model, in which a Lagrangian controls the dynamics and kinematics of the theory. Each kind of particle is described in terms of a

dynamical field that pervades space-time. The construction of the standard model proceeds following the modern method of constructing most field theories: by first postulating a set of symmetries of the system, and then by writing down the most general renormalizable Lagrangian from its particle (field) content that observes these symmetries.

The global Poincaré symmetry is postulated for all relativistic quantum field theories. It consists of the familiar translational symmetry, rotational symmetry and the inertial reference frame invariance central to the theory of special relativity. The local $SU(3) \times SU(2) \times U(1)$ gauge symmetry is an internal symmetry that essentially defines the standard model. Roughly, the three factors of the gauge symmetry give rise to the three fundamental interactions. The fields fall into different representations of the various symmetry groups of the Standard Model (see table). Upon writing the most general Lagrangian, one finds that the dynamics depend on 19 parameters, whose numerical values are established by experiment. The parameters are summarized in the table at right.

The QCD sector

The QCD sector defines the interactions between quarks and gluons, with $SU(3)$ symmetry, generated by T^a . Since leptons do not interact with gluons, they are not affected by this sector.

$$\mathcal{L}_{QCD} = \bar{U}(\partial_\mu - ig_s G_\mu^a T^a) \gamma^\mu U + \bar{D}(\partial_\mu - ig_s G_\mu^a T^a) \gamma^\mu D.$$

G_μ^a is the gluon field strength, γ^μ are the Dirac matrices, D stands for the isospin doublet section, U stands for a unitary matrix, and g_s is the strong coupling constant.

The electroweak sector

The electroweak sector is a Yang–Mills gauge theory with the symmetry group $U(1) \times SU(2)_L$,

$$\mathcal{L}_{EW} = \sum_\psi \bar{\psi} \gamma^\mu \left(i\partial_\mu - g' \frac{1}{2} Y_W B_\mu - g \frac{1}{2} \vec{\tau}_L \vec{W}_\mu \right) \psi$$

where B_μ is the $U(1)$ gauge field; Y_W is the weak hypercharge—the generator of the $U(1)$ group; $\vec{W}_\mu$ is the three-component $SU(2)$ gauge field; $\vec{\tau}_L$ are the Pauli matrices—infinitesimal generators of the $SU(2)$ group. The subscript L indicates that they only act on left fermions; g' and g are coupling constants.

The Higgs sector

In the Standard Model, the Higgs field is a complex spinor of the group $SU(2)_L$:

$$\varphi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix},$$

where the indexes $+$ and 0 indicate the electric charge (Q) of the components. The weak isospin (Y_W) of both components is 1.

Before symmetry breaking, the Higgs Lagrangian is:

$$\mathcal{L}_H = \varphi^\dagger \left(\partial_\mu - \frac{i}{2} (g' Y_W B_\mu + g \vec{\tau} \vec{W}_\mu) \right) \left(\partial_\mu + \frac{i}{2} (g' Y_W B_\mu + g \vec{\tau} \vec{W}_\mu) \right) \varphi - \frac{\lambda^2}{4} (\varphi^\dagger \varphi - v^2)^2,$$

which can also be written as:

$$\mathcal{L}_H = \left| \left(\partial_\mu + \frac{i}{2} (g' Y_W B_\mu + g \vec{\tau} \vec{W}_\mu) \right) \varphi \right|^2 - \frac{\lambda^2}{4} (\varphi^\dagger \varphi - v^2)^2.$$

Additional symmetries of the Standard Model

From the theoretical point of view, the Standard Model exhibits four additional global symmetries, not postulated at the outset of its construction, collectively denoted **accidental symmetries**, which are continuous U(1) global symmetries. The transformations leaving the Lagrangian invariant are:

$$\begin{aligned}\psi_q(x) &\rightarrow e^{i\alpha/3}\psi_q \\ E_L &\rightarrow e^{i\beta}E_L \text{ and } (e_R)^c \rightarrow e^{i\beta}(e_R)^c \\ M_L &\rightarrow e^{i\beta}M_L \text{ and } (\mu_R)^c \rightarrow e^{i\beta}(\mu_R)^c \\ T_L &\rightarrow e^{i\beta}T_L \text{ and } (\tau_R)^c \rightarrow e^{i\beta}(\tau_R)^c.\end{aligned}$$

The first transformation rule is shorthand meaning that all quark fields for all generations must be rotated by an identical phase simultaneously. The fields M_L , T_L and $(\mu_R)^c$, $(\tau_R)^c$ are the 2nd (muon) and 3rd (tau) generation analogs of E_L and $(e_R)^c$ fields.

By Noether's theorem, each symmetry above has an associated conservation law: the conservation of baryon number, electron number, muon number, and tau number. Each quark is assigned a baryon number of 1/3, while each antiquark is assigned a baryon number of -1/3. Conservation of baryon number implies that the number of quarks minus the number of antiquarks is a constant. Within experimental limits, no violation of this conservation law has been found.

Similarly, each electron and its associated neutrino is assigned an electron number of +1, while the antielectron and the associated antineutrino carry -1 electron number. Similarly, the muons and their neutrinos are assigned a muon number of +1 and the tau leptons are assigned a tau lepton number of +1. The Standard Model predicts that each of these three numbers should be conserved separately in a manner similar to the way baryon number is conserved. These numbers are collectively known as lepton family numbers (LF). Symmetry works differently for quarks than for leptons, mainly because the Standard Model predicts that neutrinos are massless. However, it was recently found that neutrinos have small masses and oscillate between flavors, signaling that the conservation of lepton family number is violated.

In addition to the accidental (but exact) symmetries described above, the Standard Model exhibits several **approximate symmetries**. These are the "SU(2) custodial symmetry" and the "SU(2) or SU(3) quark flavor symmetry."

Symmetries of the Standard Model and Associated Conservation Laws

Symmetry	Lie Group	Symmetry Type	Conservation Law
Poincaré	Translations×SO(3,1)	Global symmetry	Energy, Momentum, Angular momentum
Gauge	SU(3)×SU(2)×U(1)	Local symmetry	Color charge, Weak isospin, Electric charge, Weak hypercharge
Baryon phase	U(1)	Accidental Global symmetry	Baryon number
Electron phase	U(1)	Accidental Global symmetry	Electron number
Muon phase	U(1)	Accidental Global symmetry	Muon number
Tau phase	U(1)	Accidental Global symmetry	Tau number

Field content of the Standard Model

Field (1st generation)		Spin	Gauge group Representation	Baryon Number	Electron Number
Left-handed quark	Q_L	1/2	(3 , 2 , +1/3)	1/3	0
Left-handed up antiquark	$\bar{u}_L \equiv (u_R)^c$	1/2	($\bar{\mathbf{3}}$, 1 , -4/3)	-1/3	0
Left-handed down antiquark	$\bar{d}_L \equiv (d_R)^c$	1/2	($\bar{\mathbf{3}}$, 1 , +2/3)	-1/3	0
Left-handed lepton	L_L	1/2	(1 , 2 , -1)	0	1
Left-handed antilepton	$\bar{e}_L \equiv (e_R)^c$	1/2	(1 , 1 , +2)	0	-1
Hypercharge gauge field	B_μ	1	(1 , 1 , 0)	0	0
Isospin gauge field	W_μ	1	(1 , 3 , 0)	0	0
Gluon field	G_μ	1	(8 , 1 , 0)	0	0
Higgs field	H	0	(1 , 2 , +1)	0	0

List of standard model fermions

This table is based in part on data gathered by the Particle Data Group.^[19]

Left-handed fermions in the Standard Model

Generation 1						
Fermion (left-handed)	Symbol	Electric charge	Weak isospin	Weak hypercharge	Color charge *	Mass **
Electron	e^-	-1	-1/2	-1	1	511 keV
Positron	e^+	+1	0	+2	1	511 keV
Electron neutrino	ν_e	0	+1/2	-1	1	< 2 eV ****
Electron antineutrino	$\bar{\nu}_e$	0	0	0	1	< 2 eV ****
Up quark	u	+2/3	+1/2	+1/3	3	~ 3 MeV ***
Up antiquark	$\bar{u}$	-2/3	0	-4/3	$\bar{\mathbf{3}}$	~ 3 MeV ***
Down quark	d	-1/3	-1/2	+1/3	3	~ 6 MeV ***
Down antiquark	$\bar{d}$	+1/3	0	+2/3	$\bar{\mathbf{3}}$	~ 6 MeV ***
Generation 2						
Fermion (left-handed)	Symbol	Electric charge	Weak isospin	Weak hypercharge	Color charge *	Mass **
Muon	μ^-	-1	-1/2	-1	1	106 MeV
Antimuon	μ^+	+1	0	+2	1	106 MeV
Muon neutrino	ν_μ	0	+1/2	-1	1	< 2 eV ****
Muon antineutrino	$\bar{\nu}_\mu$	0	0	0	1	< 2 eV ****
Charm quark	c	+2/3	+1/2	+1/3	3	~ 1.337 GeV
Charm antiquark	$\bar{c}$	-2/3	0	-4/3	$\bar{\mathbf{3}}$	~ 1.3 GeV
Strange quark	s	-1/3	-1/2	+1/3	3	~ 100 MeV
Strange antiquark	$\bar{s}$	+1/3	0	+2/3	$\bar{\mathbf{3}}$	~ 100 MeV

Generation 3						
Fermion (left-handed)	Symbol	Electric charge	Weak isospin	Weak hypercharge	Color charge *	Mass **
Tau	τ^-	-1	-1/2	-1	1	1.78 GeV
Antitau	τ^+	+1	0	+2	1	1.78 GeV
Tau neutrino	ν_τ	0	+1/2	-1	1	< 2 eV ****
Tau antineutrino	$\bar{\nu}_\tau$	0	0	0	1	< 2 eV ****
Top quark	t	+2/3	+1/2	+1/3	3	171 GeV
Top antiquark	$\bar{t}$	-2/3	0	-4/3	$\bar{3}$	171 GeV
Bottom quark	b	-1/3	-1/2	+1/3	3	~ 4.2 GeV
Bottom antiquark	$\bar{b}$	+1/3	0	+2/3	$\bar{3}$	~ 4.2 GeV

Notes:

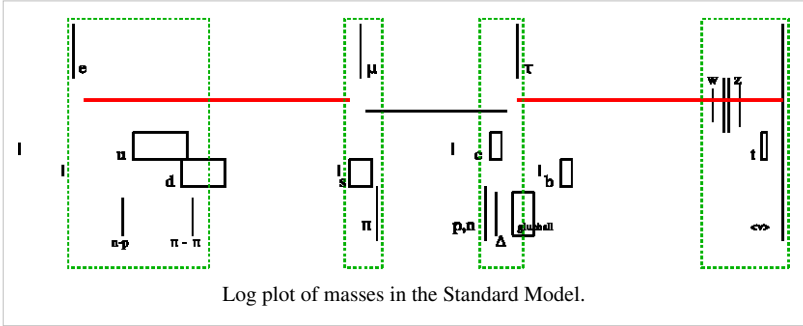
- * These are not ordinary abelian charges, which can be added together, but are labels of group representations of Lie groups.
- ** Mass is really a coupling between a left-handed fermion and a right-handed fermion. For example, the mass of an electron is really a coupling between a left-handed electron and a right-handed electron, which is the antiparticle of a left-handed positron. Also neutrinos show large mixings in their mass coupling, so it's not accurate to talk about neutrino masses in the flavor basis or to suggest a left-handed electron antineutrino.
- *** The masses of baryons and hadrons and various cross-sections are the experimentally measured quantities. Since quarks can't be isolated because of QCD confinement, the quantity here is supposed to be the mass of the quark at the renormalization scale of the QCD scale.
- **** The Standard Model assumes that neutrinos are massless. However, several contemporary experiments prove that neutrinos oscillate between their flavour states, which could not happen if all were massless.^[20] It is straightforward to extend the model to fit these data but there are many possibilities, so the mass eigenstates are still open. See Neutrino#Mass.

Tests and predictions

The Standard Model (SM) predicted the existence of the W and Z bosons, gluon, and the top and charm quarks before these particles were observed. Their predicted properties were experimentally confirmed with good precision. To give an idea of the success of the SM, the following table compares the measured masses of the W and Z bosons with the masses predicted by the SM:

Quantity	Measured (GeV)	SM prediction (GeV)
Mass of W boson	80.398 ± 0.025	80.390 ± 0.018
Mass of Z boson	91.1876 ± 0.0021	91.1874 ± 0.0021

The SM also makes several predictions about the decay of Z bosons, which have been experimentally confirmed by the Large Electron-Positron Collider at CERN.



Challenges to the standard model

There is some experimental evidence consistent with neutrinos having mass, which the Standard Model does not allow.^[21] To accommodate such findings, the Standard Model can be modified by adding a non-renormalizable interaction of lepton fields with the square of the Higgs field. This is natural in certain grand unified theories, and if new physics appears at about 10^{16} GeV, the neutrino masses are of the right order of magnitude.

Currently, there is one elementary particle predicted by the Standard Model that has yet to be observed: the Higgs boson. A major reason for building the Large Hadron Collider is that the high energies of which it is capable are expected to make the Higgs observable. However, as of August 2008, there is only indirect empirical evidence for the existence of the Higgs boson, so that its discovery cannot be claimed. Moreover, there are serious theoretical reasons for supposing that elementary scalar Higgs particles cannot exist (see Quantum triviality).

A fair amount of theoretical and experimental research has attempted to extend the Standard Model into a Unified Field Theory or a Theory of everything, a complete theory explaining all physical phenomena including constants. Inadequacies of the Standard Model that motivate such research include:

- It does not attempt to explain gravitation, and unlike for the strong and electroweak interactions of the Standard Model, there is no known way of describing general relativity, the canonical theory of gravitation, consistently in terms of quantum field theory. The reason for this is among other things that quantum field theories of gravity generally break down before reaching the Planck scale. As a consequence, we have no reliable theory for the very early universe;
- It seems rather *ad-hoc* and inelegant, requiring 19 numerical constants whose values are unrelated and arbitrary. Although the Standard Model, as it now stands, can explain why neutrinos have masses, the specifics of neutrino mass are still unclear. It is believed that explaining neutrino mass will require an additional 7 or 8 constants, which are also arbitrary parameters;
- The Higgs mechanism gives rise to the hierarchy problem if any new physics (such as quantum gravity) is present at high energy scales. In order for the weak scale to be much smaller than the Planck scale, severe fine tuning of Standard Model parameters is required;
- It should be modified so as to be consistent with the emerging "standard model of cosmology." In particular, the Standard Model cannot explain the observed amount of cold dark matter (CDM) and gives contributions to dark energy which are far too large. It is also difficult to accommodate the observed predominance of matter over antimatter (matter/antimatter asymmetry). The isotropy and homogeneity of the visible universe over large distances seems to require a mechanism like cosmic inflation, which would also constitute an extension of the Standard Model.

Currently no proposed Theory of everything has been conclusively verified.

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External links

- " Standard Model - explanation for beginners (<http://cms.web.cern.ch/cms/Physics/StandardPackage/index.html>)" *LHC*
 - " Standard Model may be found incomplete, (<http://www.newscientist.com/news/news.jsp?id=ns9999404>)" *New Scientist*.
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Topological quantum field theory

A **topological quantum field theory** (or **topological field theory** or **TQFT**) is a quantum field theory which computes topological invariants.

Although TQFTs were invented by physicists, they are also of mathematical interest, being related to, among other things, knot theory and the theory of four-manifolds in algebraic topology, and to the theory of moduli spaces in algebraic geometry. Donaldson, Jones, Witten, and Kontsevich have all won Fields Medals for work related to topological field theory.

In condensed matter physics, topological quantum field theories are the low energy effective theories of topologically ordered states, such as fractional quantum Hall states, string-net condensed states, and other strongly correlated quantum liquid states.

Overview

In a topological field theory, the correlation functions do not depend on the metric on spacetime. This means that the theory is not sensitive to changes in the shape of spacetime; if the spacetime warps or contracts, the correlation functions do not change. Consequently, they are topological invariants.

Topological field theories are not very interesting on the flat Minkowski spacetime used in particle physics. Minkowski space can be contracted to a point, so a TQFT on Minkowski space computes only trivial topological invariants. Consequently, TQFTs are usually studied on curved spacetimes, such as, for example, Riemann surfaces. Most of the known topological field theories are defined on spacetimes of dimension less than five. It seems that a few higher dimensional theories exist, but they are not very well understood.

Quantum gravity is believed to be background-independent (in some suitable sense), and TQFTs provide examples of background independent quantum field theories. This has prompted ongoing theoretical investigation of this class of models.

(Caveat: It is often said that TQFTs have only finitely many degrees of freedom. This is not a fundamental property. It happens to be true in most of the examples that physicists and mathematicians study, but it is not necessary. A topological sigma model with target infinite-dimensional projective space, if such a thing could be defined, would have countably infinitely many degrees of freedom.)

Specific models

The known topological field theories fall into two general classes: Schwarz-type TQFTs and Witten-type TQFTs. Witten TQFTs are also sometimes referred to as cohomological field theories.

Schwarz-type TQFTs

In Schwarz-type TQFTs, the correlation functions computed by the path integral are topological invariants because the path integral measure and the quantum field observables are explicitly independent of the metric. For instance, in the BF model, the spacetime is a two-dimensional manifold M , the observables are constructed from a two-form F , an auxiliary scalar B , and their derivatives. The action (which determines the path integral) is

$$S = \int_M BF$$

The spacetime metric does not appear anywhere in this theory, so the theory is explicitly topologically invariant. Another, more famous example is Chern-Simons theory, which can be used to compute knot invariants.

Witten-type TQFTs

In Witten-type topological field theories, the topological invariance is more subtle. For example the Lagrangian for the WZW model does depend explicitly on the metric, but one shows by calculation that the expectation value of the partition function and a special class of correlation functions are in fact diffeomorphism invariant.

Mathematical formulations

Atiyah-Segal axioms

Atiyah suggested a set of axioms for topological quantum field theory which was inspired by Segal's proposed axioms for conformal field theory, (Atiyah 1988). These axioms have been relatively useful for mathematical treatments of Schwarz-type QFTs, although it isn't clear that they capture the whole structure of Witten-type QFTs. The basic idea is that a TQFT is a functor from a certain category of cobordisms to the category of vector spaces.

There are in fact two different sets of axioms which could reasonably be called the Atiyah axioms. These axioms differ basically in whether or not they study a TQFT defined on a single fixed n -dimensional Riemannian / Lorentzian spacetime M or a TQFT defined on all n -dimensional spacetimes at once.

[ed. *What follows is still in rough draft form and should be regarded suspiciously.*]

The case of a fixed spacetime

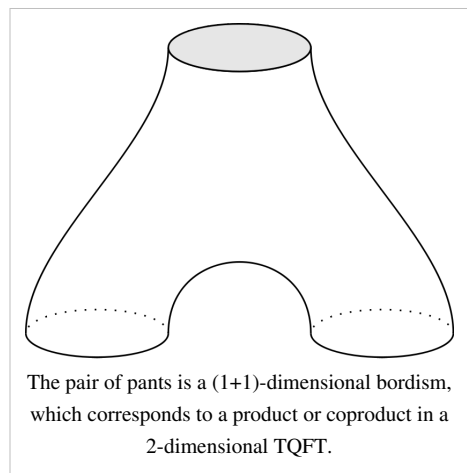
Let $Bord_M$ be the category whose morphisms are n -dimensional submanifolds of M and whose objects are connected components of the boundaries of such submanifolds. Regard two morphisms as equivalent if they are homotopic via submanifolds of M , and so form the quotient category $hBord_M$: The objects in $hBord_M$ are the objects of $Bord_M$, and the morphisms of $hBord_M$ are homotopy equivalence classes of morphisms in $Bord_M$. A TQFT on M is a symmetric monoidal functor from $hBord_M$ to the category of vector spaces.

Note that cobordisms can, if their boundaries match up, be sewn together to form a new bordism. This is the composition law for morphisms in the cobordism category. Since functors are required to preserve composition, this says that the linear map corresponding to a sewn together morphism is just the composition of the linear map for each piece.

There is an equivalence of categories between the category of 2-dimensional topological quantum field theories and the category of commutative Frobenius algebras.

All n -dimensional spacetimes at once

To consider all spacetimes at once, it is necessary to replace $hBord_M$ by a larger category. So let $Bord_n$ be the category of bordisms, i.e. the category whose morphisms are n -dimensional manifolds with boundary, and whose objects are the connected components of the boundaries of n -dimensional manifolds. (Note that any $(n-1)$ -dimensional manifold may appear as an object in $Bord_n$.) As above, regard two morphisms in $Bord_n$ as equivalent if they are homotopic, and form the quotient category $hBord_n$. $Bord_n$ is a monoidal category under the operation which takes two bordisms to the bordism made from their disjoint union. A TQFT on n -dimensional manifolds is then a functor from $hBord_n$ to the category of vector spaces, which takes disjoint unions of bordisms to the tensor product f [ed. *unfinished*]



For example, for $(1+1)$ -dimensional bordisms (2-dimensional bordisms between 1-dimensional manifolds), the map associated with a pair of pants gives a product or coproduct, depending on how the boundary components are grouped – which is commutative or cocommutative, while the map associated with a disk gives a counit (trace) or unit (scalars), depending on grouping of boundary, and thus $(1+1)$ -dimension TQFTs correspond to Frobenius algebras.

Generalizations

For some applications, it is convenient to demand extra topological structure on the morphisms, such as a choice of orientation.

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- Witten, Edward (1988), "Topological quantum field theory" ^[3], *Communications in Mathematical Physics* **117** (3): 353–386, doi:10.1007/BF01223371, MR953828

Schwarz' original paper introducing the ideas of TQFT's, in which he produces a Ray-Singer invariant from a QFT functional:

- Schwarz, Albert (1979), "The partition function of a degenerate functional" ^[4], *Communications in Mathematical Physics* **67** (1): 1–16, doi:10.1007/BF01223197

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Quantum Chromodynamics

In theoretical physics, **quantum chromodynamics (QCD)** is a theory of the strong interaction (color force), a fundamental force describing the interactions of the quarks and gluons making up hadrons (such as the proton, neutron or pion). It is the study of the $SU(3)$ Yang–Mills theory of color-charged fermions (the quarks). QCD is a quantum field theory of a special kind called a non-abelian gauge theory. It is an important part of the Standard Model of particle physics. A huge body of experimental evidence for QCD has been gathered over the years.

QCD enjoys two peculiar properties:

- **Confinement**, which means that the force between quarks does not diminish as they are separated. Because of this, it would take an infinite amount of energy to separate two quarks; they are forever bound into hadrons such as the proton and the neutron. Although analytically unproven, confinement is widely believed to be true because it explains the consistent failure of free quark searches, and it is easy to demonstrate in lattice QCD.
- **Asymptotic freedom**, which means that in very high-energy reactions, quarks and gluons interact very weakly. This prediction of QCD was first discovered in the early 1970s by David Politzer and by Frank Wilczek and David Gross. For this work they were awarded the 2004 Nobel Prize in Physics.

There is no known phase-transition line separating these two properties; confinement is dominant in low-energy scales but, as energy increases, asymptotic freedom becomes dominant.

Terminology

The word *quark* was coined by American physicist Murray Gell-Mann (b. 1929) in its present sense. It originally comes from the phrase "Three quarks for Muster Mark" in *Finnegans Wake* by James Joyce. On June 27, 1978, Gell-Mann wrote a private letter to the editor of the Oxford English Dictionary, in which he related that he had been influenced by Joyce's words: "The allusion to three quarks seemed perfect." (Originally, only three quarks had been discovered.) Gell-Mann, however, wanted to pronounce the word with (ô) not (ä), as Joyce seemed to indicate by rhyming words in the vicinity such as *Mark*. Gell-Mann got around that "by supposing that one ingredient of the line 'Three quarks for Muster Mark' was a cry of 'Three quarts for Mister . . . ' heard in H.C. Earwicker's pub," a plausible suggestion given the complex punning in Joyce's novel.^[1]

The three kinds of charge in QCD (as opposed to one in quantum electrodynamics or QED) are usually referred to as "color charge" by loose analogy to the three kinds of color (red, green and blue) perceived by humans. Other than this "clever" nomenclature, the quantum parameter "color" is completely unrelated to the everyday, familiar phenomenon of color.

Since the theory of electric charge is dubbed "electrodynamics", the Greek word "chroma" Χρώμα (meaning color) is applied to the theory of color charge, "chromodynamics".

History

With the invention of bubble chambers and spark chambers in the 1950s, experimental particle physics discovered a large and ever-growing number of particles called hadrons. It seemed that such a large number of particles could not all be fundamental. First, the particles were classified by charge and isospin by Eugene Wigner and Werner Heisenberg; then, in 1953, according to strangeness by Murray Gell-Mann and Kazuhiko Nishijima. To gain greater insight, the hadrons were sorted into groups having similar properties and masses using the *eightfold way*, invented in 1961 by Gell-Mann and Yuval Ne'eman. Gell-Mann and George Zweig, correcting an earlier approach of Shoichi Sakata, went on to propose in 1963 that the structure of the groups could be explained by the existence of three flavours of smaller particles inside the hadrons: the quarks.

Perhaps the first remark that quarks should possess an additional quantum number was made^[2] as a short footnote in the preprint of Boris Struminsky^[3] in connection with Ω^- hyperon composed of three strange quarks with parallel spins (this situation was peculiar, because since quarks are fermions, such combination is forbidden by the Pauli exclusion principle):

Three identical quarks cannot form an antisymmetric S-state. In order to realize an antisymmetric orbital S-state, it is necessary for the quark to have an additional quantum number.

– B. V. Struminsky, *Magnetic moments of barions in the quark model*, JINR-Preprint P-1939, Dubna,
Submitted on January 7, 1965

Boris Struminsky was a PhD student of Nikolay Bogolyubov. The problem considered in this preprint was suggested by Nikolay Bogolyubov, who advised Boris Struminsky in this research.^[3] In the beginning of 1965, Nikolay Bogolyubov, Boris Struminsky and Albert Tavchelimidze wrote a preprint with a more detailed discussion of the additional quark quantum degree of freedom.^[4] This work was also presented by Albert Tavchelimidze without obtaining consent of his collaborators for doing so at an international conference in Trieste (Italy), in May 1965.^[5] [6]

A similar mysterious situation was with the Δ^{++} baryon; in the quark model, it is composed of three up quarks with parallel spins. In 1965, Moo-Young Han with Yoichiro Nambu and Oscar W. Greenberg independently resolved the problem by proposing that quarks possess an additional SU(3) gauge degree of freedom, later called color charge. Han and Nambu noted that quarks might interact via an octet of vector gauge bosons: the gluons.

Since free quark searches consistently failed to turn up any evidence for the new particles, and because an elementary particle back then was *defined* as a particle which could be separated and isolated, Gell-Mann often said that quarks were merely convenient mathematical constructs, not real particles. The meaning of this statement was usually clear in context: He meant quarks are confined, but he also was implying that the strong interactions could probably not be fully described by quantum field theory.

Richard Feynman argued that high energy experiments showed quarks are real particles: he called them *partons* (since they were parts of hadrons). By particles, Feynman meant objects which travel along paths, elementary particles in a field theory.

The difference between Feynman's and Gell-Mann's approaches reflected a deep split in the theoretical physics community. Feynman thought the quarks have a distribution of position or momentum, like any other particle, and he (correctly) believed that the diffusion of parton momentum explained diffractive scattering. Although Gell-Mann believed that certain quark charges could be localized, he was open to the possibility that the quarks themselves could not be localized because space and time break down. This was the more radical approach of S-matrix theory.

James Bjorken proposed that pointlike partons would imply certain relations should hold in deep inelastic scattering of electrons and protons, which were spectacularly verified in experiments at SLAC in 1969. This led physicists to abandon the S-matrix approach for the strong interactions.

The discovery of asymptotic freedom in the strong interactions by David Gross, David Politzer and Frank Wilczek allowed physicists to make precise predictions of the results of many high energy experiments using the quantum field theory technique of perturbation theory. Evidence of gluons was discovered in three jet events at PETRA in 1979. These experiments became more and more precise, culminating in the verification of perturbative QCD at the level of a few percent at the LEP in CERN.

The other side of asymptotic freedom is confinement. Since the force between color charges does not decrease with distance, it is believed that quarks and gluons can never be liberated from hadrons. This aspect of the theory is verified within lattice QCD computations, but is not mathematically proven. One of the Millennium Prize Problems announced by the Clay Mathematics Institute requires a claimant to produce such a proof. Other aspects of non-perturbative QCD are the exploration of phases of quark matter, including the quark-gluon plasma.

The relation between the short-distance particle limit and the confining long-distance limit is one of the topics recently explored using string theory, the modern form of S-matrix theory.^[7] [8]

Theory

Some definitions

Every field theory of particle physics is based on certain symmetries of nature whose existence is deduced from observations. These can be

- local symmetries, that is the symmetry acts independently at each point in space-time. Each such symmetry is the basis of a gauge theory and requires the introduction of its own gauge bosons.
- global symmetries, which are symmetries whose operations must be simultaneously applied to all points of space-time.

QCD is a gauge theory of the $SU(3)$ gauge group obtained by taking the color charge to define a local symmetry.

Since the strong interaction does not discriminate between different flavors of quark, QCD has approximate **flavor symmetry**, which is broken by the differing masses of the quarks.

There are additional global symmetries whose definitions require the notion of chirality, discrimination between left and right-handed. If the spin of a particle has a positive projection on its direction of motion then it is called left-handed; otherwise, it is right-handed. Chirality and handedness are not the same, but become approximately equivalent at high energies.

- **Chiral** symmetries involve independent transformations of these two types of particle.
- **Vector** symmetries (also called diagonal symmetries) mean the same transformation is applied on the two chiralities.
- **Axial** symmetries are those in which one transformation is applied on left-handed particles and the inverse on the right-handed particles.

Additional remarks: duality

As mentioned, *asymptotic freedom* means that at large energy - this corresponds also to *short distances* - there is practically no interaction between the particles. This is in contrast - more precisely one would say: dual - to what one is used to, since usually one connects the absence of interactions with *large* distances. However, as already mentioned in the original paper of Franz Wegner,^[9] a solid state theorist who introduced 1971 simple gauge invariant lattice models, the high-temperature behaviour of the *original model*, e.g. the strong decay of correlations at large distances, corresponds to the low-temperature behaviour of the (usually ordered!) *dual model*, namely the asymptotic decay of non-trivial correlations, e.g. short-range deviations from almost perfect arrangements, for short distances. Here, in contrast to Wegner, we have only the dual model, which is that one described in this article.^[10]

Symmetry groups

The color group $SU(3)$ corresponds to the local symmetry whose gauging gives rise to QCD. The electric charge labels a representation of the local symmetry group $U(1)$ which is gauged to give QED: this is an abelian group. If one considers a version of QCD with N_f flavors of massless quarks, then there is a global (chiral) flavor symmetry group $SU_L(N_f) \times SU_R(N_f) \times U_B(1) \times U_A(1)$. The chiral symmetry is spontaneously broken by the QCD vacuum to the vector (L+R) $SU_V(N_f)$ with the formation of a chiral condensate. The vector symmetry, $U_B(1)$ corresponds to the baryon number of quarks and is an exact symmetry. The axial symmetry $U_A(1)$ is exact in the classical theory, but broken in the quantum theory, an occurrence called an anomaly. Gluon field configurations called instantons are closely related to this anomaly.

There are two different types of $SU(3)$ symmetry: there is the symmetry that acts on the different colors of quarks, and this is an exact gauge symmetry mediated by the gluons, and there is also a flavor symmetry which rotates different flavors of quarks to each other, or *flavor* $SU(3)$. Flavor $SU(3)$ is an approximate symmetry of the vacuum of QCD, and is not a fundamental symmetry at all. It is an accidental consequence of the small mass of the three

lightest quarks.

In the QCD vacuum there are vacuum condensates of all the quarks whose mass is less than the QCD scale. This includes the up and down quarks, and to a lesser extent the strange quark, but not any of the others. The vacuum is symmetric under SU(2) isospin rotations of up and down, and to a lesser extent under rotations of up, down and strange, or full flavor group SU(3), and the observed particles make isospin and SU(3) multiplets.

The approximate flavor symmetries do have associated gauge bosons, observed particles like the rho and the omega, but these particles are nothing like the gluons and they are not massless. They are emergent gauge bosons in an approximate string description of QCD.

Lagrangian

The dynamics of the quarks and gluons are controlled by the quantum chromodynamics Lagrangian. The gauge invariant QCD Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{QCD}} &= \bar{\psi}_i (i\gamma^\mu (D_\mu)_{ij} - m \delta_{ij}) \psi_j - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} \\ &= \bar{\psi}_i (i\gamma^\mu \partial_\mu - m) \psi_i - g G_\mu^a \bar{\psi}_i \gamma^\mu T_{ij}^a \psi_j - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu},\end{aligned}$$

where $\psi_i(x)$ is the quark field, a dynamical function of space-time, in the fundamental representation of the SU(3) gauge group, indexed by $i, j, \dots$; $G_\mu^a(x)$ are the gluon fields, also a dynamical function of space-time, in the adjoint representation of the SU(3) gauge group, indexed by $a, b, \dots$. The γ^μ are Dirac matrices connecting the spinor representation to the vector representation of the Lorentz group; and T_{ij}^a are the generators connecting the fundamental, antifundamental and adjoint representations of the SU(3) gauge group. The Gell-Mann matrices provide one such representation for the generators.

The symbol $G_{\mu\nu}^a$ represents the gauge invariant gluonic field strength tensor, analogous to the electromagnetic field strength tensor, $F^{\mu\nu}$, in Electrodynamics. It is given by

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g f^{abc} G_\mu^b G_\nu^c,$$

where f_{abc} are the structure constants of SU(3). Note that the rules to move-up or pull-down the a, b , or c indexes are *trivial*, (+.....+), so that $f^{abc} = f_{abc} = f_{bc}^a$, whereas for the μ or ν indexes one has the non-trivial *relativistic* rules, corresponding e.g. to the signature (+---). Furthermore, for mathematicians, according to this formula the gluon colour field can be represented by a SU(3)-Lie algebra-valued "curvature"-2-form $\mathbf{G} = d\tilde{\mathbf{G}} - g \tilde{\mathbf{G}} \wedge \tilde{\mathbf{G}}$, where $\tilde{\mathbf{G}}$ is a "vector potential"-1-form corresponding to $\mathbf{G}$ and $\wedge$ is the (antisymmetric) "wedge product" of this algebra, producing the "structure constants" f^{abc} . The Cartan-derivative of the field form (i.e. essentially the divergence of the field) would be zero in the absence of the "gluon terms", i.e. those $\sim g$, which represent the non-abelian character of the SU(3).

The constants m and g control the quark mass and coupling constants of the theory, subject to renormalization in the full quantum theory.

An important theoretical notion concerning the final term of the above Lagrangian is the *Wilson loop* variable. This loop variable plays a most-important role in discretized forms of the QCD (see lattice QCD), and more generally, it distinguishes confined and deconfined states of a gauge theory. It was introduced by the Nobel prize winner Kenneth G. Wilson and is treated in a separate article.

Fields

Quarks are massive spin-1/2 fermions which carry a color charge whose gauging is the content of QCD. Quarks are represented by Dirac fields in the fundamental representation **3** of the gauge group SU(3). They also carry electric charge (either -1/3 or 2/3) and participate in weak interactions as part of weak isospin doublets. They carry global quantum numbers including the baryon number, which is 1/3 for each quark, hypercharge and one of the flavor quantum numbers.

Gluons are spin-1 bosons which also carry color charges, since they lie in the adjoint representation **8** of SU(3). They have no electric charge, do not participate in the weak interactions, and have no flavor. They lie in the singlet representation **1** of all these symmetry groups.

Every quark has its own antiquark. The charge of each antiquark is exactly the opposite of the corresponding quark.

Dynamics

According to the rules of quantum field theory, and the associated Feynman diagrams, the above theory gives rise to three basic interactions: a quark may emit (or absorb) a gluon, a gluon may emit (or absorb) a gluon, and two gluons may directly interact. This contrasts with QED, in which only the first kind of interaction occurs, since photons have no charge. Diagrams involving Faddeev–Popov ghosts must be considered too.

Area law and confinement

Detailed computations with the above-mentioned Lagrangian^[11] show that the effective potential between a quark and its anti-quark in a meson contains a term $\propto r$, which represents some kind of "stiffness" of the interaction between the particle and its anti-particle at large distances, similar to the entropic elasticity of a rubber band (see below). This leads to *confinement*^[12] of the quarks to the interior of hadrons, i.e. mesons and nucleons, with typical radii R_c , corresponding to former "Bag models" of the hadrons^[13]. The order of magnitude of the "bag radius" is 1 fm ($=10^{-15}$ m). Moreover, the above-mentioned stiffness is quantitatively related to the so-called "area law" behaviour of the expectation value of the Wilson loop product P_W of the ordered coupling constants around a closed loop W ; i.e. $\langle P_W \rangle$ is proportional to the *area* enclosed by the loop. For this behaviour the non-abelian behaviour of the gauge group is essential.

Methods

Further analysis of the content of the theory is complicated. Various techniques have been developed to work with QCD. Some of them are discussed briefly below.

Perturbative QCD

This approach is based on asymptotic freedom, which allows perturbation theory to be used accurately in experiments performed at very high energies. Although limited in scope, this approach has resulted in the most precise tests of QCD to date.

Lattice QCD

Among non-perturbative approaches to QCD, the most well established one is lattice QCD. This approach uses a discrete set of space-time points (called the lattice) to reduce the analytically intractable path integrals of the continuum theory to a very difficult numerical computation which is then carried out on supercomputers like the QCDOC which was constructed for precisely this purpose. While it is a slow and resource-intensive approach, it has wide applicability, giving insight into parts of the theory inaccessible by other means. However, the numerical sign problem makes it difficult to use lattice methods to study QCD at high density and low temperature (e.g. nuclear matter or the interior of neutron stars).

1/N expansion

A well-known approximation scheme, the 1/N expansion, starts from the premise that the number of colors is infinite, and makes a series of corrections to account for the fact that it is not. Until now it has been the source of qualitative insight rather than a method for quantitative predictions. Modern variants include the AdS/CFT approach.

Effective theories

For specific problems effective theories may be written down which give qualitatively correct results in certain limits. In the best of cases, these may then be obtained as systematic expansions in some parameter of the QCD Lagrangian. One such effective field theory is chiral perturbation theory or ChPT, which is the QCD effective theory at low energies. More precisely, it is a low energy expansion based on the spontaneous chiral symmetry breaking of QCD, which is an exact symmetry when quark masses are equal to zero, but for the u,d and s quark, which have small mass, it is still a good approximate symmetry. Depending on the number of quarks which are treated as light, one uses either SU(2) ChPT or SU(3) ChPT. Other effective theories are heavy quark effective theory (which expands around heavy quark mass near infinity), and soft-collinear effective theory (which expands around large ratios of energy scales). In addition to effective theories, models like the Nambu-Jona-Lasinio model and the chiral model are often used when discussing general features.

QCD Sum Rules

Based on an Operator product expansion one can derive sets of relations that connect different observables with each other.

Experimental tests

The notion of quark flavours was prompted by the necessity of explaining the properties of hadrons during the development of the quark model. The notion of colour was necessitated by the puzzle of the Δ^{++} . This has been dealt with in the section on the history of QCD.

The first evidence for quarks as real constituent elements of hadrons was obtained in deep inelastic scattering experiments at SLAC. The first evidence for gluons came in three jet events at PETRA.

Good quantitative tests of perturbative QCD are

- the running of the QCD coupling as deduced from many observations
- scaling violation in polarized and unpolarized deep inelastic scattering
- vector boson production at colliders (this includes the Drell-Yan process)
- jet cross sections in colliders
- event shape observables at the LEP
- heavy-quark production in colliders

Quantitative tests of non-perturbative QCD are fewer, because the predictions are harder to make. The best is probably the running of the QCD coupling as probed through lattice computations of heavy-quarkonium spectra. There is a recent claim about the mass of the heavy meson B_c [14]. Other non-perturbative tests are currently at the level of 5% at best. Continuing work on masses and form factors of hadrons and their weak matrix elements are promising candidates for future quantitative tests. The whole subject of quark matter and the quark-gluon plasma is a non-perturbative test bed for QCD which still remains to be properly exploited.

Cross-relations to Solid State Physics

There are unexpected cross-relations to solid state physics. For example, the notion of gauge invariance forms the basis of the well-known Mattis spin glasses,^[15] which are systems with the usual spin degrees of freedom $s_i = \pm 1$ for $i=1,\dots,N$, with the special fixed "random" couplings $J_{i,k} = \epsilon_i J_0 \epsilon_k$. Here the ϵ_i and ϵ_k quantities can independently and "randomly" take the values ± 1 , which corresponds to a most-simple gauge transformation $(s_i \rightarrow s_i \cdot \epsilon_i \quad J_{i,k} \rightarrow \epsilon_i J_{i,k} \epsilon_k \quad s_k \rightarrow s_k \cdot \epsilon_k)$. This means that thermodynamic expectation values of measurable quantities, e.g. of the energy $\mathcal{H} := - \sum s_i J_{i,k} s_k$, are invariant.

However, here the *coupling degrees of freedom* $J_{i,k}$, which in the QCD correspond to the *gluons*, are "frozen" to fixed values (quenching). In contrast, in the QCD they "fluctuate" (annealing), and through the large number of gauge degrees of freedom the entropy plays an important role (see below).

For positive J_0 the thermodynamics of the Mattis spin glass corresponds in fact simply to a ferromagnet, just because these systems have no "frustration" at all. This term is a basic measure in spin glass theory.^[16] Quantitatively it is identical with the loop-product $P_W := J_{i,k} J_{k,l} \dots J_{n,m} J_{m,i}$ along a closed loop W . However, for a Mattis spin glass - in contrast to "genuine" spin glasses - the quantity P_W never becomes negative.

The basic notion "frustration" of the spin-glass is actually similar to the Wilson loop quantity of the QCD. The only difference is again that in the QCD one is dealing with SU(3) matrices, and that one is dealing with a "fluctuating" quantity. Energetically, perfect absence of frustration should be non-favorable and untypical for a spin glass, which means that one should add the loop-product to the Hamiltonian, by some kind of term representing a "punishment". - In the QCD the Wilson loop is essential for the Lagrangian rightaway.

The relation between the QCD and "disordered magnetic systems" (the spin glasses belong to them) were additionally stressed in a paper by Fradkin, Huberman und Shenker,^[17] which also stresses the notion of duality.

A further analogy consists in the already mentioned similarity to polymer physics, where, analogously to Wilson Loops, so-called "entangled nets" appear, which are important for the formation of the entropy-elasticity (force proportional to the length) of a rubber band. The non-abelian character of the SU(3) corresponds thereby to the non-trivial "chemical links", which glue different loop segments together, and "asymptotic freedom" means in the polymer analogy simply the fact that in the short-wave limit, i.e. for $0 \leftarrow \lambda_w \ll R_c$ (where R_c is a characteristic correlation-length for the glued loops, corresponding to the above-mentioned "bag radius", while λ_w is the wavelength of an excitation) any non-trivial correlation vanishes totally, as if the system had crystallized.^[18]

There is also a correspondence between confinement in QCD - the fact that the colour-field is only different from zero in the interior of hadrons - and the behaviour of the usual magnetic field in the theory of type-II superconductors: there the magnetism is confined to the interior of the Abrikosov flux-line lattice,^[19] i.e., the London penetration depth λ of that theory is analogous to the confinement radius R_c of quantum chromodynamics. Mathematically, this correspondence is supported by the second term, $\propto g G_\mu^a \bar{\psi}_i \gamma^\mu T_{ij}^a \psi_j$, on the r.h.s. of the Lagrangian.

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External links

- Particle data group (<http://pdg.lbl.gov/>)
- The millennium prize (<http://www.claymath.org/millennium/>) for proving confinement (<http://www.claymath.org/millennium/>)
- Ab Initio Determination of Light Hadron Masses (<http://www.sciencemag.org/cgi/content/abstract/322/5905/1224>)
- Andreas S Kronfeld (<http://www.sciencemag.org/cgi/content/summary/322/5905/1198>) *The Weight of the World Is Quantum Chromodynamics*
- Andreas S Kronfeld (http://www.iop.org/EJ/article/1742-6596/125/1/012067/jpconf8_125_012067.pdf?request-id=f9ccdf0d-ee26-4856-99fb-ce5bfef07c4c) *Quantum chromodynamics with advanced computing*
- Standard model gets right answer (http://www.sciencenews.org/view/generic/id/38788/title/Standard_model_gets_right_answer_for_proton_neutron_masses)

- Quantum Chromodynamics (<http://arxiv.org/abs/hepph/9505231>)

Quantum Geometry

In theoretical physics, **quantum geometry** is the set of new mathematical concepts generalizing the concepts of geometry whose understanding is necessary to describe the physical phenomena at very short distance scales (comparable to Planck length). At these distances, quantum mechanics has a profound effect on physics.

Each theory of quantum gravity uses the term *quantum geometry* in a slightly different fashion. String theory, a leading candidate for a quantum theory of gravity, uses the term quantum geometry to describe exotic phenomena such as T-duality and other geometric dualities, mirror symmetry, topology-changing transitions, minimal possible distance scale, and other effects that challenge our usual geometrical intuition. More technically, quantum geometry refers to the shape of the spacetime manifold as seen by D-branes which includes the quantum corrections to the metric tensor, such as the worldsheet instantons. For example, the quantum volume of a cycle is computed from the mass of a brane wrapped on this cycle.

In an alternative approach to quantum gravity called loop quantum gravity (LQG), the phrase *quantum geometry* usually refers to the formalism within LQG where the observables that capture the information about the geometry are now well defined operators on a Hilbert space. In particular, certain physical observables, such as the area, have a discrete spectrum. It has also been shown that the loop quantum geometry is non-commutative.

It is possible (but considered unlikely) that this strictly quantized understanding of geometry will be consistent with the quantum picture of geometry arising from string theory.

Another, quite successful, approach, which tries to reconstruct the geometry of space-time from "first principles" is Discrete Lorentzian quantum gravity.

External links

- Space and Time: From Antiquity to Einstein and Beyond ^[1]
- Quantum Geometry and its Applications ^[2]
- Hypercomplex Numbers in Geometry and Physics ^[3]

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[2] <http://cgpg.gravity.psu.edu/people/Ashtekar/articles/qgfinal.pdf>
[3] <http://hypercomplex.xpsweb.com/articles/221/en/pdf/main-01e.pdf>
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Loop Quantum Gravity

Loop quantum gravity (LQG), also known as **loop gravity** and **quantum geometry**, is a proposed quantum theory of spacetime which attempts to reconcile the theories of quantum mechanics and general relativity. Loop quantum gravity suggests that space can be viewed as an extremely fine fabric or network "woven" of finite quantised loops of excited gravitational fields called spin networks. When viewed over time, these spin networks are called spin foam, which should not be confused with quantum foam. A major quantum gravity contender with string theory, loop quantum gravity incorporates general relativity without requiring string theory's higher dimensions.

LQG preserves many of the important features of general relativity, while simultaneously employing quantization of both space and time at the Planck scale in the tradition of quantum mechanics. The technique of loop quantization was developed for the nonperturbative quantization of diffeomorphism-invariant gauge theory. Roughly, LQG tries to establish a quantum theory of gravity in which the very space itself, where all other physical phenomena occur, becomes quantized.

LQG is one of a family of theories called canonical quantum gravity. The LQG theory also includes matter and forces, but does not address the problem of the unification of all physical forces the way some other quantum gravity theories such as string theory do.

History of LQG

In 1986, Abhay Ashtekar reformulated Einstein's field equations of general relativity, using what have come to be known as Ashtekar variables, a particular flavor of Einstein-Cartan theory with a complex connection. In 1988, Carlo Rovelli and Lee Smolin used this formalism to introduce **the loop representation of quantum general relativity**, which was soon developed by Ashtekar, Rovelli, Smolin and many others. In the Ashtekar formulation, the fundamental objects are a rule for parallel transport (technically, a connection) and a coordinate frame (called a vierbein) at each point. Because the Ashtekar formulation was background-independent, it was possible to use Wilson loops as the basis for a nonperturbative quantization of gravity. Explicit (spatial) diffeomorphism invariance of the vacuum state plays an essential role in the regularization of the Wilson loop states.

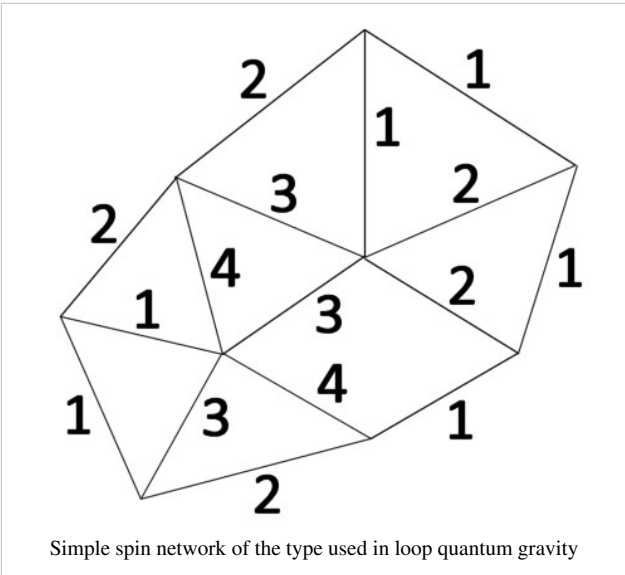
Around 1990, Rovelli and Smolin obtained an explicit basis of states of quantum geometry, which turned out to be labelled by Roger Penrose's spin networks, and showed that the **geometry is quantized**, that is, the (non-gauge-invariant) quantum operators representing area and volume have a discrete spectrum. In this context, spin networks arose as a generalization of Wilson loops necessary to deal with mutually intersecting loops. Mathematically, spin networks are related to group representation theory and can be used to construct knot invariants such as the Jones polynomial.

Key concepts of loop quantum gravity

In the framework of quantum field theory, and using the standard techniques of perturbative calculations, one finds that gravitation is non-renormalizable in contrast to the electroweak and strong interactions of the Standard Model of particle physics. This implies that there are infinitely many free parameters in the theory and thus that it cannot be predictive.

In general relativity, the Einstein field equations assign a geometry (via a metric) to space-time. Before this, there is no physical notion of distance or time measurements. In this sense, general relativity is said to be background independent. An immediate conceptual issue that arises is that the usual framework of quantum mechanics, including quantum field theory, relies on a reference (background) space-time. Therefore, one approach to finding a quantum theory of gravity is to understand how to do quantum mechanics without relying on such a background; this is the approach of the canonical quantization/loop quantum gravity/spin foam approaches.

Starting with the initial-value-formulation of general relativity (cf. the section on General relativity#Evolution equations), the result is an analogue of the Schrödinger equation called the Wheeler-deWitt equation, which some argue is ill-defined.^[1] A major break-through came with the introduction of what are now known as Ashtekar variables, which represent geometric gravity using mathematical analogues of electric and magnetic fields.^[2] The resulting candidate for a theory of quantum gravity is Loop quantum gravity, in which space is represented by a network structure called a spin network, evolving over time in discrete steps.^[3]



Though not proven, it may be impossible to quantize gravity in 3+1 dimensions without creating matter and energy artifacts.

Should LQG succeed as a quantum theory of gravity, the known matter fields will have to be incorporated into the theory *a posteriori*. Many of the approaches now being actively pursued (by Renate Loll, Jan Ambjørn, Lee Smolin, Sundance Bilson-Thompson, Laurent Freidel, Mark B. Wise and others^[4]) combine matter with geometry.

The main successes of loop quantum gravity are:

1. It is a nonperturbative quantization of 3-space geometry, with quantized area and volume operators.
2. It includes a calculation of the entropy of black holes.
3. It replaces the Big Bang spacetime singularity with a Big Bounce.

These claims are not universally accepted among the physics community, which is presently divided between different approaches to the problem of quantum gravity. LQG may possibly be viable as a refinement of either gravity or geometry. Many of the core results are rigorous mathematical physics; their physical interpretations remain speculative. Three speculative physical interpretations of LQG's core mathematical results are loop quantization, Lorentz invariance, General covariance and background independence, discussed below. Another physical test for LQG is to reproduce the physics of general relativity coupled with quantum field theory, discussed under problems.

Loop quantization

At the core of loop quantum gravity is a framework for nonperturbative quantization of diffeomorphism-invariant gauge theories, which one might call **loop quantization**. While originally developed in order to quantize vacuum general relativity in 3+1 dimensions, the formalism can accommodate arbitrary spacetime dimensionalities, fermions,^[5] an arbitrary gauge group (or even quantum group), and supersymmetry,^[6] and results in a quantization of the kinematics of the corresponding diffeomorphism-invariant gauge theory. Much work remains to be done on the dynamics, the classical limit and the correspondence principle, all of which are necessary in one way or another to make contact with experiment.

In a nutshell, loop quantization is the result of applying C*-algebraic quantization to a non-canonical algebra of gauge-invariant classical observables. *Non-canonical* means that the basic observables quantized are not generalized coordinates and their conjugate momenta. Instead, the algebra generated by spin network observables (built from holonomies) and field strength fluxes is used.

Loop quantization techniques are particularly successful in dealing with topological quantum field theories, where they give rise to state-sum/spin-foam models such as the Turaev-Viro model of 2+1 dimensional general relativity. A

much studied topological quantum field theory is the so-called BF theory in 3+1 dimensions. Since classical general relativity can be formulated as a BF theory with constraints, scientists hope that a consistent quantization of gravity may arise from the perturbation theory of BF spin-foam models.

Lorentz invariance

LQG is a quantization of a classical Lagrangian field theory which is equivalent to the usual Einstein-Cartan theory in that it leads to the same equations of motion describing general relativity with torsion. As such, it can be argued that LQG respects *local* Lorentz invariance. *Global* Lorentz invariance is broken in LQG just as in general relativity. A positive cosmological constant can be realized in LQG by replacing the Lorentz group with the corresponding quantum group.

General covariance and background independence

General covariance, also known as "diffeomorphism invariance", is the invariance of physical laws under arbitrary coordinate transformations. An example of this are the equations of general relativity, where this symmetry is one of the defining features of the theory. LQG preserves this symmetry by requiring that the physical states remain invariant under the generators of diffeomorphisms. The interpretation of this condition is well understood for purely spatial diffeomorphisms. However, the understanding of diffeomorphisms involving time (the Hamiltonian constraint) is more subtle because it is related to dynamics and the so-called problem of time in general relativity.^[7] A generally accepted calculational framework to account for this constraint is yet to be found.^{[8] [9]}

Whether or not Lorentz invariance is broken in the low-energy limit of LQG, the theory is formally background independent. The equations of LQG are not embedded in, or presuppose, space and time, except for its invariant topology. Instead, they are expected to give rise to space and time at distances which are large compared to the Planck length.

Problems

While there has been a recent proposal relating to observation of naked singularities,^[10] and doubly special relativity, as a part of a program called loop quantum cosmology, as of now there is no experimental observation for which loop quantum gravity makes a prediction not made by the Standard Model or general relativity (a problem that plagues all current theories of quantum gravity).

Making predictions from the theory of LQG has been extremely difficult computationally, also a recurring problem with modern theories in physics.

Another problem is that a crucial free parameter in the theory known as the Immirzi parameter can only be computed by demanding agreement with Bekenstein and Hawking's calculation of the black hole entropy. Loop quantum gravity predicts that the entropy of a black hole is proportional to the area of the event horizon, but does not obtain the Bekenstein-Hawking formula $S = A/4$ unless the Immirzi parameter is chosen to give this value. A prediction directly from theory would be preferable.

Presently, no semiclassical limit recovering general relativity has been shown to exist. This means it remains unproven that LQG's description of spacetime at the Planck scale has the right continuum limit, described by general relativity with possible quantum corrections. Specifically, the dynamics of the theory is encoded in the Hamiltonian constraint, but there is no candidate Hamiltonian (quantum mechanics).^[11] Other technical problems includes finding off-shell closure of the constraint algebra and physical inner product vector space, coupling to matter fields Quantum field theory, fate of the Renormalization of the graviton in Perturbation theory that lead to Ultraviolet divergence beyond 2-loops One-loop Feynman diagram in Feynman diagram.^[12] The fate of Lorentz invariance in loop quantum gravity remains an open problem.^[13]

Current LQG research directions attempt to address these known problems, and includes spinfoam models ^[14] and entropic gravity ^[15].

See also

- Loop quantum cosmology
- Heyting algebra
- Category theory
- Noncommutative geometry
- Topos theory
- C*-algebra
- Regge calculus
- Double special relativity
- Lorentz invariance in loop quantum gravity
- Immirzi parameter
- Invariance mechanics
- spin foam
- group field theory
- string-net
- Kodama state
- supersymmetry

Notes

- [1] Cf. section 3 in Kuchař 1973.
- [2] See Ashtekar 1986, Ashtekar 1987.
- [3] For a review, see Thiemann 2006; more extensive accounts can be found in Rovelli 1998, Ashtekar & Lewandowski 2004 as well as in the lecture notes Thiemann 2003.
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- A list of LQG references catered to fresh graduates (<http://sps.nus.edu.sg/~wongjian/lqg.html>)
- Loop Quantum Gravity Lectures Online (http://www.perimeterinstitute.ca/Events/Introduction_to_Quantum_Gravity/Introduction_to_Quantum_Gravity/) by Lee Smolin
- Spin networks, spin foams and loop quantum gravity (<http://jdc.math.uwo.ca/spin-foams/>)
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Quantum Algebraic Topology

In physics, **topological order** ^[1] is a new kind of order (a new kind of organization of particles) in a quantum state that is beyond the Landau symmetry-breaking description. It cannot be described by local order parameters and long range correlations. However, topological orders can be described by a new set of quantum numbers, such as ground state degeneracy, quasiparticle fractional statistics, edge states, topological entropy, etc. Roughly speaking, topological order is a pattern of long-range quantum entanglement in quantum states. States with different topological orders can change into each other only through a phase transition.

Background

Although all matter is formed by atoms, matter can have very different properties and appear in very different forms, such as solid, liquid, superfluid, magnet, etc. According to condensed matter physics and the principle of emergence, the different properties of materials originate from the different ways in which the atoms are organized in the materials. Those different organizations of the atoms (or other particles) are formally called the orders in the materials.

Atoms can organize in many ways which lead to many different orders and many different types of materials. Landau symmetry-breaking theory provides a general understanding of these different orders. It points out that different orders really correspond to different symmetries in the organizations of the constituent atoms. As a material changes from one order to another order (i.e., as the material undergoes a phase transition), what happens is that the symmetry of the organization of the atoms changes.

For example, atoms have a random distribution in a liquid, so a liquid remains the same as we displace it by an arbitrary distance. We say that a liquid has a *continuous translation symmetry*. After a phase transition, a liquid can turn into a crystal. In a crystal, atoms organize into a regular array (a lattice). A lattice remains unchanged only when we displace it by a particular distance (integer times of lattice constant), so a crystal has only *discrete translation symmetry*. The phase transition between a liquid and a crystal is a transition that reduces the continuous translation symmetry of the liquid to the discrete symmetry of the crystal. Such change in symmetry is called *symmetry breaking*. The essence of the difference between liquids and crystals is therefore that the organizations of atoms have different symmetries in the two phases.

Landau symmetry-breaking theory is a very successful theory. For a long time, physicists believed that Landau symmetry-breaking theory describes all possible orders in materials, and all possible (continuous) phase transitions.

The discovery and characterization of topological order

However, since late 1980s, it has become gradually apparent that Landau symmetry-breaking theory may not describe all possible orders. In an attempt to explain high temperature superconductivity^[2] people introduced chiral spin state.^{[3] [4]} At first, physicists still wanted to use Landau symmetry-breaking theory to describe the chiral spin state. They identified the chiral spin state as a state that breaks the time reversal and parity symmetries, but not the spin rotation symmetry. This should be the end of story according to Landau's symmetry breaking description of orders. However, it was quickly realized that there are many different chiral spin states that have exactly the same

symmetry, so symmetry alone was not enough to characterize different chiral spin states. This means that the chiral spin states contain a new kind of order that is beyond the usual symmetry description^[5]. The proposed, new kind of order was named "topological order"^[6]. (The name "topological order" is motivated by the low energy effective theory of the chiral spin states which is a topological quantum field theory (TQFT)^{[7] [8] [9]}). New quantum numbers, such as ground state degeneracy^[10] and the non-Abelian Berry's phase of degenerate ground states^[11], were introduced to characterize the different topological orders in chiral spin states. Recently, it was shown that topological orders can also be characterized by topological entropy^{[12] [13]}.

But experiments soon indicated that chiral spin states do not describe high-temperature superconductors, and the theory of topological order became a theory with no experimental realization. However, the similarity between chiral spin states and quantum Hall states allows one to use the theory of topological order to describe different quantum Hall states.^[14] Just like chiral spin states, different quantum Hall states all have the same symmetry and are beyond the Landau symmetry-breaking description. One finds that the different orders in different quantum Hall states can indeed be described by topological orders, so the topological order does have experimental realizations.

Fractional quantum Hall (FQH) states were discovered in 1982^{[15] [16]} before the introduction of the concept of topological order. But FQH states are not the first experimentally discovered topologically ordered states. Superconductors discovered in 1911 are the first, which have a $\mathbb{Z}_2$ topological order (note, however, that superconductivity does fall within the Ginzburg-Landau theory description of phase transitions - in fact the prediction of the vortex state in superconductors was one of the main successes of Ginzburg-Landau theory).^{[17] [18]}

Mechanism of topological order

A large class of topological orders is realized through a mechanism called string-net condensation^[19]. This class of topological orders can be classified by utilizing tensor category (or monoidal category) theory. One finds that string-net condensation can generate infinitely many different types of topological orders, which may indicate that there are many different new types of materials remaining to be discovered.

The collective motions of condensed strings give rise to excitations above the string-net condensed states. Those excitations turn out to be gauge bosons. The ends of strings are defects which correspond to another type of excitations. Those excitations are the gauge charges and can carry Fermi or fractional statistics.^[20]

The condensations of other extended objects such as "membranes",^[21] "brane-nets",^[22] and fractals also lead to topologically ordered phases^[23] and "quantum glassiness"^[24]

Mathematical foundation of topological order

We know that group theory is the mathematical foundation of symmetry breaking orders. What is the mathematical foundation of topological order? The string-net condensation suggests that tensor category (or monoidal category) theory may be the mathematical foundation of topological order. Quantum operator algebra is a very important mathematical tool in studying topological orders. Some also suggest that topological order is mathematically described by *extended* quantum symmetry^[25].

Applications

The materials described by Landau symmetry-breaking theory have had a substantial impact on technology. For example, Ferromagnetic materials that break spin rotation symmetry can be used as the media of digital information storage. A hard drive made of ferromagnetic materials can store gigabytes of information. Liquid crystals that break the rotational symmetry of molecules find wide application in display technology; nowadays one can hardly find a household without a liquid crystal display somewhere in it. Crystals that break translation symmetry lead to well defined electronic bands which in turn allow us to make semiconducting devices such as transistors.

Different types of Topologically orders are even richer than different types of symmetry-breaking orders. This suggests their potential for exciting, novel applications.

One theorized application would be to use topologically ordered states as media for quantum computing in a technique known as topological quantum computing. A topologically ordered state is a state with complicated non-local quantum entanglement. The non-locality means that the quantum entanglement in a topologically ordered state is distributed among many different particles. As a result, the pattern of quantum entanglements cannot be destroyed by local perturbations. This significantly reduces the effect of decoherence. This suggests that if we use different quantum entanglements in a topologically ordered state to encode quantum information, the information may last much longer.^[26] The quantum information encoded by the topological quantum entanglements can also be manipulated by dragging the topological defects around each other. This process may provide a physical apparatus for performing quantum computations.^[27] Therefore, topologically ordered states may provide natural media for both quantum memory and quantum computation. Such realizations of quantum memory and quantum computation may potentially be made fault tolerant.^[28]

Topologically ordered states in general have a special property that they contain non-trivial boundary states. In many cases, those boundary states become perfect conducting channel that can conduct electricity without generating heat.^[29] This can be another potential application of topological order in electronic devices. For example, the topological insulator^{[30] [31]} is a simple example of topological order. The boundary states of topological insulator play a key role in the detection and the application of topological insulators.

Potential impact

Why is topological order important? Landau symmetry-breaking theory is a cornerstone of condensed matter physics. It is used to define the territory of condensed matter research. The existence of topological order appears to indicate that nature is much richer than Landau symmetry-breaking theory has so far indicated. The exciting time of condensed matter physics is still ahead of us. Some suggest that topological order (or more precisely, string-net condensation) and the local bosonic (spin) models have the potential to provide a unified origin^[32] for photons, electrons and other elementary particles in our universe.^[33]

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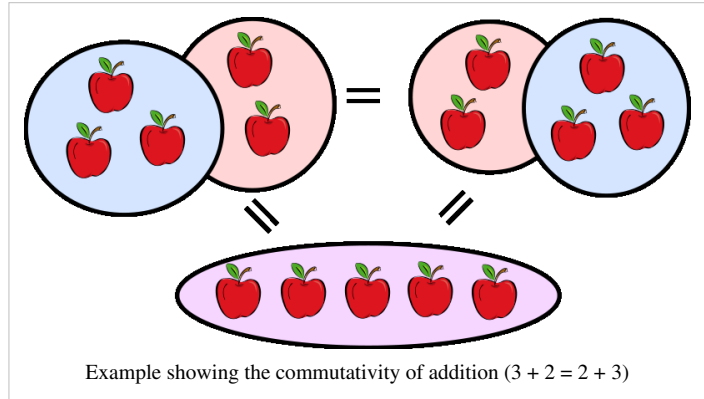
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Commutativity

In mathematics an operation is **commutative** if changing the order of the operands does not change the end result. It is a fundamental property of many binary operations, and many mathematical proofs depend on it. The commutativity of simple operations, such as multiplication and addition of numbers, was for many years implicitly assumed and the property was not named until the 19th century when mathematics started to become formalized. By contrast, division and subtraction are *not* commutative.



Common uses

The *commutative property* (or *commutative law*) is a property associated with binary operations and functions. Similarly, if the commutative property holds for a pair of elements under a certain binary operation then it is said that the two elements *commute* under that operation.

In group and set theory, many algebraic structures are called commutative when certain operands satisfy the commutative property. In higher branches of mathematics, such as analysis and linear algebra the commutativity of well known operations (such as addition and multiplication on real and complex numbers) is often used (or implicitly assumed) in proofs.^{[1] [2] [3]}

Mathematical definitions

The term "commutative" is used in several related senses.^{[4] [5]}

1. A binary operation $*$ on a set S is said to be *commutative* if:

$$\forall x, y \in S : x * y = y * x$$

- An operation that does not satisfy the above property is called *noncommutative*.

2. One says that x *commutes* with y under $*$ if:

$$x * y = y * x$$

3. A binary function $f: A \times A \rightarrow B$ is said to be *commutative* if:

$$\forall x, y \in A : f(x, y) = f(y, x)$$

History and etymology

(14)
 f et f , sont telles qu'elles donnent
 el que soit l'ordre dans lequel on les.
 ppelées *commutatives* entre elles.

; $aEz = Eaz$;

The first known use of the term was in a French Journal published in
 1814

Records of the implicit use of the commutative property go back to ancient times. The Egyptians used the commutative property of multiplication to simplify computing products.^{[6] [7]} Euclid is known to have assumed the commutative property of multiplication in his book *Elements*.^[8] Formal uses of the commutative property arose in the late 18th and early 19th centuries, when mathematicians began to work on a theory of functions. Today the commutative property is a well known and basic property used in most branches of mathematics.

The first recorded use of the term *commutative* was in a memoir by François Servois in 1814,^{[9] [10]} which used the word *commutatives* when describing functions that have what is now called the commutative property. The word is a combination of the French word *commuter* meaning "to substitute or switch" and the suffix *-ative* meaning "tending to" so the word literally means "tending to substitute or switch." The term then appeared in English in *Philosophical Transactions of the Royal Society* in 1844.^[9]

Related properties

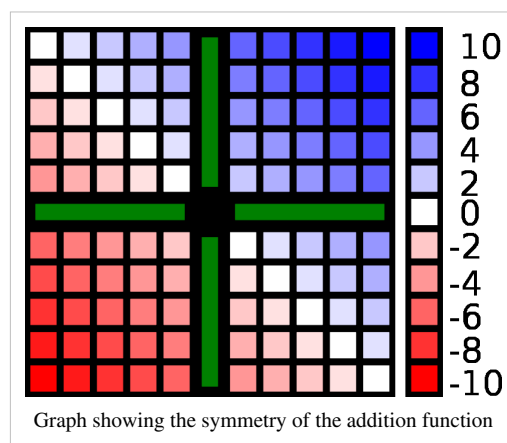
Associativity

The associative property is closely related to the commutative property. The associative property of an expression containing two or more occurrences of the same operator states that the order in which operations are performed does not affect the final result, as long as the order of terms is not changed. In contrast, the commutative property states that the order of the terms does not affect the final result.

Symmetry

Symmetry can be directly linked to commutativity. When a commutative operator is written as a binary function then the resulting function is symmetric across the line $y = x$. As an example, if we let a function f represent addition (a commutative operation) so that $f(x,y) = x + y$ then f is a symmetric function which can be seen in the image on the right.

For binary relations, a symmetric relation is analogous to a commutative operation, in that if a relation R is symmetric, then $aRb \Leftrightarrow bRa$.



Examples

Commutative operations in everyday life

- Putting on socks resembles a commutative operation, since which sock is put on first is unimportant. Either way, the end result (having both socks on), is the same.
- The commutativity of addition is observed when paying for an item with cash. Regardless of the order in which the bills handed over, they always give the same total.

Commutative operations in mathematics

Two well-known examples of commutative binary operations are:^[4]

- The addition of real numbers, which is commutative since

$$\forall(y, z) \in \mathbb{R} : y + z = z + y$$

For example $4 + 5 = 5 + 4$, since both expressions equal 9.

- The multiplication of real numbers, which is commutative since

$$\forall y, z \in \mathbb{R} : yz = zy$$

For example, $3 \times 5 = 5 \times 3$, since both expressions equal 15.

- Further examples of commutative binary operations include addition and multiplication of complex numbers, addition and scalar multiplication of vectors, and intersection and union of sets.

Noncommutative operations in everyday life

- Concatenation, the act of joining character strings together, is a noncommutative operation. For example

$$EA + T = EAT \neq TEA = T + EA$$

- Washing and drying clothes resembles a noncommutative operation; washing and then drying produces a markedly different result to drying and then washing.
- The twists of the Rubik's Cube are noncommutative. This is studied in group theory.

Noncommutative operations in mathematics

- Subtraction is noncommutative since $0 - 1 \neq 1 - 0$
- Division is noncommutative since $1/2 \neq 2/1$
- Infinite addition is not (necessarily) commutative:

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots \leq 1$$

whereas

$$1 + 1 - 1 + 1 + 1 - 1 + 1 + 1 - 1 + \dots = \infty$$

- Matrix multiplication is noncommutative since

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

- The vector product (or cross product) of two vectors in three dimensions is anti-commutative, i.e., $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$.

Some noncommutative binary operations are:^[11]

Mathematical structures and commutativity

- A commutative semigroup is a set endowed with a total, associative and commutative operation.
- If the operation additionally has an identity element, we have a commutative monoid
- An abelian group, or *commutative group* is a group whose group operation is commutative.^[2]
- A commutative ring is a ring whose multiplication is commutative. (Addition in a ring is always commutative.)^[12]
- In a field both addition and multiplication are commutative.^[13]

Non-commuting operators in quantum mechanics

In quantum mechanics as formulated by Schrödinger, physical variables are represented by linear operators such as x (meaning multiply by x), and d/dx . These two operators do not commute as may be seen by considering the effect of their products $x(d/dx)$ and $(d/dx)x$ on a one-dimensional wave function $\psi(x)$:

$$x \frac{d}{dx} \psi = x\psi' \neq \frac{d}{dx} x\psi = \psi + x\psi'$$

According to the uncertainty principle of Heisenberg, if the two operators representing a pair of variables do not commute, then that pair of variables are mutually complementary which means that they cannot be simultaneously measured or known precisely. For example, the position and the linear momentum of a particle are represented respectively (in the x -direction) by the operators x and $(h/2\pi i)d/dx$ (where h is Planck's constant). This is the same example except for the constant $(h/2\pi i)$, so again the operators do not commute and the physical meaning is that the position and linear momentum in a given direction are complementary.

Notes

- [1] Axler, p.2
- [2] Gallian, p.34
- [3] p. 26,87
- [4] Krowne, p.1
- [5] Weisstein, *Commutate*, p.1
- [6] Lumpkin, p.11
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- [8] O'Conner and Robertson, *Real Numbers*
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Noncommutative quantum field theory

In mathematical physics, **noncommutative quantum field theory** (or quantum field theory on noncommutative spacetime) is an application of noncommutative mathematics to the spacetime of quantum field theory that is an outgrowth of noncommutative geometry and index theory in which the coordinate functions^[1] are noncommutative. One commonly studied version of such theories has the "canonical" commutation relation:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

which means that (with any given set of axes), it is impossible to accurately measure the position of a particle with respect to more than one axis. In fact, this leads to an uncertainty relation for the coordinates analogous to the Heisenberg uncertainty principle.

Various lower limits have been claimed for the noncommutative scale, (i.e. how accurately positions can be measured) but there is currently no experimental evidence in favour of such theory or grounds for ruling them out.

One of the novel features of noncommutative field theories is the UV/IR mixing^[2] phenomenon in which the physics at high energies affects the physics at low energies which does not occur in quantum field theories in which the coordinates commute.

Other features include violation of Lorentz invariance due to the preferred direction of noncommutativity. Relativistic invariance can however be retained in the sense of twisted Poincaré invariance of the theory^[3]. The causality condition is modified from that of the commutative theories.

History and motivation

Heisenberg was the first to suggest extending noncommutativity to the coordinates as a possible way of removing the infinite quantities appearing in field theories before the renormalization procedure was developed and had gained acceptance. The first paper on the subject was published in 1947 by Hartland Snyder. The success of the renormalization method resulted in little attention being paid to the subject for some time. In the 1980s, mathematicians, most notably Alain Connes, developed noncommutative geometry. Among other things, this work generalized the notion of differential structure to a noncommutative setting. This led to an operator algebraic description of noncommutative space-times, and the development of a Yang-Mills theory on a noncommutative torus.

The particle physics community became interested in the noncommutative approach because of a paper by Nathan Seiberg and Edward Witten.^[4] They argued in the context of string theory that the coordinate functions of the endpoints of open strings constrained to a D-brane in the presence of a constant Neveu-Schwartz B-field -- equivalent to a constant magnetic field on the brane -- would satisfy the noncommutative algebra set out above. The implication is that a quantum field theory on noncommutative spacetime can be interpreted as a low energy limit of the theory of open strings.

A paper by Sergio Doplicher, Klaus Fredenhagen and John Roberts^[5] set out another motivation for the possible noncommutativity of space-time. Their arguments goes as follows: According to general relativity, when the energy density grows sufficiently large, a black hole is formed. On the other hand according to the Heisenberg uncertainty principle, a measurement of a space-time separation causes an uncertainty in momentum inversely proportional to the extent of the separation. Thus energy whose scale corresponds to the uncertainty in momentum is localized in the system within a region corresponding to the uncertainty in position. When the separation is small enough, the Schwarzschild radius of the system is reached and a black hole is formed, which prevents any information from escaping the system. Thus there is a lower bound for the measurement of length. A sufficient condition for preventing gravitational collapse can be expressed as an uncertainty relation for the coordinates. This relation can in turn be derived from a commutation relation for the coordinates.

Footnotes

- [1] It is possible to have a noncommuting time coordinate, but this causes many problems such as the violation of unitarity of the S-matrix. Hence most research is restricted to so-called "space-space" noncommutativity. There have been attempts to avoid these problems by redefining the perturbation theory. However, string theory derivations of noncommutative coordinates excludes time-space noncommutativity.
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Noncommutative standard model

In theoretical particle physics, the **non-commutative Standard Model**, mainly due to the French mathematician Alain Connes, uses his noncommutative geometry to devise an extension of the Standard Model to include a modified form of general relativity. This unification implies a few constraints on the parameters of the Standard Model. Under an additional assumption, known as the "big desert" hypothesis, one of these constraints determines the mass of the Higgs boson to be around 170 GeV, comfortably within the range of the Large Hadron Collider. Recent Tevatron experiments exclude a Higgs mass of 158 to 175 GeV at the 95% confidence level.^[1]

Background

Current physical theory features four elementary forces: the gravitational force, the electromagnetic force, the weak force, and the strong force. Gravity has an elegant and experimentally precise theory: Einstein's general relativity. It is based on Riemannian geometry and interprets the gravitational force as curvature of space-time. Its Lagrangian formulation requires only two empirical parameters, the gravitational constant and the cosmological constant.

The other three forces also have a Lagrangian theory, called the Standard Model. Its underlying idea is that they are mediated by the exchange of spin-1 particles, the so-called gauge bosons. The one responsible for electromagnetism is the photon. The weak force is mediated by the W and Z bosons; the strong force, by gluons. The gauge Lagrangian is much more complicated than the gravitational one: at present, it involves some 30 real parameters, a number that could increase. What is more, the gauge Lagrangian must also contain a spin 0 particle, the Higgs boson, to give mass to the spin 1/2 and spin 1 particles. This particle has yet to be observed, and if it is not detected at the Large Hadron Collider in Geneva, the consistency of the Standard Model is in doubt.

Alain Connes has generalized Bernhard Riemann's geometry to noncommutative geometry. It describes spaces with curvature and uncertainty. Historically, the first example of such a geometry is quantum mechanics, which introduced Heisenberg's uncertainty relation by turning the classical observables of position and momentum into noncommuting operators. Noncommutative geometry is still sufficiently similar to Riemannian geometry that

Connes was able to rederive general relativity. In doing so, he obtained the gauge Lagrangian as a companion of the gravitational one, a truly geometric unification of all four fundamental interactions. Connes has thus devised a fully geometric formulation of the Standard Model, where all the parameters are geometric invariants of a noncommutative space. A result is that parameters like the electron mass are now analogous to purely mathematical constants like π .

Notes

[1] The TEVNPH Working Group (<http://arxiv.org/abs/1007.4587>)

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External links

- Alain Connes official website (<http://www.alainconnes.org/>) with downloadable papers. (<http://www.alainconnes.org/en/downloads.php>)
- Alain Connes's Standard Model. (<http://resonaances.blogspot.com/2007/02/alain-connes-standard-model.html>)

Nonabelian Gauge Theory

In physics, a **gauge theory** is a type of field theory in which the Lagrangian is invariant under a continuous group of local transformations.

The term *gauge* refers to redundant degrees of freedom in the Lagrangian. The transformations between possible gauges, called *gauge transformations*, form a Lie group which is referred to as the *symmetry group* or the *gauge group* of the theory. Associated with any Lie group is the Lie algebra of group generators. For each group generator there necessarily arises a corresponding vector field called the *gauge field*. Gauge fields are included in the Lagrangian to ensure its invariance under the local group transformations (called *gauge invariance*). When such a theory is quantized, the quanta of the gauge fields are called *gauge bosons*. If the symmetry group is non-commutative, the gauge theory is referred to as *non-abelian*, the usual example being the Yang–Mills theory.

Gauge theories are important as the successful field theories explaining the dynamics of elementary particles. Quantum electrodynamics is an abelian gauge theory with the symmetry group $U(1)$ and has one gauge field, the electromagnetic field, with the photon being the gauge boson. The Standard Model is a non-abelian gauge theory with the symmetry group $U(1) \times SU(2) \times SU(3)$ and has a total of twelve gauge bosons: the photon, three weak bosons and eight gluons.

Many powerful theories in physics are described by Lagrangians which are invariant under some symmetry transformation groups. When they are invariant under a transformation identically performed at *every* point in the space in which the physical processes occur, they are said to have a global symmetry. The requirement of local symmetry, the cornerstone of gauge theories, is a stricter constraint. In fact, a global symmetry is just a local symmetry whose group's parameters are fixed in space-time. Gauge symmetries can be viewed as analogues of the equivalence principle of general relativity in which each point in spacetime is allowed a choice of local reference (coordinate) frame. Both symmetries reflect a redundancy in the description of a system.

Historically, these ideas were first stated in the context of classical electromagnetism and later in general relativity. However, the modern importance of gauge symmetries appeared first in the relativistic quantum mechanics of electrons — quantum electrodynamics, elaborated on below. Today, gauge theories are useful in condensed matter, nuclear and high energy physics among other subfields.

History and importance

The earliest field theory having a gauge symmetry was Maxwell's formulation of electrodynamics in 1864. The importance of this symmetry remained unnoticed in the earliest formulations. Similarly unnoticed, Hilbert had derived the Einstein field equations by postulating the invariance of the action under a general coordinate transformation. Later Hermann Weyl, in an attempt to unify general relativity and electromagnetism, conjectured (incorrectly, as it turned out) that *Eichinvarianz* or invariance under the change of scale (or "gauge") might also be a local symmetry of general relativity. After the development of quantum mechanics, Weyl, Vladimir Fock and Fritz London modified gauge by replacing the scale factor with a complex quantity and turned the scale transformation into a change of phase — a $U(1)$ gauge symmetry. This explained the electromagnetic field effect on the wave function of a charged quantum mechanical particle. This was the first widely recognised gauge theory, popularised by Pauli in the 1940s.^[1]

In 1954, attempting to resolve some of the great confusion in elementary particle physics, Chen Ning Yang and Robert Mills introduced **non-abelian gauge theories** as models to understand the strong interaction holding together nucleons in atomic nuclei. (Ronald Shaw, working under Abdus Salam, independently introduced the same notion in his doctoral thesis.) Generalizing the gauge invariance of electromagnetism, they attempted to construct a theory based on the action of the (non-abelian) $SU(2)$ symmetry group on the isospin doublet of protons and neutrons. This is similar to the action of the $U(1)$ group on the spinor fields of quantum electrodynamics. In particle physics the

emphasis was on using **quantized gauge theories**.

This idea later found application in the quantum field theory of the weak force, and its unification with electromagnetism in the electroweak theory. Gauge theories became even more attractive when it was realized that non-abelian gauge theories reproduced a feature called asymptotic freedom. Asymptotic freedom was believed to be an important characteristic of strong interactions. This motivated searching for a strong force gauge theory. This theory, now known as quantum chromodynamics, is a gauge theory with the action of the $SU(3)$ group on the color triplet of quarks. The Standard Model unifies the description of electromagnetism, weak interactions and strong interactions in the language of gauge theory.

In the 1970s, Sir Michael Atiyah began studying the mathematics of solutions to the classical Yang–Mills equations. In 1983, Atiyah's student Simon Donaldson built on this work to show that the differentiable classification of smooth 4-manifolds is very different from their classification up to homeomorphism. Michael Freedman used Donaldson's work to exhibit exotic $\mathbf{R}^4$ s, that is, exotic differentiable structures on Euclidean 4-dimensional space. This led to an increasing interest in gauge theory for its own sake, independent of its successes in fundamental physics. In 1994, Edward Witten and Nathan Seiberg invented gauge-theoretic techniques based on supersymmetry which enabled the calculation of certain topological invariants. These contributions to mathematics from gauge theory have led to a renewed interest in this area.

The importance of gauge theories for physics stems from the tremendous success of the mathematical formalism in providing a unified framework to describe the quantum field theories of electromagnetism, the weak force and the strong force. This theory, known as the Standard Model, accurately describes experimental predictions regarding three of the four fundamental forces of nature, and is a gauge theory with the gauge group $SU(3) \times SU(2) \times U(1)$. Modern theories like string theory, as well as some formulations of general relativity, are, in one way or another, gauge theories.

See Pickering for more about the history of gauge and quantum field theories.^[2]

Description

Global and local symmetries

In physics, the mathematical description of any physical situation usually contains excess degrees of freedom; the same physical situation is equally well described by many equivalent mathematical configurations. For instance, in Newtonian dynamics, if two configurations are related by a Galilean transformation—an inertial change of reference frame—they represent the same physical situation. These transformations form a group of "symmetries" of the theory, and a physical situation corresponds not to an individual mathematical configuration but to a class of configurations related to one another by this symmetry group. This idea can be generalized to include local as well as global symmetries, analogous to much more abstract "changes of coordinates" in a situation where there is no preferred "inertial" coordinate system that covers the entire physical system. A **gauge theory** is a mathematical model that has symmetries of this kind, together with a set of techniques for making physical predictions consistent with the symmetries of the model.

Example of global symmetry

When a quantity occurring in the mathematical configuration is not just a number but has some geometrical significance, such as a velocity or an axis of rotation, its representation as numbers arranged in a vector or matrix is also changed by a coordinate transformation. For instance, if one description of a pattern of fluid flow states that the fluid velocity in the neighborhood of $(x=1, y=0)$ is 1 m/s in the positive x direction, then a description of the same situation in which the coordinate system has been rotated clockwise by 90 degrees will state that the fluid velocity in the neighborhood of $(x=0, y=1)$ is 1 m/s in the positive y direction. The coordinate transformation has affected both the coordinate system used to identify the *location* of the measurement and the basis in which its *value* is expressed.

As long as this transformation is performed globally (affecting the coordinate basis in the same way at every point), the effect on values that represent the *rate of change* of some quantity along some path in space and time as it passes through point P is the same as the effect on values that are truly local to P .

Use of fiber bundles to describe local symmetries

In order to adequately describe physical situations in more complex theories, it is often necessary to introduce a "coordinate basis" for some of the objects of the theory that do not have this simple relationship to the coordinates used to label points in space and time. (In mathematical terms, the theory involves a fiber bundle in which the fiber at each point of the base space consists of possible coordinate bases for use when describing the values of objects at that point.) In order to spell out a mathematical configuration, one must choose a particular coordinate basis at each point (a *local section* of the fiber bundle) and express the values of the objects of the theory (usually "fields" in the physicist's sense) using this basis. Two such mathematical configurations are equivalent (describe the same physical situation) if they are related by a transformation of this abstract coordinate basis (a change of local section, or *gauge transformation*).

In most gauge theories, the set of possible transformations of the abstract gauge basis at an individual point in space and time is a finite-dimensional Lie group. The simplest such group is $U(1)$, which appears in the modern formulation of quantum electrodynamics (QED) via its use of complex numbers. QED is generally regarded as the first, and simplest, physical gauge theory. The set of possible gauge transformations of the entire configuration of a given gauge theory also forms a group, the *gauge group* of the theory. An element of the gauge group can be parameterized by a smoothly varying function from the points of spacetime to the (finite-dimensional) Lie group, whose value at each point represents the action of the gauge transformation on the fiber over that point.

A gauge transformation with constant parameter at every point in space and time is analogous to a rigid rotation of the geometric coordinate system; it represents a global symmetry of the gauge representation. As in the case of a rigid rotation, this gauge transformation affects expressions that represent the rate of change along a path of some gauge-dependent quantity in the same way as those that represent a truly local quantity. A gauge transformation whose parameter is *not* a constant function is referred to as a local symmetry; its effect on expressions that involve a derivative is qualitatively different from that on expressions that don't. (This is analogous to a non-inertial change of reference frame, which can produce a Coriolis effect.)

Gauge fields

The "gauge covariant" version of a gauge theory accounts for this effect by introducing a gauge field (in mathematical language, an Ehresmann connection) and formulating all rates of change in terms of the covariant derivative with respect to this connection. The gauge field becomes an essential part of the description of a mathematical configuration. A configuration in which the gauge field can be eliminated by a gauge transformation has the property that its field strength (in mathematical language, its curvature) is zero everywhere; a gauge theory is *not* limited to these configurations. In other words, the distinguishing characteristic of a gauge theory is that the gauge field does not merely compensate for a poor choice of coordinate system; there is generally no gauge transformation that makes the gauge field vanish.

When analyzing the dynamics of a gauge theory, the gauge field must be treated as a dynamical variable, similarly to other objects in the description of a physical situation. In addition to its interaction with other objects via the covariant derivative, the gauge field typically contributes energy in the form of a "self-energy" term. One can obtain the equations for the gauge theory by:

- starting from a naïve ansatz without the gauge field (in which the derivatives appear in a "bare" form);
- listing those global symmetries of the theory that can be characterized by a continuous parameter (generally an abstract equivalent of a rotation angle);
- computing the correction terms that result from allowing the symmetry parameter to vary from place to place; and

- reinterpreting these correction terms as couplings to one or more gauge fields, and giving these fields appropriate self-energy terms and dynamical behavior.

This is the sense in which a gauge theory "extends" a global symmetry to a local symmetry, and closely resembles the historical development of the gauge theory of gravity known as **general relativity**.

Physical experiments

Gauge theories are used to model the results of physical experiments, essentially by:

- limiting the universe of possible configurations to those consistent with the information used to set up the experiment, and then
- computing the probability distribution of the possible outcomes that the experiment is designed to measure.

The mathematical descriptions of the "setup information" and the "possible measurement outcomes" (loosely speaking, the "boundary conditions" of the experiment) are generally not expressible without reference to a particular coordinate system, including a choice of gauge. (If nothing else, one assumes that the experiment has been adequately isolated from "external" influence, which is itself a gauge-dependent statement.) Mishandling gauge dependence in boundary conditions is a frequent source of anomalies in gauge theory calculations, and gauge theories can be broadly classified by their approaches to anomaly avoidance.

Continuum theories

The two gauge theories mentioned above (continuum electrodynamics and general relativity) are examples of continuum field theories. The techniques of calculation in a continuum theory implicitly assume that:

- given a completely fixed choice of gauge, the boundary conditions of an individual configuration can in principle be completely described;
- given a completely fixed gauge and a complete set of boundary conditions, the principle of least action determines a unique mathematical configuration (and therefore a unique physical situation) consistent with these bounds;
- the likelihood of possible measurement outcomes can be determined by:
 - establishing a probability distribution over all physical situations determined by boundary conditions that are consistent with the setup information,
 - establishing a probability distribution of measurement outcomes for each possible physical situation, and
 - convolving these two probability distributions to get a distribution of possible measurement outcomes consistent with the setup information; and
- fixing the gauge introduces no anomalies in the calculation, due either to gauge dependence in describing partial information about boundary conditions or to incompleteness of the theory.

These assumptions are close enough to valid, across a wide range of energy scales and experimental conditions, to allow these theories to make accurate predictions about almost all of the phenomena encountered in daily life, from light, heat, and electricity to eclipses and spaceflight. They fail only at the smallest and largest scales (due to omissions in the theories themselves) and when the mathematical techniques themselves break down (most notably in the case of turbulence and other chaotic phenomena).

Quantum field theories

Other than these "classical" continuum field theories, the most widely known gauge theories are quantum field theories, including quantum electrodynamics and the Standard Model of elementary particle physics. The starting point of a quantum field theory is much like that of its continuum analog: a gauge-covariant action integral which characterizes "allowable" physical situations according to the principle of least action. However, continuum and quantum theories differ significantly in how they handle the excess degrees of freedom represented by gauge transformations. Continuum theories, and most pedagogical treatments of the simplest quantum field theories, use a

gauge fixing prescription to reduce the orbit of mathematical configurations that represent a given physical situation to a smaller orbit related by a smaller gauge group (the global symmetry group, or perhaps even the trivial group).

More sophisticated quantum field theories, in particular those which involve a non-abelian gauge group, break the gauge symmetry within the techniques of perturbation theory by introducing additional fields (the Faddeev–Popov ghosts) and counterterms motivated by anomaly cancellation, in an approach known as BRST quantization. While these concerns are in one sense highly technical, they are also closely related to the nature of measurement, the limits on knowledge of a physical situation, and the interactions between incompletely specified experimental conditions and incompletely understood physical theory. The mathematical techniques that have been developed in order to make gauge theories tractable have found many other applications, from solid-state physics and crystallography to low-dimensional topology.

Classical gauge theory

Classical electromagnetism

Historically, the first example of gauge symmetry to be discovered was classical electromagnetism. In static electricity, one can either discuss the electric field, $\mathbf{E}$, or its corresponding electric potential, V . Knowledge of one makes it possible to find the other, except that potentials differing by a constant, $V \rightarrow V + C$, correspond to the same electric field. This is because the electric field relates to *changes* in the potential from one point in space to another, and the constant C would cancel out when subtracting to find the change in potential. In terms of vector calculus, the electric field is the gradient of the potential, $\mathbf{E} = -\nabla V$. Generalizing from static electricity to electromagnetism, we have a second potential, the vector potential $\mathbf{A}$, with

$$\begin{aligned}\mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A} .\end{aligned}$$

The general gauge transformations now become not just $V \rightarrow V + C$ but

$$\begin{aligned}\mathbf{A} &\rightarrow \mathbf{A} + \nabla f \\ V &\rightarrow V - \frac{\partial f}{\partial t} ,\end{aligned}$$

where f is any function that depends on position and time. The fields remain the same under the gauge transformation, and therefore Maxwell's equations are still satisfied. That is, Maxwell's equations have a gauge symmetry.

An example: Scalar $O(n)$ gauge theory

The remainder of this section requires some familiarity with classical or quantum field theory, and the use of Lagrangians.

Definitions in this section: gauge group, gauge field, interaction Lagrangian, gauge boson.

The following illustrates how local gauge invariance can be "motivated" heuristically starting from global symmetry properties, and how it leads to an interaction between fields which were originally non-interacting.

Consider a set of n non-interacting scalar fields, with equal masses m . This system is described by an action which is the sum of the (usual) action for each scalar field φ_i

$$\mathcal{S} = \int d^4x \sum_{i=1}^n \left[\frac{1}{2} \partial_\mu \varphi_i \partial^\mu \varphi_i - \frac{1}{2} m^2 \varphi_i^2 \right] .$$

The Lagrangian (density) can be compactly written as

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi)^T \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^T \Phi$$

by introducing a vector of fields

$$\Phi = (\varphi_1, \varphi_2, \dots, \varphi_n)^T.$$

The term ∂_μ is Einstein notation for the partial derivative of Φ in each of the four dimensions. It is now transparent that the Lagrangian is invariant under the transformation

$$\Phi \mapsto \Phi' = G\Phi$$

whenever G is a *constant* matrix belonging to the n -by- n orthogonal group $O(n)$. This is seen to preserve the Lagrangian since the derivative of Φ will transform identically to Φ and both quantities appear inside dot products in the Lagrangian (orthogonal transformations preserve the dot product).

$$(\partial_\mu \Phi) \mapsto (\partial_\mu \Phi)' = G\partial_\mu \Phi$$

This characterizes the *global* symmetry of this particular Lagrangian, and the symmetry group is often called the **gauge group**; the mathematical term is **structure group**, especially in the theory of G -structures. Incidentally, Noether's theorem implies that invariance under this group of transformations leads to the conservation of the *current*

$$J_\mu^a = i\partial_\mu \Phi^T T^a \Phi$$

where the T^a matrices are generators of the $SO(n)$ group. There is one conserved current for every generator.

Now, demanding that this Lagrangian should have *local* $O(n)$ -invariance requires that the G matrices (which were earlier constant) should be allowed to become functions of the space-time coordinates x .

Unfortunately, the G matrices do not "pass through" the derivatives, when $G = G(x)$,

$$\partial_\mu (G\Phi) \neq G(\partial_\mu \Phi)$$

The failure of the derivative to commute with " G " introduces an additional term (in keeping with the product rule) which spoils the invariance of the Lagrangian. In order to rectify this we define a new derivative operator such that the derivative of Φ will again transform identically with Φ

$$(D_\mu \Phi)' = G D_\mu \Phi.$$

This new "derivative" is called a covariant derivative and takes the form

$$D_\mu = \partial_\mu + gA_\mu$$

Where g is called the coupling constant – a quantity defining the strength of an interaction. After a simple calculation we can see that the **gauge field** $A(x)$ must transform as follows

$$A'_\mu = GA_\mu G^{-1} - \frac{1}{g}(\partial_\mu G)G^{-1}$$

The gauge field is an element of the Lie algebra, and can therefore be expanded as

$$A_\mu = \sum_a A_\mu^a T^a$$

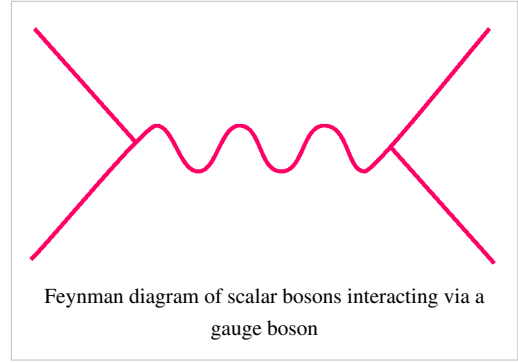
There are therefore as many gauge fields as there are generators of the Lie algebra.

Finally, we now have a *locally gauge invariant* Lagrangian

$$\mathcal{L}_{\text{loc}} = \frac{1}{2}(D_\mu \Phi)^T D^\mu \Phi - \frac{1}{2}m^2 \Phi^T \Phi.$$

Pauli calls *gauge transformation of the first type* to the one applied to fields as Φ , while the compensating transformation in A is said to be a *gauge transformation of the second type*.

The difference between this Lagrangian and the original *globally gauge-invariant* Lagrangian is seen to be the **interaction Lagrangian**



$$\mathcal{L}_{\text{int}} = \frac{g}{2} \Phi^T A_\mu^T \partial^\mu \Phi + \frac{g}{2} (\partial_\mu \Phi)^T A^\mu \Phi + \frac{g^2}{2} (A_\mu \Phi)^T A^\mu \Phi.$$

This term introduces interactions between the n scalar fields just as a consequence of the demand for local gauge invariance. However, to make this interaction physical and not completely arbitrary, the mediator $A(x)$ needs to propagate in space. That is dealt with in the next section by adding yet another term, $\mathcal{L}_{\text{gf}}$, to the Lagrangian. In the quantized version of the obtained classical field theory, the quanta of the gauge field $A(x)$ are called gauge bosons. The interpretation of the interaction Lagrangian in quantum field theory is of scalar bosons interacting by the exchange of these gauge bosons.

The Yang–Mills Lagrangian for the gauge field

The picture of a classical gauge theory developed in the previous section is almost complete, except for the fact that to define the covariant derivatives D , one needs to know the value of the gauge field $A(x)$ at all space-time points. Instead of manually specifying the values of this field, it can be given as the solution to a field equation. Further requiring that the Lagrangian which generates this field equation is locally gauge invariant as well, one possible form for the gauge field Lagrangian is (conventionally) written as

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2} \text{Tr}(F^{\mu\nu} F_{\mu\nu})$$

with

$$F_{\mu\nu} = \frac{1}{ig} [D_\mu, D_\nu]$$

and the trace being taken over the vector space of the fields. This is called the **Yang–Mills action**. Other gauge invariant actions also exist (e.g. nonlinear electrodynamics, Born–Infeld action, Chern–Simons model, theta term etc.).

Note that in this Lagrangian term there is no field whose transformation counterweighs the one of A . Invariance of this term under gauge transformations is a particular case of *a priori* classical (geometrical) symmetry. This symmetry must be restricted in order to perform quantization, the procedure being denominated gauge fixing, but even after restriction, gauge transformations may be possible.^[3]

The complete Lagrangian for the gauge theory is now

$$\mathcal{L} = \mathcal{L}_{\text{loc}} + \mathcal{L}_{\text{gf}} = \mathcal{L}_{\text{global}} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{gf}}$$

An example: Electrodynamics

As a simple application of the formalism developed in the previous sections, consider the case of electrodynamics, with only the electron field. The bare-bones action which generates the electron field's Dirac equation is

$$\mathcal{S} = \int \bar{\psi}(i\hbar c \gamma^\mu \partial_\mu - mc^2)\psi \, d^4x.$$

The global symmetry for this system is

$$\psi \mapsto e^{i\theta}\psi.$$

The gauge group here is $U(1)$, just the phase angle of the field, with a constant θ .

"Local"ising this symmetry implies the replacement of θ by $\theta(x)$.

An appropriate covariant derivative is then

$$D_\mu = \partial_\mu - i\frac{e}{\hbar}A_\mu.$$

Identifying the "charge" e with the usual electric charge (this is the origin of the usage of the term in gauge theories), and the gauge field $A(x)$ with the four-vector potential of electromagnetic field results in an interaction Lagrangian

$$\mathcal{L}_{\text{int}} = \frac{e}{\hbar}\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) = J^\mu(x)A_\mu(x).$$

where $J^\mu(x)$ is the usual four vector electric current density. The gauge principle is therefore seen to naturally introduce the so-called *minimal coupling* of the electromagnetic field to the electron field.

Adding a Lagrangian for the gauge field $A_\mu(x)$ in terms of the field strength tensor exactly as in electrodynamics, one obtains the Lagrangian which is used as the starting point in quantum electrodynamics.

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\hbar c \gamma^\mu D_\mu - mc^2)\psi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}.$$

See also: Dirac equation, Maxwell's equations, Quantum electrodynamics

Mathematical formalism

Gauge theories are usually discussed in the language of differential geometry. Mathematically, a *gauge* is just a choice of a (local) section of some principal bundle. A **gauge transformation** is just a transformation between two such sections.

Although gauge theory is dominated by the study of connections (primarily because it's mainly studied by high-energy physicists), the idea of a connection is not central to gauge theory in general. In fact, a result in general gauge theory shows that affine representations (i.e. affine modules) of the gauge transformations can be classified as sections of a jet bundle satisfying certain properties. There are representations which transform covariantly pointwise (called by physicists gauge transformations of the first kind), representations which transform as a connection form (called by physicists gauge transformations of the second kind, an affine representation) and other more general representations, such as the B field in BF theory. There are more general nonlinear representations (realizations), but are extremely complicated. Still, nonlinear sigma models transform nonlinearly, so there are applications.

If there is a principal bundle P whose base space is space or spacetime and structure group is a Lie group, then the sections of P form a principal homogeneous space of the group of gauge transformations.

connection (gauge connection) define this principal bundle, yielding a covariant derivative ∇ in each associated vector bundle. If a local frame is chosen (a local basis of sections), then this covariant derivative is represented by the connection form A , a Lie algebra-valued 1-form which is called the **gauge potential** in physics. This is evidently not an intrinsic but a frame-dependent quantity. The curvature form F is constructed from a connection form, a Lie algebra-valued 2-form which is an intrinsic quantity, by

$$\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$$

where d stands for the exterior derivative and $\wedge$ stands for the wedge product. ($\mathbf{A}$ is an element of the vector space spanned by the generators T^a , and so the components of $\mathbf{A}$ do not commute with one another. Hence the wedge product $\mathbf{A} \wedge \mathbf{A}$ does not vanish.)

Infinitesimal gauge transformations form a Lie algebra, which is characterized by a smooth Lie algebra valued scalar, ε . Under such an infinitesimal gauge transformation,

$$\delta_\varepsilon \mathbf{A} = [\varepsilon, \mathbf{A}] - d\varepsilon$$

where $[\cdot, \cdot]$ is the Lie bracket.

One nice thing is that if $\delta_\varepsilon X = \varepsilon X$, then $\delta_\varepsilon DX = \varepsilon DX$ where D is the covariant derivative

$$DX \stackrel{\text{def}}{=} dX + \mathbf{A}X.$$

Also, $\delta_\varepsilon \mathbf{F} = \varepsilon \mathbf{F}$, which means $\mathbf{F}$ transforms covariantly.

Not all gauge transformations can be generated by infinitesimal gauge transformations in general. An example is when the base manifold is a compact manifold without boundary such that the homotopy class of mappings from that manifold to the Lie group is nontrivial. See instanton for an example.

The *Yang–Mills action* is now given by

$$\frac{1}{4g^2} \int \text{Tr}[*F \wedge F]$$

where $*$ stands for the Hodge dual and the integral is defined as in differential geometry.

A quantity which is **gauge-invariant** i.e. invariant under gauge transformations is the Wilson loop, which is defined over any closed path, γ , as follows:

$$\chi^{(\rho)} \left(\mathcal{P} \left\{ e^{\int_\gamma \mathbf{A}} \right\} \right)$$

where χ is the character of a complex representation ρ and $\mathcal{P}$ represents the path-ordered operator.

Quantization of gauge theories

Gauge theories may be quantized by specialization of methods which are applicable to any quantum field theory. However, because of the subtleties imposed by the gauge constraints (see section on Mathematical formalism, above) there are many technical problems to be solved which do not arise in other field theories. At the same time, the richer structure of gauge theories allow simplification of some computations: for example Ward identities connect different renormalization constants.

Methods and aims

The first gauge theory to be quantized was quantum electrodynamics (QED). The first methods developed for this involved gauge fixing and then applying canonical quantization. The Gupta–Bleuler method was also developed to handle this problem. Non-abelian gauge theories are now handled by a variety of means. Methods for quantization are covered in the article on quantization.

The main point to quantization is to be able to compute quantum amplitudes for various processes allowed by the theory. Technically, they reduce to the computations of certain correlation functions in the vacuum state. This involves a renormalization of the theory.

When the running coupling of the theory is small enough, then all required quantities may be computed in perturbation theory. Quantization schemes intended to simplify such computations (such as canonical quantization) may be called **perturbative quantization schemes**. At present some of these methods lead to the most precise experimental tests of gauge theories.

However, in most gauge theories, there are many interesting questions which are non-perturbative. Quantization schemes suited to these problems (such as lattice gauge theory) may be called **non-perturbative quantization**

schemes. Precise computations in such schemes often require supercomputing, and are therefore less well-developed currently than other schemes.

Anomalies

Some of the symmetries of the classical theory are then seen not to hold in the quantum theory — a phenomenon called an **anomaly**. Among the most well known are:

- The scale anomaly, which gives rise to a *running coupling constant*. In QED this gives rise to the phenomenon of the Landau pole. In Quantum Chromodynamics (QCD) this leads to asymptotic freedom.
- The chiral anomaly in either chiral or vector field theories with fermions. This has close connection with topology through the notion of instantons. In QCD this anomaly causes the decay of a pion to two photons.
- The gauge anomaly, which must cancel in any consistent physical theory. In the electroweak theory this cancellation requires an equal number of quarks and leptons.

Pure gauge

A pure gauge is the set of field configurations obtained by a gauge transformation on the null field configuration. So it is a particular "gauge orbit" in the field configuration's space.

In the abelian case, where $A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu f(x)$, the pure gauge is the set of field configurations $A'_\mu(x) = \partial_\mu f(x)$ for all $f(x)$.

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List of quantum field theories

List of quantum field theories:

- Chern-Simons model
 - Chiral model
 - Gross-Neveu
 - Kondo model
 - Lower dimensional quantum field theory
 - Minimal model
 - Nambu-Jona-Lasinio
 - Noncommutative quantum field theory
 - Nonlinear sigma model
 - Phi to the fourth
 - Quantum chromodynamics
 - Quantum electrodynamics
 - Quantum flavordynamics
 - Quantum Yang-Mills theory
 - Schwinger model
 - Sine-Gordon
 - Standard model
 - String Theory
 - Thirring model
 - Toda field theory
 - Topological quantum field theory
 - Wess-Zumino model
 - Wess-Zumino-Witten model
 - Yang-Mills
 - Yang-Mills-Higgs model
 - Yukawa model
-

Noncommutative geometry

Noncommutative geometry (NCG), is a branch of mathematics concerned with geometric approach to noncommutative algebras, and with construction of *spaces* which are locally presented by noncommutative algebras of functions (possibly in some generalized sense). A noncommutative algebra is here an associative algebra in which the multiplication is not commutative, that is, for which xy does not always equal yx ; or more generally an algebraic structure in which one of the principal binary operations is not commutative; one also allows additional structures, e.g. topology or norm to be possibly carried by the noncommutative algebra of functions. The leading direction in noncommutative geometry has been laid by French mathematician Alain Connes since his involvement from about 1979.

Motivation

Main motivation is to extend the commutative duality between spaces and functions to the noncommutative setting. In mathematics, there is a close relationship between *spaces*, which are geometric in nature, and the numerical functions on them. In general, such functions will form a commutative ring. For instance, one may take the ring $C(X)$ of continuous complex-valued functions on a topological space X . In many important cases (e.g., if X is a compact Hausdorff space), we can recover X from $C(X)$, and therefore it makes some sense to say that X has *commutative geometry*.

More specifically, in topology, compact Hausdorff topological spaces can be reconstructed from the Banach algebra of functions on the space (Gel'fand-Neimark). In commutative algebraic geometry, algebraic schemes are locally prime spectra of commutative unital rings (A. Grothendieck), and schemes can be reconstructed from the categories of quasicoherent sheaves of modules on them (P. Gabriel-A. Rosenberg). For Grothendieck topologies, the cohomological properties of a site are invariant of the corresponding category of sheaves of sets viewed abstractly as a topos (A. Grothendieck). In all these cases, a space is reconstructed from the algebra of functions or its categorified version—some category of sheaves on that space.

Functions on a topological space can be multiplied and added pointwise hence they form a commutative algebra; in fact these operations are local in the topology of the base space, hence the functions form a sheaf of commutative rings over the base space.

The dream of noncommutative geometry is to generalize this duality to the duality between

- noncommutative algebras, or sheaves of noncommutative algebras, or sheaf-like noncommutative algebraic or operator-algebraic structures
- and geometric entities of certain kind,

and interact between the algebraic and geometric description of those via this duality.

Regarding that the commutative rings correspond to usual affine schemes, and commutative C^* -algebras to usual topological spaces, the extension to noncommutative rings and algebras requires non-trivial generalization of topological spaces, as "non-commutative spaces". For this reason, some talk about non-commutative topology, though the term has also other meanings.

Applications in mathematical physics

Some applications in particle physics are described on the entries Noncommutative standard model and Noncommutative quantum field theory. Sudden rise in interest in noncommutative geometry in physics, follows after the speculations of its role in M-theory made in 1997^[1].

Motivation from ergodic theory

Some of the theory developed by Alain Connes to handle noncommutative geometry at a technical level has roots in older attempts, in particular in ergodic theory. The proposal of George Mackey to create a *virtual subgroup* theory, with respect to which ergodic group actions would become homogeneous spaces of an extended kind, has by now been subsumed.

Non-commutative C*-algebras, von Neumann algebras

(The formal duals of) non-commutative C*-algebras are often now called non-commutative spaces. This is by analogy with the Gelfand representation, which shows that commutative C*-algebras are dual to locally compact Hausdorff spaces. In general, one can associate to any C*-algebra S a topological space $\hat{S}$; see spectrum of a C*-algebra.

For the duality between σ -finite measure spaces and commutative von Neumann algebras, noncommutative von Neumann algebras are called *non-commutative measure spaces*.

Non-commutative differentiable manifolds

A smooth Riemannian manifold M is a topological space with a lot of extra structure. From its algebra of continuous functions $C(M)$ we only recover M topologically. The algebraic invariant that recovers the Riemannian structure is a spectral triple. It is constructed from a smooth vector bundle E over M , e.g. the exterior algebra bundle. The Hilbert space $L^2(M, E)$ of square integrable sections of E carries a representation of $C(M)$ by multiplication operators, and we consider an unbounded operator D in $L^2(M, E)$ with compact resolvent (e.g. the signature operator), such that the commutators $[D, f]$ are bounded whenever f is smooth. A recent deep theorem states that M as a Riemannian manifold can be recovered from this data.

This suggests that one might define a noncommutative Riemannian manifold as a spectral triple (A, H, D) , consisting of a representation of a C*-algebra A on a Hilbert space H , together with an unbounded operator D on H , with compact resolvent, such that $[D, a]$ is bounded for all a in some dense subalgebra of A . Research in spectral triples is very active, and many examples of noncommutative manifolds have been constructed.

Non-commutative affine and projective schemes

In analogy to the duality between affine schemes and commutative rings, we define a category of **noncommutative affine schemes** as the dual of the category of associative unital rings. There are certain analogues of Zariski topology in that context so that one can glue such affine schemes to more general objects.

There are also generalizations of the Cone and of the Proj of a commutative graded ring, mimicking a Serre's theorem on Proj. Namely the category of quasicoherent sheaves of $\mathcal{O}$ -modules on a Proj of a commutative graded algebra is equivalent to the category of graded modules over the ring localized on Serre's subcategory of graded modules of finite length; there is also analogous theorem for coherent sheaves when the algebra is Noetherian. This theorem is extended as a definition of **noncommutative projective geometry** by Michael Artin and J. J. Zhang^[2], who add also some general ring-theoretic conditions (e.g. Artin-Schelter regularity).

Many properties of projective schemes extend to this context. For example, there exist an analog of the celebrated Serre duality for noncommutative projective schemes of Artin and Zhang^[3].

A. L. Rosenberg has created a rather general relative concept of **noncommutative quasicompact scheme** (over a base category), abstracting the Grothendieck's study of morphisms of schemes and covers in terms of categories of quasicoherent sheaves and flat localization functors^[4]. There is also another interesting approach via localization theory, due to Fred Van Oystaeyen, Luc Willaert and Alain Verschoeren, where the main concept is that of a **schematic algebra**^[5].

Invariants for noncommutative spaces

Some of the motivating questions of the theory are concerned with extending known topological invariants to formal duals of noncommutative (operator) algebras and other replacements and candidates for noncommutative spaces. One of the main starting points of the Alain Connes' direction in noncommutative geometry is his spectacular discovery (and independently by Boris Tsygan) of a very important new homology theory associated to noncommutative associative algebras and noncommutative operator algebras, namely the cyclic homology and its relations to the algebraic K-theory (primarily via Connes-Chern character map).

The theory of characteristic classes of smooth manifolds has been extended to spectral triples, employing the tools of operator K-theory and cyclic cohomology. Several generalizations of now classical index theorems allow for effective extraction of numerical invariants from spectral triples. The fundamental characteristic class in cyclic cohomology, the JLO cocycle, generalizes the classical Chern character.

Examples of non-commutative spaces

- In Weyl quantization, the symplectic phase space of classical mechanics is deformed into a non-commutative phase space generated by the position and momentum operators.
- The standard model of particle physics is another example of a noncommutative geometry, cf noncommutative standard model.
- The noncommutative torus, deformation of the function algebra of the ordinary torus, can be given the structure of a spectral triple. This class of examples has been studied intensively and still functions as a test case for more complicated situations.
- Snyder space^[6]
- Noncommutative algebras arising from foliations.
- Examples related to dynamical systems arising from number theory, such as the Gauss shift on continued fractions, give rise to noncommutative algebras that appear to have interesting noncommutative geometries.

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- Lectures on Noncommutative Geometry (<http://arxiv.org/abs/math/0506603>) by Victor Ginzburg
- Very Basic Noncommutative Geometry (<http://arxiv.org/abs/math/0408416>) by Masoud Khalkhali
- Lectures on Arithmetic Noncommutative Geometry (<http://arxiv.org/abs/math.qa/0409520>) by Matilde Marcolli
- Noncommutative Geometry for Pedestrians (<http://arxiv.org/abs/gr-qc/9906059>) by J. Madore
- An informal introduction to the ideas and concepts of noncommutative geometry (<http://arxiv.org/abs/math-ph/0612012>) by Thierry Masson (an easier introduction that is still rather technical)
- Noncommutative geometry on arxiv.org (<http://xstructure.inr.ac.ru/x-bin/subthemes3.py?level=2&index1=-173391&skip=0>)
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Quantum gravity

Quantum gravity (QG) is the field of theoretical physics attempting to unify quantum mechanics with general relativity in a self-consistent manner, or more precisely, to formulate a self-consistent theory which reduces to ordinary quantum mechanics in the limit of weak gravity (potentials much less than c^2) and which reduces to Einsteinian general relativity in the limit of large actions (action much larger than reduced Planck's constant). The theory must be able to predict the outcome of situations where both quantum effects and strong-field gravity are important (at the Planck scale, unless large extra dimension conjectures are correct). Motivation for quantizing gravity comes from the remarkable success of the quantum theories of the other three fundamental interactions. Although some quantum gravity theories such as string theory and other so-called theories of everything attempt to unify gravity with the other fundamental forces, others such as loop quantum gravity make no such attempt; they simply quantize the gravitational field while keeping it separate from the other forces.

Observed physical phenomena in the early 21st century can be described well by quantum mechanics or general relativity, without needing both. This can be thought of as due to an extreme separation of mass scales at which they are important. Quantum effects are usually important only for the "very small", that is, for objects no larger than typical molecules. General relativistic effects, on the other hand, show up only for the "very large" bodies such as collapsed stars. (Planets' gravitational fields, as of 2009, are well-described by linearized gravity; so strong-field effects—any effects of gravity beyond lowest nonvanishing order in φ/c^2 —have not been observed even in the gravitational fields of planets and main sequence stars). There is a lack of experimental evidence relating to quantum gravity and classical physics adequately describes the observed effects of gravity over a range of 50 orders of magnitude of mass, i.e. for masses of objects from about 10^{-23} to 10^{30} kg.

Overview

Much of the difficulty in meshing these theories at all energy scales comes from the different assumptions that these theories make on how the universe works. Quantum field theory depends on particle fields embedded in the flat space-time of special relativity. General relativity models gravity as a curvature within space-time that changes as a gravitational mass moves. Historically, the most obvious way of combining the two (such as treating gravity as simply another particle field) ran quickly into what is known as the renormalization problem. In the old-fashioned understanding of renormalization, gravity particles would attract each other and adding together all of the interactions results in many infinite values which cannot easily be cancelled out mathematically to yield sensible, finite results. This is in contrast with quantum electrodynamics where, while the series still do not converge, the interactions sometimes evaluate to infinite results, but those are few enough in number to be removable via renormalization.

Effective field theories

Quantum gravity can be treated as an effective field theory. Effective quantum field theories come with some high-energy cutoff, beyond which we do not expect that the theory provides a good description of nature. The "infinities" then become large but finite quantities proportional to this finite cutoff scale, and correspond to processes that involve very high energies near the fundamental cutoff. These quantities can then be absorbed into an infinite collection of coupling constants, and at energies well below the fundamental cutoff of the theory, to any desired precision; only a finite number of these coupling constants need to be measured in order to make legitimate quantum-mechanical predictions. This same logic works just as well for the highly successful theory of low-energy pions as for quantum gravity. Indeed, the first quantum-mechanical corrections to graviton-scattering and Newton's law of gravitation have been explicitly computed^[1] (although they are so astronomically small that we may never be able to measure them). In fact, gravity is in many ways a much better quantum field theory than the Standard Model, since it appears to be valid all the way up to its cutoff at the Planck scale. (By comparison, the Standard Model is

expected to start to break down above its cutoff at the much smaller scale of around 1000 GeV.)

While confirming that quantum mechanics and gravity are indeed consistent at reasonable energies, it is clear that near or above the fundamental cutoff of our effective quantum theory of gravity (the cutoff is generally assumed to be of order the Planck scale), a new model of nature will be needed. Specifically, the problem of combining quantum mechanics and gravity becomes an issue only at very high energies, and may well require a totally new kind of model.

Quantum gravity theory for the highest energy scales

The general approach to deriving a quantum gravity theory that is valid at even the highest energy scales is to assume that such a theory will be simple and elegant and, accordingly, to study symmetries and other clues offered by current theories that might suggest ways to combine them into a comprehensive, unified theory. One problem with this approach is that it is unknown whether quantum gravity will actually conform to a simple and elegant theory, as it should resolve the dual conundrums of special relativity with regard to the uniformity of acceleration and gravity, and general relativity with regard to spacetime curvature.

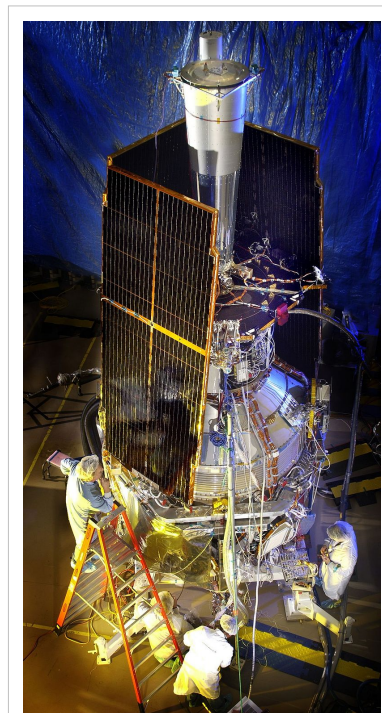
Such a theory is required in order to understand problems involving the combination of very high energy and very small dimensions of space, such as the behavior of black holes, and the origin of the universe.

Quantum mechanics and general relativity

The graviton

At present, one of the deepest problems in theoretical physics is harmonizing the theory of general relativity, which describes gravitation, and applies to large-scale structures (stars, planets, galaxies), with quantum mechanics, which describes the other three fundamental forces acting on the atomic scale. This problem must be put in the proper context, however. In particular, contrary to the popular claim that quantum mechanics and general relativity are fundamentally incompatible, one can demonstrate that the structure of general relativity essentially follows inevitably from the quantum mechanics of interacting theoretical spin-2 massless particles ^{[2] [3] [4] [5] [6]} (called gravitons).

While there is no concrete proof of the existence of gravitons, quantized theories of matter may necessitate their existence. Supporting this theory is the observation that all other fundamental forces have one or more messenger particles, *except* gravity, leading researchers to believe that at least one most likely does exist; they have dubbed these hypothetical particles *gravitons*. Many of the accepted notions of a unified theory of physics since the 1970s, including string theory, superstring theory, M-theory, loop quantum gravity, all assume, and to some degree depend upon, the existence of the graviton. Many researchers view the detection of the graviton as vital to validating their work.



Gravity Probe B (GP-B) has measured spacetime curvature near Earth to test related models in application of Einstein's general theory of relativity.

The dilaton

The dilaton made its first appearance in Kaluza-Klein theory, a five-dimensional theory that combined gravitation and electromagnetism. Generally, it appears in string theory. More recently, it has appeared in the lower-dimensional many-bodied gravity problem^[7] based on the field theoretic approach of Roman Jackiw. The impetus arose from the fact that complete analytical solutions for the metric of a covariant N -body system have proven elusive in General Relativity. To simplify the problem, the number of dimensions was lowered to $(1+1)$ namely one spatial dimension and one temporal dimension. This model problem, known as $R=T$ theory^[8] (as opposed to the general $G=T$ theory) was amenable to exact solutions in terms of a generalization of the Lambert W function. It was also found that the field equation governing the dilaton (derived from differential geometry) was none other than the Schrödinger equation and consequently amenable to quantization.^[9] Thus, one had a theory which combined gravity, quantization and even the electromagnetic interaction, promising ingredients of a fundamental physical theory. It is worth noting that the outcome revealed a previously unknown and already existing *natural link* between general relativity and quantum mechanics. However, this theory needs to be generalized in $(2+1)$ or $(3+1)$ dimensions although, in principle, the field equations are amenable to such generalization. It is not yet clear what field equation will govern the dilaton in higher dimensions. This is further complicated by the fact that gravitons can propagate in $(3+1)$ dimensions and consequently that would imply gravitons and dilatons exist in the real world. Moreover, detection of the dilaton is expected to be even more elusive than the graviton. However, since this approach allows for the combination of gravitational, electromagnetic and quantum effects, their coupling could potentially lead to a means of vindicating the theory, through cosmology and perhaps even *experimentally*.

Nonrenormalizability of gravity

General relativity, like electromagnetism, is a classical field theory. One might expect that, as with electromagnetism, there should be a corresponding quantum field theory.

However, gravity is nonrenormalizable.^[10] For a quantum field theory to be well-defined according to this understanding of the subject, it must be asymptotically free or asymptotically safe. The theory must be characterized by a choice of *finitely many* parameters, which could, in principle, be set by experiment. For example, in quantum electrodynamics, these parameters are the charge and mass of the electron, as measured at a particular energy scale.

On the other hand, in quantizing gravity, there are *infinitely many independent parameters* needed to define the theory. For a given choice of those parameters, one could make sense of the theory, but since we can never do infinitely many experiments to fix the values of every parameter, we do not have a meaningful physical theory:

- At low energies, the logic of the renormalization group tells us that, despite the unknown choices of these infinitely many parameters, quantum gravity will reduce to the usual Einstein theory of general relativity.
- On the other hand, if we could probe very high energies where quantum effects take over, then *every one* of the infinitely many unknown parameters would begin to matter, and we could make no predictions at all.

As explained below, there is a way around this problem by treating QG as an effective field theory.

Any meaningful theory of quantum gravity that makes sense and is predictive at all energy scales must have some deep principle that reduces the infinitely many unknown parameters to a finite number that can then be measured.

- One possibility is that normal perturbation theory is not a reliable guide to the renormalizability of the theory, and that there really *is* a UV fixed point for gravity. Since this is a question of non-perturbative quantum field theory, it is difficult to find a reliable answer, but some people still pursue this option.
- Another possibility is that there are new symmetry principles that constrain the parameters and reduce them to a finite set. This is the route taken by string theory, where all of the excitations of the string essentially manifest themselves as new symmetries.

QG as an effective field theory

In an effective field theory, all but the first few of the infinite set of parameters in a non-renormalizable theory are suppressed by huge energy scales and hence can be neglected when computing low-energy effects. Thus, at least in the low-energy regime, the model is indeed a predictive quantum field theory.^[1] (A very similar situation occurs for the very similar effective field theory of low-energy pions.) Furthermore, many theorists agree that even the Standard Model should really be regarded as an effective field theory as well, with "nonrenormalizable" interactions suppressed by large energy scales and whose effects have consequently not been observed experimentally.

Recent work^[1] has shown that by treating general relativity as an effective field theory, one can actually make legitimate predictions for quantum gravity, at least for low-energy phenomena. An example is the well-known calculation of the tiny first-order quantum-mechanical correction to the classical Newtonian gravitational potential between two masses.

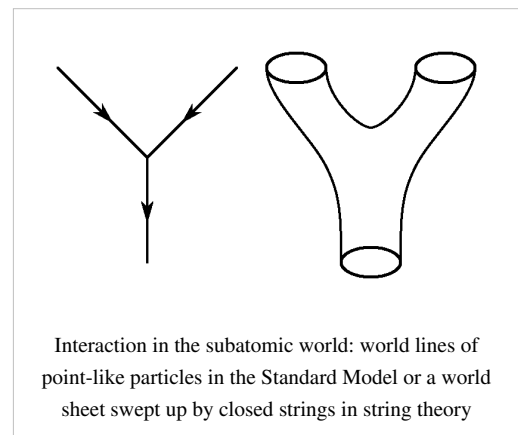
Spacetime background dependence

A fundamental lesson of general relativity is that there is no fixed spacetime background, as found in Newtonian mechanics and special relativity; the spacetime geometry is dynamic. While easy to grasp in principle, this is the hardest idea to understand about general relativity, and its consequences are profound and not fully explored, even at the classical level. To a certain extent, general relativity can be seen to be a relational theory,^[1] in which the only physically relevant information is the relationship between different events in space-time.

On the other hand, quantum mechanics has depended since its inception on a fixed background (non-dynamic) structure. In the case of quantum mechanics, it is time that is given and not dynamic, just as in Newtonian classical mechanics. In relativistic quantum field theory, just as in classical field theory, Minkowski spacetime is the fixed background of the theory.

String theory

String theory started out as a generalization of quantum field theory where instead of point particles, string-like objects propagate in a fixed spacetime background. Although string theory had its origins in the study of quark confinement and not of quantum gravity, it was soon discovered that the string spectrum contains the graviton, and that "condensation" of certain vibration modes of strings is equivalent to a modification of the original background. In this sense, string perturbation theory exhibits exactly the features one would expect of a perturbation theory that may exhibit a strong dependence on asymptotics (as seen, for example, in the AdS/CFT correspondence) which is a weak form of background dependence.



Background independent theories

Loop quantum gravity is the fruit of an effort to formulate a background-independent quantum theory.

Topological quantum field theory provided an example of background-independent quantum theory, but with no local degrees of freedom, and only finitely many degrees of freedom globally. This is inadequate to describe gravity in 3+1 dimensions which has local degrees of freedom according to general relativity. In 2+1 dimensions, however, gravity is a topological field theory, and it has been successfully quantized in several different ways, including spin networks.

Semi-classical quantum gravity

Quantum field theory on curved (non-Minkowskian) backgrounds, while not a full quantum theory of gravity, has shown many promising early results. In an analogous way to the development of quantum electrodynamics in the early part of the 20th century (when physicists considered quantum mechanics in classical electromagnetic fields), the consideration of quantum field theory on a curved background has led to predictions such as black hole radiation.

Phenomena such as the Unruh effect, in which particles exist in certain accelerating frames but not in stationary ones, do not pose any difficulty when considered on a curved background (the Unruh effect occurs even in flat Minkowskian backgrounds). The vacuum state is the state with least energy (and may or may not contain particles). See Quantum field theory in curved spacetime for a more complete discussion.

Points of tension

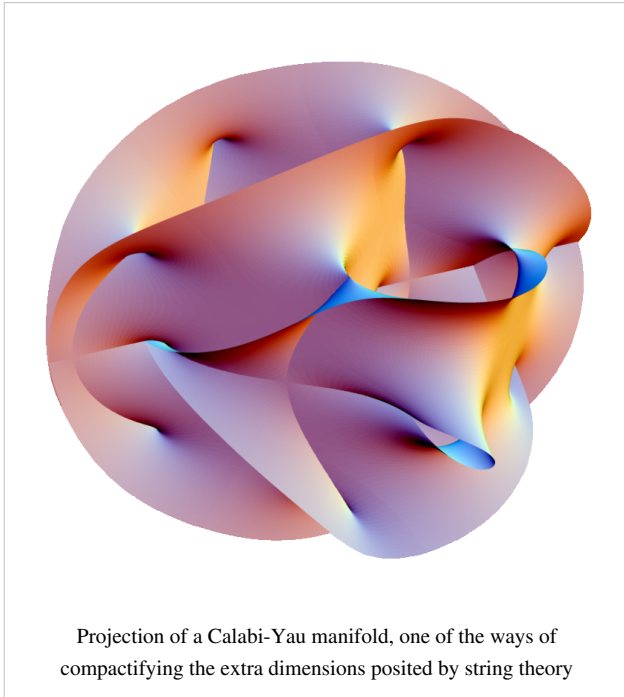
There are other points of tension between quantum mechanics and general relativity.

- First, classical general relativity breaks down at singularities, and quantum mechanics becomes inconsistent with general relativity in the neighborhood of singularities (however, no one is certain that classical general relativity applies near singularities in the first place).
- Second, it is not clear how to determine the gravitational field of a particle, since under the Heisenberg uncertainty principle of quantum mechanics its location and velocity cannot be known with certainty. The resolution of these points may come from a better understanding of general relativity.^[12]
- Third, there is the Problem of Time in quantum gravity. Time has a different meaning in quantum mechanics and general relativity and hence there are subtle issues to resolve when trying to formulate a theory which combines the two.^[13]

Candidate theories

There are a number of proposed quantum gravity theories.^[14] Currently, there is still no complete and consistent quantum theory of gravity, and the candidate models still need to overcome major formal and conceptual problems. They also face the common problem that, as yet, there is no way to put quantum gravity predictions to experimental tests, although there is hope for this to change as future data from cosmological observations and particle physics experiments becomes available.^{[15] [16]}

String theory



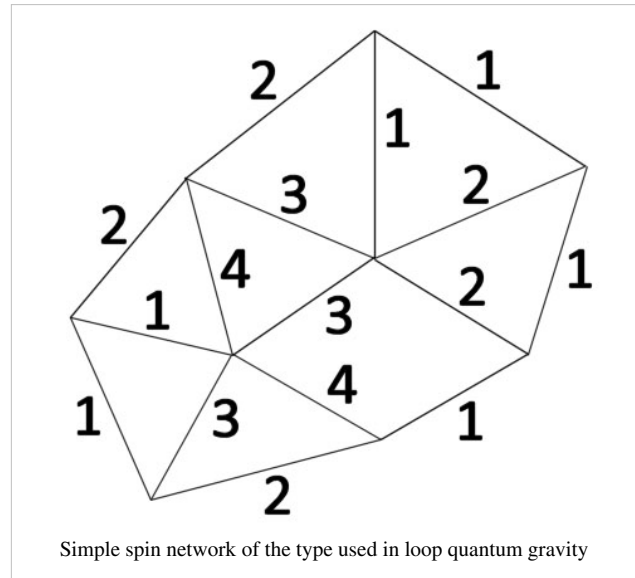
One suggested starting point is ordinary quantum field theories which, after all, are successful in describing the other three basic fundamental forces in the context of the standard model of elementary particle physics. However, while this leads to an acceptable effective (quantum) field theory of gravity at low energies,^[17] gravity turns out to be much more problematic at higher energies. Where, for ordinary field theories such as quantum electrodynamics, a technique known as renormalization is an integral part of deriving predictions which take into account higher-energy contributions,^[18] gravity turns out to be nonrenormalizable: at high energies, applying the recipes of ordinary quantum field theory yields models that are devoid of all predictive power.^[19]

One attempt to overcome these limitations is to replace ordinary quantum field theory, which is based on the classical concept of a point particle, with a quantum theory of one-dimensional extended objects: string theory.^[20] At the energies reached in current experiments, these strings are indistinguishable from point-like particles, but, crucially, different modes of oscillation of one and the same type of fundamental string appear as particles with different (electric and other) charges. In this way, string theory promises to be a unified description of all particles and interactions.^[21] The theory is successful in that one mode will always correspond to a graviton, the messenger particle of gravity; however, the price to pay are unusual features such as six extra dimensions of space in addition to the usual three for space and one for time.^[22]

In what is called the second superstring revolution, it was conjectured that both string theory and a unification of general relativity and supersymmetry known as supergravity^[23] form part of a hypothesized eleven-dimensional model known as M-theory, which would constitute a uniquely defined and consistent theory of quantum gravity.^[24] ^[25] As presently understood, however, string theory is consistent with an estimated 10^{500} equally possible solutions (the so-called "string landscape").

Loop quantum gravity

Another approach to quantum gravity starts with the canonical quantization procedures of quantum theory. Starting with the initial-value-formulation of general relativity (cf. the section on evolution equations, above), the result is an analogue of the Schrödinger equation: the Wheeler–DeWitt equation, which some argue is ill-defined.^[26] A major break-through came with the introduction of what are now known as Ashtekar variables, which represent geometric gravity using mathematical analogues of electric and magnetic fields.^{[27] [28]} The resulting candidate for a theory of quantum gravity is Loop quantum gravity, in which space is represented by a network structure called a spin network, evolving over time in discrete steps.^{[29] [30] [31] [32]}



Other approaches

There are a number of other approaches to quantum gravity. The approaches differ depending on which features of general relativity and quantum theory are accepted unchanged, and which features are modified.^{[33] [34]} Examples include:

- Supergravity
- Path-integral based models of quantum cosmology^[35]
- Causal Dynamical Triangulation^[36]
- Regge calculus
- Causal sets^[37]
- Asymptotic safety
- Twistor models^[38]
- Noncommutative geometry.
- String-nets giving rise to gapless helicity ± 2 excitations with no other gapless excitations^[39]
- Acoustic metric and other analog models of gravity
- MacDowell–Mansouri action
- Group field theory^[40]

Weinberg-Witten theorem

In quantum field theory, the Weinberg-Witten theorem places some constraints on theories of composite gravity/emergent gravity. However, recent developments attempt to show that if locality is only approximate and the holographic principle is correct, the Weinberg-Witten theorem would not be valid.

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External links

- Quantum gravity on Scholarpedia (http://www.scholarpedia.org/article/Quantum_gravity)
- Quantum Gravity (http://www.perimeterinstitute.ca/Outreach/What_We_Research/Quantum_Gravity/) Perimeter Institute for Theoretical Physics
- Prima facie questions in quantum gravity (<http://arxiv.org/abs/gr-qc/9310031>) Chris J. Isham
- Quantum Gravity (<http://plato.stanford.edu/entries/quantum-gravity/>) in Stanford Encyclopedia of Philosophy (<http://plato.stanford.edu>)
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- Quantum gravity articles (<http://xstructure.inr.ac.ru/x-bin/subthemes3.py?level=3&index1=-12474&skip=0>)
- Quantum Gravity pages by Lee Smolin (<http://www.qgravity.org/>) (no longer active, archived at <http://web.archive.org/web/20060428001322/http://www.qgravity.org/>)

String Theories

String theory is a developing theory in particle physics that attempts to reconcile quantum mechanics and general relativity.^[1] It is a contender for the theory of everything (TOE), a manner of describing the known fundamental forces and matter in a mathematically complete system. The theory has yet to make testable experimental predictions, which a theory must do in order to be considered a part of science.

String theory mainly posits that the electrons and quarks within an atom are not 0-dimensional objects, but rather 1-dimensional oscillating lines ("strings"). The earliest string model, the bosonic string, incorporated only bosons, although this view developed to the superstring theory, which posits that a connection (a "supersymmetry") exists between bosons and fermions. String theories also require the existence of several extra, unobservable, dimensions to the universe, in addition to the usual four spacetime dimensions.

The theory has its origins in the dual resonance model (1969). Since that time, the term *string theory* has developed to incorporate any of a group of related superstring theories. Five major string theories were formulated. The main differences between each of them were the number of dimensions in which the strings developed and their characteristics, however all appeared to be correct. In the mid 1990s a unification of all previous superstring theories, called M-theory, was proposed, which asserted that strings are really 1-dimensional slices of a 2-dimensional membrane vibrating in 11-dimensional space.

As a result of the many properties and principles shared by these approaches (such as the holographic principle), their mutual logical consistency, and the fact that some easily include the standard model of particle physics, some mathematical physicists (e.g. Witten, Maldacena and Susskind) believe that string theory is a step towards the correct fundamental description of nature.^{[2] [3] [4] [5]} Nevertheless, other prominent physicists (e.g. Feynman and Glashow) have criticized string theory for not providing any quantitative experimental predictions.^{[6] [7]}

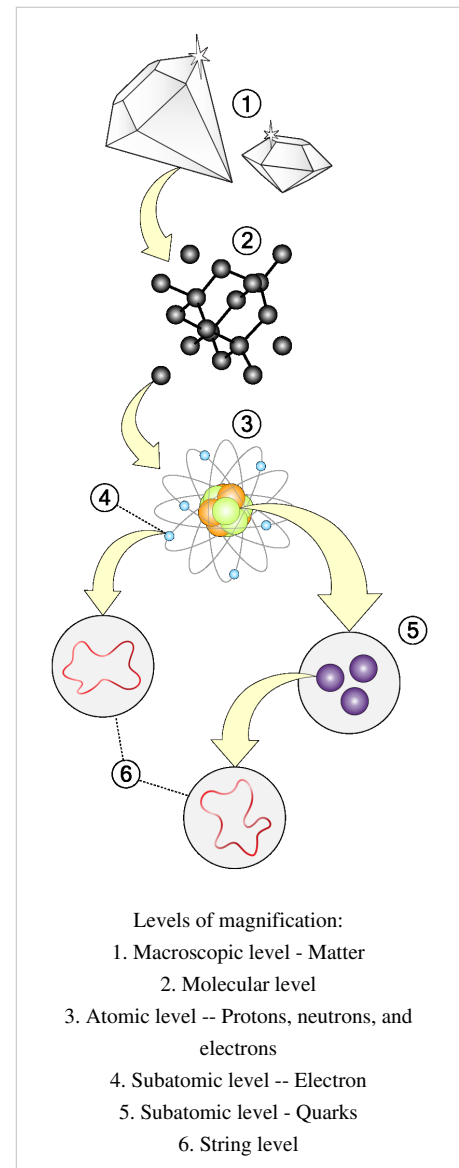
Overview

String theory posits that the electrons and quarks within an atom are not 0-dimensional objects, but 1-dimensional strings. These strings can move and vibrate, giving the observed particles their flavor, charge, mass and spin. String theories also include objects more general than strings, called branes. The word *brane*, derived from "membrane", refers to a variety of interrelated objects, such as D-branes, black p-branes and Neveu-Schwarz 5-branes. These are extended objects that are charged sources for differential form generalizations of the vector potential electromagnetic field. These objects are related to one another by a variety of dualities. Black hole-like black p-branes are identified with D-branes, which are endpoints for strings, and this identification is called Gauge-gravity duality. Research on this equivalence has led to new insights on quantum chromodynamics, the fundamental theory of the strong nuclear force.^{[8] [9] [10] [11]} The strings make closed loops unless they encounter D-branes, where they can open up into 1-dimensional lines. The endpoints of the string cannot break off the D-brane, but they can slide around on it.

Since the string theory is widely believed to be a consistent theory of quantum gravity, many hope that it correctly describes our universe, making it a theory of everything. There are known configurations which describe all the observed fundamental forces and matter but with a zero cosmological constant and some new fields.^[12] There are other configurations with different values of the cosmological constant, which are metastable but long-lived. This leads many to believe that there is at least one metastable solution which is quantitatively identical with the standard model, with a small cosmological constant, which contains dark matter and a plausible mechanism for cosmic inflation. It is not yet known whether string theory has such a solution, nor how much freedom the theory allows to choose the details.

The full theory does not yet have a satisfactory definition in all circumstances, since the scattering of strings is most straightforwardly defined by a perturbation theory. The complete quantum mechanics of high dimensional branes is not easily defined, and the behavior of string theory in cosmological settings (time-dependent backgrounds) is not fully worked out. It is also not clear if there is any principle by which string theory selects its vacuum state, the spacetime configuration which determines the properties of our universe (see string theory landscape).

Like any other quantum theory of gravity, it is widely believed that testing the theory directly would require prohibitively expensive feats of engineering. Although direct experimental testing of string theory involves grand explorations and development in engineering, there are several indirect experiments that may prove partial truth to string theory.



Basic properties

String theory can be formulated in terms of an action principle, either the Nambu-Goto action or the Polyakov action, which describes how strings move through space and time. In the absence of external interactions, string dynamics are governed by tension and kinetic energy, which combine to produce oscillations. The quantum mechanics of strings implies these oscillations take on discrete vibrational modes, the spectrum of the theory.

On distance scales larger than the string radius, each oscillation mode behaves as a different species of particle, with its mass, spin and charge determined by the string's dynamics. Splitting and recombination of strings correspond to particle emission and absorption, giving rise to the interactions between particles.

An analogy for strings' modes of vibration is a guitar string's production of multiple but distinct musical notes. In the analogy, different notes correspond to different particles. The only difference is the guitar is only 2-dimensional; you can strum it up, and down. In actuality the guitar strings would be every dimension, and the strings could vibrate in any direction, meaning that the particles could move through not only our dimension, but other dimensions as well.

String theory includes both *open* strings, which have two distinct endpoints, and *closed* strings making a complete loop. The two types of string behave in slightly different ways, yielding two different spectra. For example, in most

string theories, one of the closed string modes is the graviton, and one of the open string modes is the photon. Because the two ends of an open string can always meet and connect, forming a closed string, there are no string theories without closed strings.

The earliest string model, the bosonic string, incorporated only bosons. This model describes, in low enough energies, a quantum gravity theory, which also includes (if open strings are incorporated as well) gauge fields such as the photon (or, more generally, any gauge theory). However, this model has problems. Most importantly, the theory has a fundamental instability, believed to result in the decay (at least partially) of spacetime itself. Additionally, as the name implies, the spectrum of particles contains only bosons, particles which, like the photon, obey particular rules of behavior. Roughly speaking, bosons are the constituents of radiation, but not of matter, which is made of fermions. Investigating how a string theory may include fermions in its spectrum led to the invention of supersymmetry, a mathematical relation between bosons and fermions. String theories which include fermionic vibrations are now known as superstring theories; several different kinds have been described, but all are now thought to be different limits of M-theory.

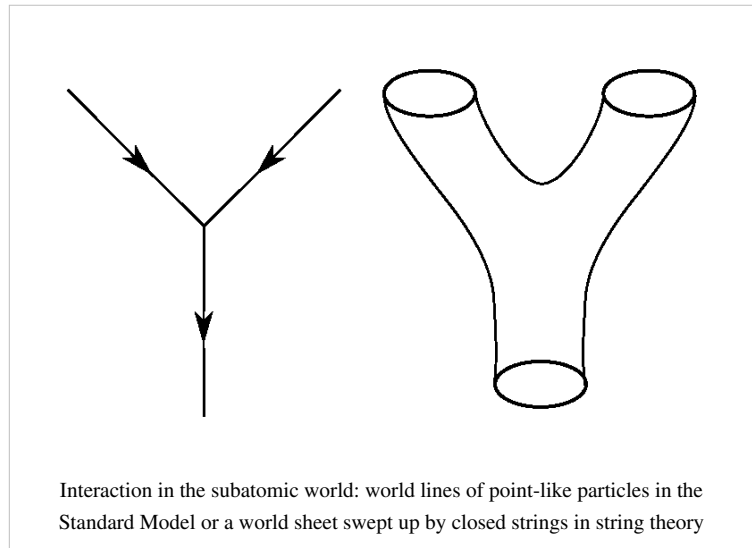
Some qualitative properties of quantum strings can be understood in a fairly simple fashion. For example, quantum strings have tension, much like regular strings made of twine; this tension is considered a fundamental parameter of the theory. The tension of a quantum string is closely related to its size. Consider a closed loop of string, left to move through space without external forces. Its tension will tend to contract it into a smaller and smaller loop. Classical intuition suggests that it might shrink to a single point, but this would violate Heisenberg's uncertainty principle. The characteristic size of the string loop will be a balance between the tension force, acting to make it small, and the uncertainty effect, which keeps it "stretched". Consequently, the minimum size of a string is related to the string tension.

World-sheet

A point-like particle's motion may be described by drawing a graph of its position (in one or two dimensions of space) against time. The resulting picture depicts the worldline of the particle (its 'history') in spacetime. By analogy, a similar graph depicting the progress of a *string* as time passes by can be obtained; the string (a one-dimensional object — a small line — by itself) will trace out a surface (a two-dimensional manifold), known as the worldsheet. The different string modes (representing different particles, such as photon or graviton) are surface waves on this manifold.

A closed string looks like a small loop, so its worldsheet will look like a pipe or, more generally, a Riemann surface (a two-dimensional oriented manifold) with no boundaries (i.e. no edge). An open string looks like a short line, so its worldsheet will look like a strip or, more generally, a Riemann surface with a boundary.

Strings can split and connect. This is reflected by the form of their worldsheet (more accurately, by its topology). For example, if a closed string splits, its worldsheet will look like a single pipe splitting (or connected) to two pipes (often referred to as a *pair of pants* — see drawing at right). If a closed string splits and its two parts later reconnect, its worldsheet will look like a single pipe splitting to two and then reconnecting, which also looks like a torus connected to two pipes (one representing the ingoing string, and the other — the outgoing one). An open string doing the same thing will have its worldsheet looking like a ring connected to two strips.



Note that the process of a string splitting (or strings connecting) is a global process of the worldsheet, not a local one: locally, the worldsheet looks the same everywhere and it is not possible to determine a single point on the worldsheet where the splitting occurs. Therefore these processes are an integral part of the theory, and are described by the same dynamics that controls the string modes.

In some string theories (namely, closed strings in Type I and some versions of the bosonic string), strings can split and reconnect in an opposite orientation (as in a Möbius strip or a Klein bottle). These theories are called *unoriented*. Formally, the worldsheet in these theories is a non-orientable surface.

Dualities

Before the 1990s, string theorists believed there were five distinct superstring theories: open type I, closed type I, closed type IIA, closed type IIB, and the two flavors of heterotic string theory ($SO(32)$ and $E_8 \times E_8$).^[13] The thinking was that out of these five candidate theories, only one was the actual correct theory of everything, and that theory was the one whose low energy limit, with ten spacetime dimensions compactified down to four, matched the physics observed in our world today. It is now believed that this picture was incorrect and that the five superstring theories are connected to one another as if they are each a special case of some more fundamental theory (thought to be M-theory). These theories are related by transformations that are called dualities. If two theories are related by a duality transformation, it means that the first theory can be transformed in some way so that it ends up looking just like the second theory. The two theories are then said to be dual to one another under that kind of transformation. Put differently, the two theories are mathematically different descriptions of the same phenomena.

These dualities link quantities that were also thought to be separate. Large and small distance scales, as well as strong and weak coupling strengths, are quantities that have always marked very distinct limits of behavior of a physical system in both classical field theory and quantum particle physics. But strings can obscure the difference between large and small, strong and weak, and this is how these five very different theories end up being related. T-duality relates the large and small distance scales between string theories, whereas S-duality relates strong and weak coupling strengths between string theories. U-duality links T-duality and S-duality.

String theories		
Type	Spacetime dimensions	Details
Bosonic	26	Only bosons, no fermions, meaning only forces, no matter, with both open and closed strings; major flaw: a particle with imaginary mass, called the tachyon, representing an instability in the theory.
I	10	Supersymmetry between forces and matter, with both open and closed strings; no tachyon; group symmetry is $SO(32)$
IIA	10	Supersymmetry between forces and matter, with only closed strings bound to D-branes; no tachyon; massless fermions are non-chiral
IIB	10	Supersymmetry between forces and matter, with only closed strings bound to D-branes; no tachyon; massless fermions are chiral
HO	10	Supersymmetry between forces and matter, with closed strings only; no tachyon; heterotic, meaning right moving and left moving strings differ; group symmetry is $SO(32)$
HE	10	Supersymmetry between forces and matter, with closed strings only; no tachyon; heterotic, meaning right moving and left moving strings differ; group symmetry is $E_8 \times E_8$

Note that in the type IIA and type IIB string theories closed strings are allowed to move everywhere throughout the ten-dimensional spacetime (called the *bulk*), while open strings have their ends attached to D-branes, which are membranes of lower dimensionality (their dimension is odd — 1, 3, 5, 7 or 9 — in type IIA and even — 0, 2, 4, 6 or 8 — in type IIB, including the time direction).

Extra dimensions

Number of dimensions

An intriguing feature of string theory is that it involves the prediction of extra dimensions. The number of dimensions is not fixed by any consistency criterion, but flat spacetime solutions do exist in the so-called "critical dimension". Cosmological solutions exist in a wider variety of dimensionalities, and these different dimensions—more precisely different values of the "effective central charge", a count of degrees of freedom which reduces to dimensionality in weakly curved regimes—are related by dynamical transitions.^[14]

One such theory is the 11-dimensional M-theory, which requires spacetime to have eleven dimensions,^[15] as opposed to the usual three spatial dimensions and the fourth dimension of time. The original string theories from the 1980s describe special cases of M-theory where the eleventh dimension is a very small circle or a line, and if these formulations are considered as fundamental, then string theory requires ten dimensions. But the theory also describes universes like ours, with four observable spacetime dimensions, as well as universes with up to 10 flat space dimensions, and also cases where the position in some of the dimensions is not described by a real number, but by a completely different type of mathematical quantity. So the notion of spacetime dimension is not fixed in string theory: it is best thought of as different in different circumstances.^[16]

Nothing in Maxwell's theory of electromagnetism or Einstein's theory of relativity makes this kind of prediction; these theories require physicists to insert the number of dimensions "by both hands", and this number is fixed and independent of potential energy. String theory allows one to relate the number of dimensions to scalar potential energy. Technically, this happens because a gauge anomaly exists for every separate number of predicted dimensions, and the gauge anomaly can be counteracted by including nontrivial potential energy into equations to solve motion. Furthermore, the absence of potential energy in the "critical dimension" explains why flat spacetime solutions are possible.

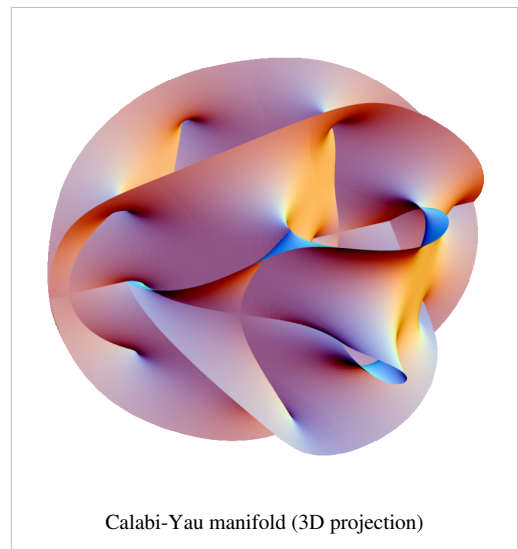
This can be better understood by noting that a photon included in a consistent theory (technically, a particle carrying a force related to an unbroken gauge symmetry) must be massless. The mass of the photon which is predicted by

string theory depends on the energy of the string mode which represents the photon. This energy includes a contribution from the Casimir effect, namely from quantum fluctuations in the string. The size of this contribution depends on the number of dimensions since for a larger number of dimensions; there are more possible fluctuations in the string position. Therefore, the photon in flat spacetime will be massless—and the theory consistent—only for a particular number of dimensions.^[17] When the calculation is done, the critical dimensionality is not four as one may expect (three axes of space and one of time). The subset of X is equal to the relation of photon fluctuations in a linear dimension. Flat space string theories are 26-dimensional in the bosonic case, while superstring and M-theories turn out to involve 10 or 11 dimensions for flat solutions. In bosonic string theories, the 26 dimensions come from the Polyakov equation.^[18] Starting from any dimension greater than four, it is necessary to consider how these are reduced to four dimensional spacetime.

Compact dimensions

Two different ways have been proposed to resolve this apparent contradiction. The first is to compactify the extra dimensions; i.e., the 6 or 7 extra dimensions are so small as to be undetectable by present day experiments.

To retain a high degree of supersymmetry, these compactification spaces must be very special, as reflected in their holonomy. A 6-dimensional manifold must have $SU(3)$ structure, a particular case (torsionless) of this being $SU(3)$ holonomy, making it a Calabi-Yau space, and a 7-dimensional manifold must have G_2 structure, with G_2 holonomy again being a specific, simple, case. Such spaces have been studied in attempts to relate string theory to the 4-dimensional Standard Model, in part due to the computational simplicity afforded by the assumption of supersymmetry. More recently, progress has been made constructing more realistic compactifications without the degree of symmetry of Calabi-Yau or G_2 manifolds.



A standard analogy for this is to consider multidimensional space as a garden hose. If the hose is viewed from a sufficient distance, it appears to have only one dimension, its length. Indeed, think of a ball just small enough to enter the hose. Throwing such a ball inside the hose, the ball would move more or less in one dimension; in any experiment we make by throwing such balls in the hose, the only important movement will be one-dimensional, that is, along the hose. However, as one approaches the hose, one discovers that it contains a second dimension, its circumference. Thus, an ant crawling inside it would move in two dimensions (and a fly flying in it would move in three dimensions). This "extra dimension" is only visible within a relatively close range to the hose, or if one "throws in" small enough objects. Similarly, the extra compact dimensions are only "visible" at extremely small distances, or by experimenting with particles with extremely small wavelengths (of the order of the compact dimension's radius), which in quantum mechanics means very high energies (see wave-particle duality).

Brane-world scenario

Another possibility is that we are "stuck" in a 3+1 dimensional (i.e. three spatial dimensions plus the time dimension) subspace of the full universe. If such sub-spacetimes are exceptional sets within a larger-dimensional one, there typically exist properly localized matter and Yang-Mills gauge fields.^[19] These "exceptional sets" are ubiquitous in Calabi-Yau n -folds and may be described as subspaces without local deformations, akin to a crease in a sheet of paper or a crack in a crystal, the neighborhood of which is markedly different from the exceptional subspace itself. However, until the work of Randall and Sundrum,^[20] it was not known that gravity too can be properly localized to a sub-spacetime; their proof that it can made such cosmological scenarios realistic. In addition,

spacetime may well be stratified, containing strata of various dimensions so that we may be inhabiting a 3+1 dimensional stratum; such geometries occur naturally in Calabi-Yau compactifications.^[21] Such sub-spacetimes are supposed to be D-branes, hence such models are known as a brane-world theories.

Effect of the hidden dimensions

In either case, gravity acting in the hidden dimensions affects other non-gravitational forces such as electromagnetism. In fact, Kaluza's early work demonstrated that general relativity in five dimensions actually predicts the existence of electromagnetism. However, because of the nature of Calabi-Yau manifolds, no new forces appear from the small dimensions, but their shape has a profound effect on how the forces between the strings appear in our four-dimensional universe. In principle, therefore, it is possible to deduce the nature of those extra dimensions by requiring consistency with the standard model, but this is not yet a practical possibility. It is also possible to extract information regarding the hidden dimensions by precision tests of gravity, but so far these have only put upper limitations on the size of such hidden dimensions.

D-branes

Another key feature of string theory is the existence of D-branes. These are membranes of different dimensionality (anywhere from a zero dimensional membrane — which is in fact a point — and up, including 2-dimensional membranes, 3-dimensional volumes and so on).

D-branes are defined by the fact that worldsheet boundaries are attached to them. D-branes have mass, since they emit and absorb closed strings which describe gravitons, and — in superstring theories — charge as well, since they couple to open strings which describe gauge interactions.

From the point of view of open strings, D-branes are objects to which the ends of open strings are attached. The open strings attached to a D-brane are said to "live" on it, and they give rise to gauge theories "living" on it (since one of the open string modes is a gauge boson such as the photon). In the case of one D-brane there will be one type of a gauge boson and we will have an Abelian gauge theory (with the gauge boson being the photon). If there are multiple parallel D-branes there will be multiple types of gauge bosons, giving rise to a non-Abelian gauge theory.

D-branes are thus gravitational sources, on which a gauge theory "lives". This gauge theory is coupled to gravity (which is said to exist in the *bulk*), so that normally each of these two different viewpoints is incomplete.

Gauge-gravity duality

Gauge-gravity duality is a conjectured duality between a quantum theory of gravity in certain cases and gauge theory in a lower number of dimensions. This means that each predicted phenomenon and quantity in one theory has an analogue in the other theory, with a "dictionary" translating from one theory to the other.

Description of the duality

In certain cases the gauge theory on the D-branes is decoupled from the gravity living in the bulk; thus open strings attached to the D-branes are not interacting with closed strings. Such a situation is termed a *decoupling limit*.

In those cases, the D-branes have two independent alternative descriptions. As discussed above, from the point of view of closed strings, the D-branes are gravitational sources, and thus we have a gravitational theory on spacetime with some background fields. From the point of view of open strings, the physics of the D-branes is described by the appropriate gauge theory. Therefore in such cases it is often conjectured that the gravitational theory on spacetime with the appropriate background fields is dual (i.e. physically equivalent) to the gauge theory on the boundary of this spacetime (since the subspace filled by the D-branes is the boundary of this spacetime). So far, this duality has not been proven in any cases, so there is also disagreement among string theorists regarding how strong the duality applies to various models.

Examples and intuition

The best known example and the first one to be studied is the duality between Type IIB superstring on $AdS^5 \times S^5$ (a product space of a five-dimensional Anti de Sitter space and a five-sphere) on one hand, and $N = 4$ supersymmetric Yang-Mills theory on the four-dimensional boundary of the Anti de Sitter space (either a flat four-dimensional spacetime $R^{3,1}$ or a three-sphere with time $S^3 \times R$). This is known as the AdS/CFT correspondence,^{[22] [23] [24] [25]} a name often used for Gauge / gravity duality in general.

This duality can be thought of as follows: suppose there is a spacetime with a gravitational source, for example an extremal black hole.^[26] When particles are far away from this source, they are described by closed strings (i.e. a gravitational theory, or usually supergravity). As the particles approach the gravitational source, they can still be described by closed strings; alternatively, they can be described by objects similar to QCD strings,^{[27] [28] [29]} which are made of gauge bosons (gluons) and other gauge theory degrees of freedom.^[30] So if one is able (in a *decoupling limit*) to describe the gravitational system as two separate regions — one (the *bulk*) far away from the source, and the other close to the source — then the latter region can also be described by a gauge theory on D-branes. This latter region (close to the source) is termed the *near-horizon limit*, since usually there is an event horizon around (or at) the gravitational source.

In the gravitational theory, one of the directions in spacetime is the radial direction, going from the gravitational source and away (towards the bulk). The gauge theory lives only on the D-brane itself, so it does not include the radial direction: it lives in a spacetime with one less dimension compared to the gravitational theory (in fact, it lives on a spacetime identical to the boundary of the near-horizon gravitational theory). Let us understand how the two theories are still equivalent:

The physics of the near-horizon gravitational theory involves only on-shell states (as usual in string theory), while the field theory includes also off-shell correlation function. The on-shell states in the near-horizon gravitational theory can be thought of as describing only particles arriving from the bulk to the near-horizon region and interacting there between themselves. In the gauge theory these are "projected" onto the boundary, so that particles which arrive at the source from different directions will be seen in the gauge theory as (off-shell) quantum fluctuations far apart from each other, while particles arriving at the source from almost the same direction in space will be seen in the gauge theory as (off-shell) quantum fluctuations close to each other. Thus the angle between the arriving particles in the gravitational theory translates to the distance scale between quantum fluctuations in the gauge theory. The angle between arriving particles in the gravitational theory is related to the radial distance from the gravitational source at which the particles interact: the larger the angle, the closer the particles have to get to the source in order to interact with each other. On the other hand, the scale of the distance between quantum fluctuations in a quantum field theory is related (inversely) to the energy scale in this theory, so small radius in the gravitational theory translates to low energy scale in the gauge theory (i.e. the IR regime of the field theory) while large radius in the gravitational theory translates to high energy scale in the gauge theory (i.e. the UV regime of the field theory).

A simple example to this principle is that if in the gravitational theory there is a setup in which the dilaton field (which determines the strength of the coupling) is decreasing with the radius, then its dual field theory will be asymptotically free, i.e. its coupling will grow weaker in high energies.

Contact with experiment

The mathematics of string theory may lead to new insights on quantum chromodynamics, a gauge theory which is the fundamental theory of the strong nuclear force. To this end, it is hoped that a gravitational theory dual to quantum chromodynamics will be found.^[31]

A mathematical technique from string theory (the AdS/CFT correspondence) has been used to describe qualitative features of quark-gluon plasma behavior in relativistic heavy-ion collisions;^{[11] [8] [9] [10]} the physics, however, is strictly that of standard quantum chromodynamics, which has been quantitatively modeled by lattice QCD methods with good results.^[32]

Other potential forms of evidence for string theory have been proposed. One would be the discovery of large cosmic strings in space, formed when the Big Bang "stretched" some strings to astronomical proportions. Other theories, however, predict cosmic strings with a different physical basis (topological defects in various fields). Novel phenomena consistent with versions of string theory may be observed with the newly built Large Hadron Collider. One would be indications of an anomalously large strength of gravity on a microscopic scale, which would be consistent with branes interacting across extra dimensions through leakage of gravitons from one to the other. The discovery of supersymmetry could also be considered evidence since string theory was the first theory to require it, though other theories incorporate supersymmetry as well. The absence of supersymmetric particles at energies accessible to the LHC would not necessarily disprove string theory, since supersymmetry could exist but still be outside the accelerator's range.

Problems and controversy

Although string theory comes from physics, some say that string theory's current untestable status means that it should be classified as more of a mathematical framework for building models as opposed to a physical theory.^[33] Some go further, and say that string theory as a theory of everything is a failure.^{[34] [35]} This led to a public debate in 2007,^{[36] [37]} with one journalist expressing this opinion:

"For more than a generation, physicists have been chasing a will-o'-the-wisp called string theory. The beginning of this chase marked the end of what had been three-quarters of a century of progress. Dozens of string-theory conferences have been held, hundreds of new Ph.D.s have been minted, and thousands of papers have been written. Yet, for all this activity, not a single new testable prediction has been made, not a single theoretical puzzle has been solved. In fact, there is no theory so far—just a set of hunches and calculations suggesting that a theory might exist. And, even if it does, this theory will come in such a bewildering number of versions that it will be of no practical use: a Theory of Nothing."

—Jim Holt^[38]

Is string theory predictive?

String theory as a theory of everything has been criticized as unscientific because it is so difficult to test by experiments. The controversy concerns two properties:

1. It is widely believed that any theory of quantum gravity would require extremely high energies to probe directly, higher by orders of magnitude than those that current experiments such as the Large Hadron Collider^[39] can reach.
2. String theory as it is currently understood has a huge number of equally possible solutions, called string vacua,^[40] and these vacua might be sufficiently diverse to explain almost any phenomena we might observe at lower energies.

If these properties are true, string theory as a theory of everything would have little or no predictive power for low energy particle physics experiments.^{[41] [42]} Because the theory is so difficult to test, some theoretical physicists have asked if it can even be called a scientific theory. Notable critics include Peter Woit, Lee Smolin, Philip Warren

Anderson,^[43] Sheldon Glashow,^[44] Lawrence Krauss,^[45] and Carlo Rovelli.^[46]

All string theory models are quantum mechanical, Lorentz invariant, unitary and contain Einstein's General Relativity as a low energy limit.^[47] Therefore to falsify string theory, it would suffice to falsify quantum mechanics, Lorentz invariance, or general relativity.^[48] Hence string theory is falsifiable and meets the definition of scientific theory according to the Popperian criterion. However to constitute a convincing potential verification of string theory, a prediction should be specific to it, not shared by any quantum field theory model or by General Relativity.

One such unique prediction is *string harmonics*: at sufficiently high energies—probably near the quantum gravity scale—the string-like nature of particles would become obvious. There should be heavier copies of all particles corresponding to higher vibrational states of the string. But it is not clear how high these energies are. In the most likely case, they would be 10^{14} times higher than those accessible in the newest particle accelerator, the LHC, making this prediction impossible to test with any particle accelerator in the foreseeable future.

Swampland

In response to these concerns, Cumrun Vafa and others have challenged the idea that string theory is compatible with anything. They propose that most possible theories of low energy physics lie in the swampland. The swampland is the collection of theories which could be true if gravity wasn't an issue, but which are not compatible with string theory. An example of a theory in the swampland is quantum electrodynamics in the limit of very small electron charge. This limit is perfectly fine in quantum field theory — in fact, in this limit, the perturbation theory becomes better and better. But in string theory, at the moment the charge of the lightest charged particle becomes less than the mass in natural units, the theory becomes inconsistent.

The reason is that two such charged massive particles will attract each other gravitationally more than they repel each other electrostatically, and could be used to form black holes. If there are no light charged particles, these black holes could not decay efficiently, barring improbable conspiracies or remnants. From the study of examples, and from the analysis of black-hole evaporation, it is now accepted that theories with a small charge quantum must come with light charged particles. This is only true within string theory—there is no such restriction in quantum field theory. This means that the discovery of a new gauge group with a small quantum of charge and only heavy charged particles would falsify string theory. Since this argument is very general—relying only on black-hole evaporation and the holographic principle, it has been suggested that this prediction would be true of any consistent holographic theory of quantum gravity, although the phrase "consistent holographic theory of quantum gravity" might very well be synonymous with "String Theory".

It is notable that all the gross features of the Standard model can be embedded within String theory, so that the standard model is not in the swampland. This includes features such as non-abelian gauge groups and chiral fermions which are hard to incorporate in non-string theories of quantum gravity.

Background independence

A separate and older criticism of string theory is that it is background-dependent — string theory describes perturbative expansions about fixed spacetime backgrounds. Although the theory has some background-independence — topology change is an established process in string theory, and the exchange of gravitons is equivalent to a change in the background — mathematical calculations in the theory rely on preselecting a background as a starting point. This is because, like many quantum field theories, much of string theory is still only formulated perturbatively, as a divergent series of approximations. Although nonperturbative techniques have progressed considerably — including conjectured complete definitions in spacetimes satisfying certain asymptotics — a full non-perturbative definition of the theory is still lacking. Some see background independence as a fundamental requirement of a theory of quantum gravity, particularly since general relativity is already background independent. Some hope that M-theory, or a non-perturbative treatment of string theory (string field theory was thought to be non-perturbative in the 1980s) have a background-independent formulation.

Supersymmetry breaking

A central problem for applications is that the best understood backgrounds of string theory preserve much of the supersymmetry of the underlying theory, which results in time-invariant spacetimes: currently string theory cannot deal well with time-dependent, cosmological backgrounds. However, several models have been proposed to explain supersymmetry breaking, most notably the KKLT model,^[40] which incorporates branes and fluxes to make a metastable compactification.

String theory landscape

The vacuum structure of the theory, called the string theory landscape (or the anthropic portion of string theory vacua), is not well understood. String theory contains an infinite number of distinct meta-stable vacua, and perhaps 10^{500} of these or more correspond to a universe roughly similar to ours — with four dimensions, a high planck scale, gauge groups, and chiral fermions. Each of these corresponds to a different possible universe, with a different collection of particles and forces.^[40] What principle, if any, can be used to select among these vacua is an open issue. While there are no continuous parameters in the theory, there is a very large set of possible universes, which may be radically different from each other.

Some physicists believe this is a good thing, because it may allow a natural anthropic explanation of the observed values of physical constants, in particular the small value of the cosmological constant.^[49] ^[50] The argument is that most universes contain values for physical constants which do not lead to habitable universes (at least for humans), and so we happen to live in the most "friendly" universe. This principle is already employed to explain the existence of life on earth as the result of a life-friendly orbit around the medium-sized sun among an infinite number of possible orbits (as well as a relatively stable location in the galaxy). However, the cosmological version of the anthropic principle remains highly controversial because it would be difficult if not impossible to Popper falsify; so many do not accept it as scientific.

Other testability criteria

Many physicists strongly oppose the idea that string theory is not falsifiable, among them Sylvester James Gates: "So, the next time someone tells you that string theory is not testable, remind them of the AdS/CFT connection ..."^[51] AdS/CFT relates string theory to gauge theory, and allows contact with low energy experiments in quantum chromodynamics. This type of string theory, which only describes the strong interactions, is much less controversial today than string theories of everything (although two decades ago, it was the other way around).

In addition, Gates points out that the grand unification natural in string theories of everything requires that the coupling constants of the four forces meet at one point under renormalization group rescaling. This is also a falsifiable statement, but it is not restricted to string theory, but is shared by grand unified theories.^[52] The LHC will be used both for testing AdS/CFT, and to check if the electroweakstrong unification does happen as predicted.^[53]

History

Some of the structures reintroduced by string theory arose for the first time much earlier as part of the program of classical unification started by Albert Einstein. The first person to add a fifth dimension to general relativity was German mathematician Theodor Kaluza in 1919, who noted that gravity in five dimensions describes both gravity and electromagnetism in four. In 1926, the Swedish physicist Oskar Klein gave a physical interpretation of the unobservable extra dimension--- it is wrapped into a small circle. Einstein introduced a non-symmetric metric tensor, while much later Brans and Dicke added a scalar component to gravity. These ideas would be revived within string theory, where they are demanded by consistency conditions.

String theory was originally developed during the late 1960s and early 1970s as a never completely successful theory of hadrons, the subatomic particles like the proton and neutron which feel the strong interaction. In the 1960s,

Geoffrey Chew and Steven Frautschi discovered that the mesons make families called Regge trajectories with masses related to spins in a way that was later understood by Yoichiro Nambu, Holger Bech Nielsen and Leonard Susskind to be the relationship expected from rotating strings. Chew advocated making a theory for the interactions of these trajectories which did not presume that they were composed of any fundamental particles, but would construct their interactions from self-consistency conditions on the S-matrix. The S-matrix approach was started by Werner Heisenberg in the 1940s as a way of constructing a theory which did not rely on the local notions of space and time, which Heisenberg believed break down at the nuclear scale. While the scale was off by many orders of magnitude, the approach he advocated was ideally suited for a theory of quantum gravity.

Working with experimental data, R. Dolen, D. Horn and C. Schmid^[54] developed some sum rules for hadron exchange. When a particle and antiparticle scatter, virtual particles can be exchanged in two qualitatively different ways. In the s-channel, the two particles annihilate to make temporary intermediate states which fall apart into the final state particles. In the t-channel, the particles exchange intermediate states by emission and absorption. In field theory, the two contributions add together, one giving a continuous background contribution, the other giving peaks at certain energies. In the data, it was clear that the peaks were stealing from the background--- the authors interpreted this as saying that the t-channel contribution was dual to the s-channel one, meaning both described the whole amplitude and included the other.

The result was widely advertised by Murray Gell-Mann, leading Gabriele Veneziano to construct a scattering amplitude which had the property of Dolen-Horn-Schmid duality, later renamed world-sheet duality. The amplitude needed poles where the particles appear, on straight line trajectories, and there is a special mathematical function whose poles are evenly spaced on half the real line--- the Gamma function--- which was widely used in Regge theory. By manipulating combinations of Gamma functions, Veneziano was able to find a consistent scattering amplitude with poles on straight lines, with mostly positive residues, which obeyed duality and had the appropriate Regge scaling at high energy. The amplitude could fit near-beam scattering data as well as other Regge type fits, and had a suggestive integral representation which could be used for generalization.

Over the next years, hundreds of physicists worked to complete the bootstrap program for this model, with many surprises. Veneziano himself discovered that for the scattering amplitude to describe the scattering of a particle which appears in the theory, an obvious self-consistency condition, the lightest particle must be a tachyon. Miguel Virasoro and Joel Shapiro found a different amplitude now understood to be that of closed strings, while Ziro Koba and Holger Nielsen generalized Veneziano's integral representation to multiparticle scattering. Veneziano and Sergio Fubini introduced an operator formalism for computing the scattering amplitudes which was a forerunner of world-sheet conformal theory, while Virasoro understood how to remove the poles with wrong-sign residues using a constraint on the states. Claud Lovelace calculated a loop amplitude, and noted that there is an inconsistency unless the dimension of the theory is 26. Charles Thorn, Peter Goddard and Richard Brower went on to prove that there are no wrong-sign propagating states in dimensions less than or equal to 26.

In 1969 Yoichiro Nambu, Holger Bech Nielsen and Leonard Susskind recognized that the theory could be given a description in space and time in terms of strings. The scattering amplitudes were derived systematically from the action principle by Peter Goddard, Jeffrey Goldstone, Claudio Rebbi and Charles Thorn, giving a space-time picture to the vertex operators introduced by Veneziano and Fubini and a geometrical interpretation to the Virasoro conditions.

In 1970, Pierre Ramond added fermions to the model, which led him to formulate a two-dimensional supersymmetry to cancel the wrong-sign states. John Schwarz and André Neveu added another sector to the fermi theory a short time later. In the fermion theories, the critical dimension was 10. Stanley Mandelstam formulated a world sheet conformal theory for both the bose and fermi case, giving a two-dimensional field theoretic path-integral to generate the operator formalism. Michio Kaku and Keiji Kikkawa gave a different formulation of the bosonic string, as a string field theory, with infinitely many particle types and with fields taking values not on points, but on loops and curves.

In 1974, Tamiaki Yoneya discovered that all the known string theories included a massless spin-two particle which obeyed the correct Ward identities to be a graviton. John Schwarz and Joel Scherk came to the same conclusion and made the bold leap to suggest that string theory was a theory of gravity, not a theory of hadrons. They reintroduced Kaluza–Klein theory as a way of making sense of the extra dimensions. At the same time, quantum chromodynamics was recognized as the correct theory of hadrons, shifting the attention of physicists and apparently leaving the bootstrap program in the dustbin of history.

String theory eventually made it out of the dustbin, but for the following decade all work on the theory was completely ignored. Still, the theory continued to develop at a steady pace thanks the work of a handful of devotees. Ferdinando Gliozzi, Joel Scherk, and David Olive realized in 1976 that the original Ramond and Neveu Schwarz-strings were separately inconsistent and needed to be combined. The resulting theory did not have a tachyon, and was proven to have space-time supersymmetry by John Schwarz and Michael Green in 1981. The same year, Alexander Polyakov gave the theory a modern path integral formulation, and went on to develop conformal field theory extensively. In 1979, Daniel Friedan showed that the equations of motions of string theory, which are generalizations of the Einstein equations of General Relativity, emerge from the Renormalization group equations for the two-dimensional field theory. Schwarz and Green discovered T-duality, and constructed two different superstring theories--- IIA and IIB related by T-duality, and type I theories with open strings. The consistency conditions had been so strong, that the entire theory was nearly uniquely determined, with only a few discrete choices.

In the early 1980s, Edward Witten discovered that most theories of quantum gravity could not accommodate chiral fermions like the neutrino. This led him, in collaboration with Luis Alvarez-Gaumé to study violations of the conservation laws in gravity theories with anomalies, concluding that type I string theories were inconsistent. Green and Schwarz discovered a contribution to the anomaly that Witten and Alvarez-Gaumé had missed, which restricted the gauge group of the type I string theory to be $SO(32)$. In coming to understand this calculation, Edward Witten became convinced that string theory was truly a consistent theory of gravity, and he became a high-profile advocate. Following Witten's lead, between 1984 and 1986, hundreds of physicists started to work in this field, and this is sometimes called the first superstring revolution.

During this period, David Gross, Jeffrey Harvey, Emil Martinec, and Ryan Rohm discovered heterotic strings. The gauge group of these closed strings was two copies of E_8 , and either copy could easily and naturally include the standard model. Philip Candelas, Gary Horowitz, Andrew Strominger and Edward Witten found that the Calabi-Yau manifolds are the compactifications which preserve a realistic amount of supersymmetry, while Lance Dixon and others worked out the physical properties of orbifolds, distinctive geometrical singularities allowed in string theory. Cumrun Vafa generalized T-duality from circles to arbitrary manifolds, creating the mathematical field of mirror symmetry. David Gross and Vipul Periwal discovered that string perturbation theory was divergent in a way that suggested that new non-perturbative objects were missing.

In the 1990s, Joseph Polchinski discovered that the theory requires higher-dimensional objects, called D-branes and identified these with the black-hole solutions of supergravity. These were understood to be the new objects suggested by the perturbative divergences, and they opened up a new field with rich mathematical structure. It quickly became clear that D-branes and other p-branes, not just strings, formed the matter content of the string theories, and the physical interpretation of the strings and branes was revealed--- they are a type of black hole. Leonard Susskind had incorporated the holographic principle of Gerardus 't Hooft into string theory, identifying the long highly-excited string states with ordinary thermal black hole states. As suggested by 't Hooft, the fluctuations of the black hole horizon, the world-sheet or world-volume theory, describes not only the degrees of freedom of the black hole, but all nearby objects too.

In 1995, at the annual conference of string theorists at the University of Southern California (USC), Edward Witten gave a speech on string theory that essentially united the five string theories that existed at the time, and giving birth to a new 11-dimensional theory called M-theory. M-theory was also foreshadowed in the work of Paul Townsend at

approximately the same time. The flurry of activity which began at this time is sometimes called the second superstring revolution.

During this period, Tom Banks, Willy Fischler, Stephen Shenker and Leonard Susskind formulated a full holographic description of M-theory on IIA D0 branes,^[55] the first definition of string theory that was fully non-perturbative and a concrete mathematical realization of the holographic principle. Andrew Strominger and Cumrun Vafa calculated the entropy of certain configurations of D-branes and found agreement with the semi-classical answer for extreme charged black holes. Petr Horava and Edward Witten found the eleven-dimensional formulation of the heterotic string theories, showing that orbifolds solve the chirality problem. Witten noted that the effective description of the physics of D-branes at low energies is by a supersymmetric gauge theory, and found geometrical interpretations of mathematical structures in gauge theory that he and Nathan Seiberg had earlier discovered in terms of the location of the branes.

In 1997 Juan Maldacena noted that the low energy excitations of a theory near a black hole consist of objects close to the horizon, which for extreme charged black holes looks like an anti de Sitter space. He noted that in this limit the gauge theory describes the string excitations near the branes. So he hypothesized that string theory on a near-horizon extreme-charged black-hole geometry, an anti-deSitter space times a sphere with flux, is equally well described by the low-energy limiting gauge theory, the $N=4$ supersymmetric Yang-Mills theory. This hypothesis, which is called the AdS/CFT correspondence, was further developed by Steven Gubser, Igor Klebanov and Alexander Polyakov, and by Edward Witten, and it is now well-accepted. It is a concrete realization of the holographic principle, which has far-reaching implications for black holes, locality and information in physics, as well as the nature of the gravitational interaction. Through this relationship, string theory has been shown to be related to gauge theories like quantum chromodynamics and this has led to more quantitative understanding of the behavior of hadrons, bringing string theory back to its roots.

See also

- List of string theory topics
- Conformal field theory
- F-theory
- Fuzzballs
- Little string theory
- Loop quantum gravity
- Relationship between string theory and quantum field theory
- String cosmology
- Supergravity
- Zeta function regularization
- *The Elegant Universe*

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- Spinning the Superweb: Essays on the History of Superstring Theory (<http://www.spinningthesuperweb.blogspot.com>) – A Science Studies' approach to the history of string theory (an elementary knowledge of string theory is required).

Superstring Theories

Superstring theory is an attempt to explain all of the particles and fundamental forces of nature in one theory by modelling them as vibrations of tiny supersymmetric strings. Superstring theory is a shorthand for **supersymmetric string theory** because unlike bosonic string theory, it is the version of string theory that incorporates fermions and supersymmetry. So far the theory has made no quantitative experimental predictions, so that the theory could be falsified (tested)^{[1] [2]}.

Background

The deepest problem in theoretical physics is harmonizing the theory of general relativity, which describes gravitation and applies to large-scale structures (stars, galaxies, super clusters), with quantum mechanics, which describes the other three fundamental forces acting on the atomic scale.

The development of a quantum field theory of a force invariably results in infinite (and therefore useless) probabilities. Physicists have developed mathematical techniques (renormalization) to eliminate these infinities which work for three of the four fundamental forces – electromagnetic, strong nuclear and weak nuclear forces – but not for gravity. The development of a quantum theory of gravity must therefore come about by different means than those used for the other forces.

Basic idea

The basic idea is that the fundamental constituents of reality are strings of the Planck length (about 10^{-33} cm) which vibrate at resonant frequencies. Every string in theory has a unique resonance, or harmonic. Different harmonics determine different fundamental forces. The tension in a string is on the order of the Planck force (10^{44} newtons). The graviton (the proposed messenger particle of the gravitational force), for example, is predicted by the theory to be a string with wave amplitude zero. Another key insight provided by the theory is that no measurable differences can be detected between strings that wrap around dimensions smaller than themselves and those that move along

larger dimensions (i.e., effects in a dimension of size R equal those whose size is $1/R$). Singularities are avoided because the observed consequences of "Big Crunches" never reach zero size. In fact, should the universe begin a "big crunch" sort of process, string theory dictates that the universe could never be smaller than the size of a string, at which point it would actually begin expanding.

Extra dimensions

See also: Why does consistency require 10 dimensions?

Our physical space is observed to have only three large dimensions and—taken together with duration as the fourth dimension—a physical theory must take this into account. However, nothing prevents a theory from including more than 4 dimensions. In the case of string theory, consistency requires spacetime to have 10 ($3+1+6$) dimensions. The conflict between observation and theory is resolved by making the unobserved dimensions compactified.

Our minds have difficulty visualizing higher dimensions because we can only move in three spatial dimensions. One way of dealing with this limitation is not to try to visualize higher dimensions at all, but just to think of them as extra numbers in the equations that describe the way the world works. This opens the question of whether these 'extra numbers' can be investigated directly in any experiment (which must show different results in 1, 2, or $2+1$ dimensions to a human scientist). This, in turn, raises the question of whether models that rely on such abstract modelling (and potentially impossibly huge experimental apparatuses) can be considered scientific. Six-dimensional Calabi–Yau shapes can account for the additional dimensions required by superstring theory. The theory states that every point in space (or whatever we had previously considered a point) is in fact a very small manifold where each extra dimension has a size on the order of the Planck length.

Superstring theory is not the first theory to propose extra spatial dimensions; the Kaluza-Klein theory had done so previously. Modern string theory relies on the mathematics of folds, knots, and topology, which were largely developed after Kaluza and Klein, and has made physical theories relying on extra dimensions much more credible.

Number of superstring theories

Theoretical physicists were troubled by the existence of five separate string theories. A possible solution for this dilemma was suggested at the beginning of what is called the second superstring revolution in the 1990s, which suggests that the five string theories might be different limits of a single underlying theory, called M-theory. Unfortunately, however, to this date this remains a conjecture.

String theories							
Type	Spacetime dimensions	SUSY generators	chiral	open strings?	heterotic compactification?	gauge group	tachyon
Bosonic (closed)	26	$N = 0$	no	no	no	none	yes
Bosonic (open)	26	$N = 0$	no	yes	no	$U(1)$	yes
I	10	$N = (1,0)$	yes	yes	no	$SO(32)$	no
IIA	10	$N = (1,1)$	no	no	no	$U(1)$	no
IIB	10	$N = (2,0)$	yes	no	no	none	no
HO	10	$N = (1,0)$	yes	no	yes	$SO(32)$	no
HE	10	$N = (1,0)$	yes	no	yes	$E_8 \times E_8$	no
M-theory	11	$N = 1$	no	no	no	none	no

The five consistent superstring theories are:

- The type I string has one supersymmetry in the ten-dimensional sense (16 supercharges). This theory is special in the sense that it is based on unoriented open and closed strings, while the rest are based on oriented closed strings.
- The type II string theories have two supersymmetries in the ten-dimensional sense (32 supercharges). There are actually two kinds of type II strings called type IIA and type IIB. They differ mainly in the fact that the IIA theory is non-chiral (parity conserving) while the IIB theory is chiral (parity violating).
- The heterotic string theories are based on a peculiar hybrid of a type I superstring and a bosonic string. There are two kinds of heterotic strings differing in their ten-dimensional gauge groups: the heterotic $E_8 \times E_8$ string and the heterotic $SO(32)$ string. (The name heterotic $SO(32)$ is slightly inaccurate since among the $SO(32)$ Lie groups, string theory singles out a quotient $Spin(32)/Z_2$ that is not equivalent to $SO(32)$.)

Chiral gauge theories can be inconsistent due to anomalies. This happens when certain one-loop Feynman diagrams cause a quantum mechanical breakdown of the gauge symmetry. The anomalies were canceled out via the Green–Schwarz mechanism.

Please note that the number of superstring theories given above is only a high-level classification; the actual number of mathematically distinct theories which are compatible with observation and would therefore have to be examined to find the one that correctly describes nature is currently believed to be at least 10^{500} (a one with five hundred zeroes). This has given rise to the concern that superstring theories, despite the alluring simplicity of their basic principles, are, in fact, not simple at all, and according to the principle of Occam's razor perhaps alternative physical theories going beyond the Standard Model should be explored. This is aggravated by the fact that it is exceedingly hard to make predictions from any superstring theory which can be falsified by experiment, and in fact no current superstring theory makes any falsifiable prediction.

Integrating general relativity and quantum mechanics

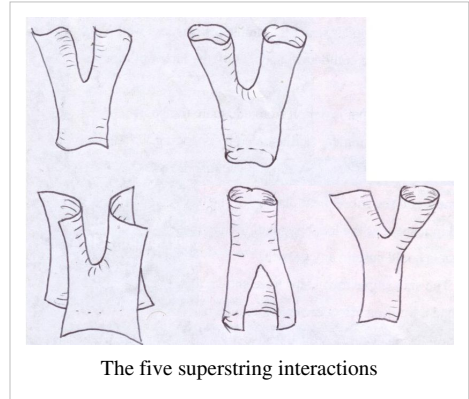
General relativity typically deals with situations involving large mass objects in fairly large regions of spacetime whereas quantum mechanics is generally reserved for scenarios at the atomic scale (small spacetime regions). The two are very rarely used together, and the most common case in which they are combined is in the study of black holes. Having "peak density", or the maximum amount of matter possible in a space, and very small area, the two must be used in synchrony in order to predict conditions in such places; yet, when used together, the equations fall apart, spitting out impossible answers, such as imaginary distances and less than one dimension.

The major problem with their congruence is that, at Planck scale (a fundamental small unit of length) lengths, general relativity predicts a smooth, flowing surface, while quantum mechanics predicts a random, warped surface, neither of which are anywhere near compatible. Superstring theory resolves this issue, replacing the classical idea of point particles with loops. These loops have an average diameter of the Planck length, with extremely small variances, which completely ignores the quantum mechanical predictions of Planck-scale length dimensional warping.

The five superstring interactions

There are five ways open and closed strings can interact. An interaction in superstring theory is a topology changing event. Since superstring theory has to be a local theory to obey causality the topology change must only occur at a single point. If C represents a closed string and O an open string, then the five interactions are OOO, CCC, OOOO, CO and COO.

All open superstring theories also contain closed superstrings since closed superstrings can be seen from the fifth interaction, and they are unavoidable. Although all these interactions are possible, in practice the most used superstring model is the closed heterotic E8xE8 superstring which only has closed strings and so only the second interaction (CCC) is needed.



The five superstring interactions

The mathematics

The single most important equation in (first quantized bosonic) string theory is the N-point scattering amplitude. This treats the incoming and outgoing strings as points, which in string theory are tachyons, with momentum k_i , which connect to a string world surface at the surface points z_i . It is given by the following functional integral which integrates (sums) over all possible embeddings of this 2D surface in 27 dimensions.

$$A_N = \int D\mu \int D[X] \exp \left(-\frac{1}{4\pi\alpha} \int \partial_z X_\mu(z, \bar{z}) \partial_{\bar{z}} X^\mu(z, \bar{z}) dz^2 + i \sum_{i=1}^N k_{i\mu} X^\mu(z_i, \bar{z}_i) \right)$$

The functional integral can be done because it is a Gaussian to become:

$$A_N = \int D\mu \prod_{0 < i < j < N+1} |z_i - z_j|^{2\alpha k_i \cdot k_j}$$

This is integrated over the various points z_i . Special care must be taken because two parts of this complex region may represent the same point on the 2D surface and you don't want to integrate over them twice. Also you need to make sure you are not integrating multiple times over different parameterizations of the surface. When this is taken into account it can be used to calculate the 4-point scattering amplitude (the 3-point amplitude is simply a delta function):

$$A_4 = \frac{\Gamma(-1 + \frac{1}{2}(k_1 + k_2)^2) \Gamma(-1 + \frac{1}{2}(k_2 + k_3)^2)}{\Gamma(-2 + \frac{1}{2}((k_1 + k_2)^2 + (k_2 + k_3)^2))}$$

Which is a beta function. It was this beta function which was apparently found before full string theory was developed. With superstrings the equations contain not only the 10D space-time coordinates X but also the Grassmann coordinates θ . Since there are various ways this can be done this leads to different string theories.

When integrating over surfaces such as the torus, we end up with equations in terms of theta functions and elliptic functions such as the Dedekind eta function. This is smooth everywhere, which it has to be to make physical sense, only when raised to the 24th power. This is the origin of needing 26 dimensions of space-time for bosonic string theory. The extra two dimensions arise as degrees of freedom of the string surface.

D-branes

D-branes are membrane-like objects in 10D string theory. They can be thought of as occurring as a result of a Kaluza-Klein compactification of 11D M-Theory which contains membranes. Because compactification of a geometric theory produces extra vector fields the D-branes can be included in the action by adding an extra $U(1)$ vector field to the string action.

$$\partial_z \rightarrow \partial_z + iA_z(z, \bar{z})$$

In **type I** open string theory, the ends of open strings are always attached to D-brane surfaces. A string theory with more gauge fields such as $SU(2)$ gauge fields would then correspond to the compactification of some higher dimensional theory above 11 dimensions which is not thought to be possible to date.

Why five superstring theories?

For a 10 dimensional supersymmetric theory we are allowed a 32-component Majorana spinor. This can be decomposed into a pair of 16-component Majorana-Weyl (chiral) spinors. There are then various ways to construct an invariant depending on whether these two spinors have the same or opposite chiralities:

Superstring model	Invariant
Heterotic	$\partial_z X^\mu - i\bar{\theta}_L \Gamma^\mu \partial_z \theta_L$
IIA	$\partial_z X^\mu - i\bar{\theta}_L \Gamma^\mu \partial_z \theta_L - i\bar{\theta}_R \Gamma^\mu \partial_z \theta_R$
IIB	$\partial_z X^\mu - i\bar{\theta}_L^1 \Gamma^\mu \partial_z \theta_L^1 - i\bar{\theta}_L^2 \Gamma^\mu \partial_z \theta_L^2$

The heterotic superstrings come in two types $SO(32)$ and $E_8 \times E_8$ as indicated above and the type I superstrings include open strings.

Beyond superstring theory

It is commonly believed that the five superstring theories are approximated to a theory in higher dimensions possibly involving membranes. Unfortunately because the action for this involves quartic terms and higher so is not Gaussian the functional integrals are very difficult to solve and so this has confounded the top theoretical physicists. Edward Witten has popularised the concept of a theory in 11 dimensions M-Theory involving membranes interpolating from the known symmetries of superstring theory. It may turn out that there exist membrane models or other non-membrane models in higher dimensions which may become acceptable when new unknown symmetries of nature are found, such as noncommutative geometry for example. It is thought, however, that 16 is probably the maximum since $O(16)$ is a maximal subgroup of E_8 the largest exceptional lie group and also is more than large enough to contain the Standard Model. Quartic integrals of the non-functional kind are easier to solve so there is hope for the future. This is the series solution which is always convergent when a is non-zero and negative:

$$\int_{-\infty}^{\infty} \exp(ax^4 + bx^3 + cx^2 + dx + f) dx = e^f \sum_{n,m,p=0}^{\infty} \frac{b^{4n}}{(4n)!} \frac{c^{2m}}{(2m)!} \frac{d^{4p}}{(4p)!} \frac{\Gamma(3n + m + p + \frac{1}{4})}{a^{3n+m+p+\frac{1}{4}}}$$

In the case of membranes the series would correspond to sums of various membrane interactions that are not seen in string theory.

Compactification

Investigating theories of higher dimensions often involves looking at the 10 dimensional superstring theory and interpreting some of the more obscure results in terms of compactified dimensions. For example D-branes are seen as compactified membranes from 11D M-Theory. Theories of higher dimensions such as 12D F-theory and beyond will produce other effects such as gauge terms higher than $U(1)$. The components of the extra vector fields (A) in the D-brane actions can be thought of as extra coordinates (X) in disguise. However, the *known* symmetries including

supersymmetry currently restrict the spinors to have 32-components which limits the number of dimensions to 11 (or 12 if you include two time dimensions.) Some commentators (e.g. John Baez et al.) have speculated that the exceptional lie groups E_6 , E_7 and E_8 having maximum orthogonal subgroups $O(10)$, $O(12)$ and $O(16)$ may be related to theories in 10, 12 and 16 dimensions; 10 dimensions corresponding to string theory and the 12 and 16 dimensional theories being yet undiscovered but would be theories based on 3-branes and 7-branes respectively. However this is a minority view within the string community. Since E_7 is in some sense F_4 quaternified and E_8 is F_4 octonified, then the 12 and 16 dimensional theories, if they did exist, may involve the noncommutative geometry based on the quaternions and octonions respectively. From the above discussion, it can be seen that physicists have many ideas for extending superstring theory beyond the current 10 dimensional theory, but so far none have been successful.

Kac–Moody algebras

Since strings can have an infinite number of modes, the symmetry used to describe string theory is based on infinite dimensional Lie algebras. Some Kac–Moody algebras that have been considered as symmetries for M-Theory have been E_{10} and E_{11} and their supersymmetric extensions.

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Group Representations and Symmetry

Symmetry

Symmetry (from the Greek: "συμμετρειν" = to measure together), generally conveys two primary meanings. The first is an imprecise sense of harmonious or aesthetically pleasing proportionality and balance;^{[1] [2]} such that it reflects beauty or perfection. The second meaning is a precise and well-defined concept of balance or "patterned self-similarity" that can be demonstrated or proved according to the rules of a formal system: by geometry, through physics or otherwise.

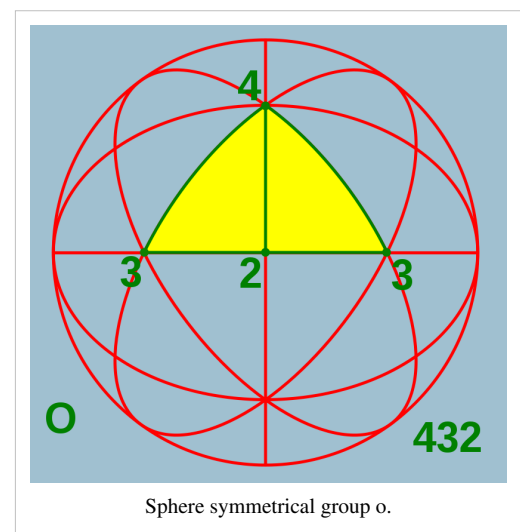
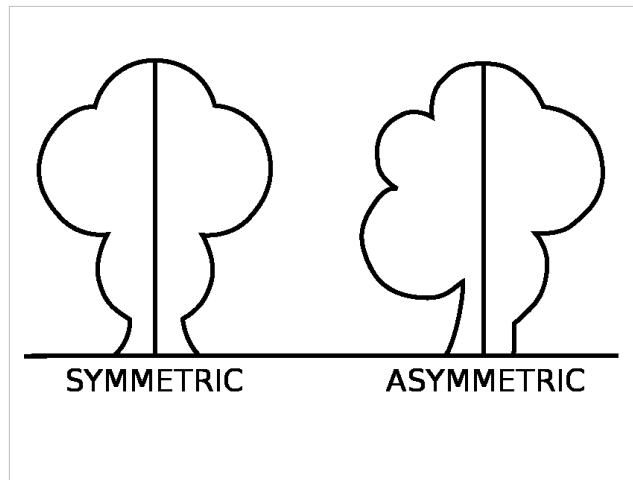
Although the meanings are distinguishable in some contexts, both meanings of "symmetry" are related and discussed in parallel.^{[2] [3]}

The "precise" notions of symmetry have various measures and operational definitions. For example, symmetry may be observed:

- with respect to the passage of time;
- as a spatial relationship;
- through geometric transformations such as scaling, reflection, and rotation;
- through other kinds of functional transformations;^[4] and
- as an aspect of abstract objects, theoretic models, language, music and even knowledge itself.^{[5] [6]}

This article describes these notions of symmetry from four perspectives. The first is that of symmetry in geometry, which is the most familiar type of symmetry for many people. The second perspective is the more general meaning of symmetry in mathematics as a whole. The third perspective describes symmetry as it relates to science and technology. In this context, symmetries underlie some of the most profound results found in modern physics, including aspects of space and time. Finally, a fourth perspective discusses symmetry in the humanities, covering its rich and varied use in history, architecture, art, and religion.

The opposite of symmetry is asymmetry.



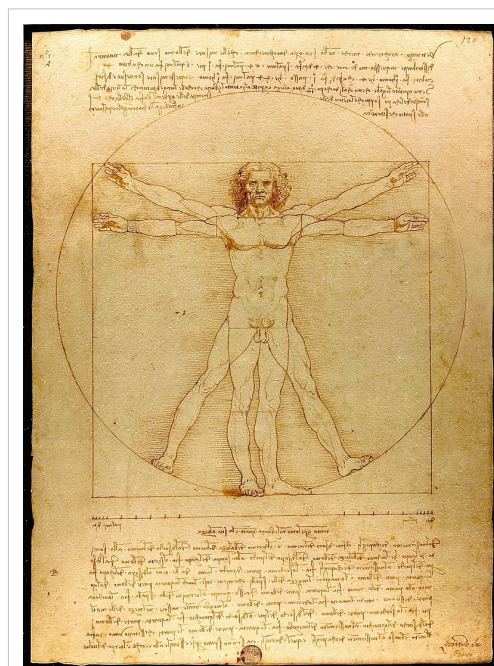
Symmetry in geometry

The most familiar type of symmetry for many people is geometrical symmetry. Formally, this means symmetry under a sub-group of the Euclidean group of isometries in two or three dimensional Euclidean space. These isometries consist of reflections, rotations, translations and combinations of these basic operations.

Reflection symmetry

Reflection symmetry, mirror symmetry, mirror-image symmetry, or bilateral symmetry is symmetry with respect to reflection.

In 1D, there is a point of symmetry. In 2D there is an axis of symmetry, in 3D a plane of symmetry. An object or figure which is indistinguishable from its transformed image is called mirror symmetric (see mirror image).



Leonardo da Vinci's *Vitruvian Man* (ca. 1487) is often used as a representation of symmetry in the human body and, by extension, the natural universe.

The axis of symmetry of a two-dimensional figure is a line such that, if a perpendicular is constructed, any two points lying on the perpendicular at equal distances from the axis of symmetry are identical. Another way to think about it is that if the shape were to be folded in half over the axis, the two halves would be identical: the two halves are each other's mirror image. Thus a square has four axes of symmetry, because there are four different ways to fold it and have the edges all match. A circle has infinitely many axes of symmetry, for the same reason.

If the letter T is reflected along a vertical axis, it appears the same. Note that this is sometimes called horizontal symmetry, and sometimes vertical symmetry. One can better use an unambiguous formulation, e.g. "T has a vertical symmetry axis" or "T has left-right symmetry."

The triangles with this symmetry are isosceles, the quadrilaterals with this symmetry are the kites and the isosceles trapezoids.

For each line or plane of reflection, the symmetry group is isomorphic with C_s (see point groups in three dimensions), one of the three types of order two (involutions), hence algebraically C_2 . The fundamental domain is a half-plane or half-space.

Bilateria (bilateral animals, including humans) are more or less symmetric with respect to the sagittal plane.

In certain contexts there is rotational symmetry anyway. Then mirror-image symmetry is equivalent with inversion symmetry; in such contexts in modern physics the term P-symmetry is used for both (P stands for parity).

For more general types of reflection there are corresponding more general types of reflection symmetry. Examples:

- with respect to a non-isometric affine involution (an oblique reflection in a line, plane, etc.).
- with respect to circle inversion

Rotational symmetry

Rotational symmetry is symmetry with respect to some or all rotations in m -dimensional Euclidean space. Rotations are direct isometries, i.e., isometries preserving orientation. Therefore a symmetry group of rotational symmetry is a subgroup of $E^+(m)$.

Symmetry with respect to all rotations about all points implies translational symmetry with respect to all translations, and the symmetry group is the whole $E^+(m)$. This does not apply for objects because it makes space homogeneous, but it may apply for physical laws.

For symmetry with respect to rotations about a point we can take that point as origin. These rotations form the special orthogonal group $SO(m)$, the group of $m \times m$ orthogonal matrices with determinant 1. For $m=3$ this is the rotation group.

In another meaning of the word, the rotation group of an object is the symmetry group within $E^+(n)$, the group of direct isometries; in other words, the intersection of the full symmetry group and the group of direct isometries. For chiral objects it is the same as the full symmetry group.

Laws of physics are $SO(3)$ -invariant if they do not distinguish different directions in space. Because of Noether's theorem, rotational symmetry of a physical system is equivalent to the angular momentum conservation law. See also rotational invariance.

Translational symmetry

Translational symmetry leaves an object invariant under a discrete or continuous group of translations $T_a(p) = p + a$.

Glide reflection symmetry

A glide reflection symmetry (in 3D: a glide plane symmetry) means that a reflection in a line or plane combined with a translation along the line / in the plane, results in the same object. It implies translational symmetry with twice the translation vector.

The symmetry group is isomorphic with $\mathbf{Z}$.

Rotoreflexion symmetry

In 3D, roto-reflection or improper rotation in the strict sense is rotation about an axis, combined with reflection in a plane perpendicular to that axis. As symmetry groups with regard to a roto-reflection we can distinguish:

- the angle has no common divisor with 360° , the symmetry group is not discrete
- $2n$ -fold roto-reflection (angle of $180^\circ/n$) with symmetry group S_{2n} of order $2n$ (not to be confused with symmetric groups, for which the same notation is used; abstract group C_{2n}); a special case is $n=1$, inversion, because it does not depend on the axis and the plane, it is characterized by just the point of inversion.
- C_{nh} (angle of $360^\circ/n$); for odd n this is generated by a single symmetry, and the abstract group is C_{2n} , for even n this is not a basic symmetry but a combination. See also point groups in three dimensions.

Helical symmetry

Helical symmetry is the kind of symmetry seen in such everyday objects as springs, Slinky toys, drill bits, and augers. It can be thought of as rotational



A drill bit with helical symmetry.

symmetry along with translation along the axis of rotation, the screw axis). The concept of helical symmetry can be visualized as the tracing in three-dimensional space that results from rotating an object at an even angular speed while simultaneously moving at another even speed along its axis of rotation (translation). At any one point in time, these two motions combine to give a *coiling angle* that helps define the properties of the tracing. When the tracing object rotates quickly and translates slowly, the coiling angle will be close to 0° . Conversely, if the rotation is slow and the translation is speedy, the coiling angle will approach 90° .

Three main classes of helical symmetry can be distinguished based on the interplay of the angle of coiling and translation symmetries along the axis:

- **Infinite helical symmetry.** If there are no distinguishing features along the length of a helix or helix-like object, the object will have infinite symmetry much like that of a circle, but with the additional requirement of translation along the long axis of the object to return it to its original appearance. A helix-like object is one that has at every point the regular angle of coiling of a helix, but which can also have a cross section of indefinitely high complexity, provided only that precisely the same cross section exists (usually after a rotation) at every point along the length of the object. Simple examples include evenly coiled springs, slinkies, drill bits, and augers. Stated more precisely, an object has infinite helical symmetries if for any small rotation of the object around its central axis there exists a point nearby (the translation distance) on that axis at which the object will appear exactly as it did before. It is this infinite helical symmetry that gives rise to the curious illusion of movement along the length of an auger or screw bit that is being rotated. It also provides the mechanically useful ability of such devices to move materials along their length, provided that they are combined with a force such as gravity or friction that allows the materials to resist simply rotating along with the drill or auger.
- **n -fold helical symmetry.** If the requirement that every cross section of the helical object be identical is relaxed, additional lesser helical symmetries become possible. For example, the cross section of the helical object may change, but still repeats itself in a regular fashion along the axis of the helical object. Consequently, objects of this type will exhibit a symmetry after a rotation by some fixed angle θ and a translation by some fixed distance, but will not in general be invariant for any rotation angle. If the angle (rotation) at which the symmetry occurs divides evenly into a full circle (360°), the result is the helical equivalent of a regular polygon. This case is called *n -fold helical symmetry*, where $n = 360^\circ / \theta$, see e.g. double helix. This concept can be further generalized to include cases where $m\theta$ is a multiple of 360° —that is, the cycle does eventually repeat, but only after more than one full rotation of the helical object.
- **Non-repeating helical symmetry.** This is the case in which the angle of rotation θ required to observe the symmetry is irrational. The angle of rotation never repeats exactly no matter how many times the helix is rotated. Such symmetries are created by using a non-repeating point group in two dimensions. DNA is an example of this type of non-repeating helical symmetry.

Non-isometric symmetries

A wider definition of geometric symmetry allows operations from a larger group than the Euclidean group of isometries. Examples of larger geometric symmetry groups are:

- The group of similarity transformations, i.e. affine transformations represented by a matrix A that is a scalar times an orthogonal matrix. Thus dilations are added, self-similarity is considered a symmetry.
- The group of affine transformations represented by a matrix A with determinant 1 or -1, i.e. the transformations which preserve area; this adds e.g. oblique reflection symmetry.
- The group of all bijective affine transformations.
- The group of Möbius transformations which preserve cross-ratios.

In Felix Klein's Erlangen program, each possible group of symmetries defines a geometry in which objects that are related by a member of the symmetry group are considered to be equivalent. For example, the Euclidean group

defines Euclidean geometry, whereas the group of Möbius transformations defines projective geometry.

Scale symmetry and fractals

Scale symmetry refers to the idea that if an object is expanded or reduced in size, the new object has the same properties as the original. Scale symmetry is notable for the fact that it does *not* exist for most physical systems, a point that was first discerned by Galileo. Simple examples of the lack of scale symmetry in the physical world include the difference in the strength and size of the legs of elephants versus mice, and the observation that if a candle made of soft wax was enlarged to the size of a tall tree, it would immediately collapse under its own weight.

A more subtle form of scale symmetry is demonstrated by fractals. As conceived by Benoît Mandelbrot, fractals are a mathematical concept in which the structure of a complex form looks similar or even exactly the same no matter what degree of magnification is used to examine it. A coast is an example of a naturally occurring fractal, since it retains roughly comparable and similar-appearing complexity at every level from the view of a satellite to a microscopic examination of how the water laps up against individual grains of sand. The branching of trees, which enables children to use small twigs as stand-ins for full trees in dioramas, is another example.

This similarity to naturally occurring phenomena provides fractals with an everyday familiarity not typically seen with mathematically generated functions. As a consequence, they can produce strikingly beautiful results such as the Mandelbrot set. Intriguingly, fractals have also found a place in CG, or computer-generated movie effects, where their ability to create very complex curves with fractal symmetries results in more realistic virtual worlds.

Symmetry in mathematics

In formal terms, we say that a mathematical object is *symmetric* with respect to a given mathematical operation, if, when applied to the object, this operation preserves some property of the object. The set of operations that preserve a given property of the object form a group. Two objects are symmetric to each other with respect to a given group of operations if one is obtained from the other by some of the operations (and vice versa).

Mathematical model for symmetry

The set of all symmetry operations considered, on all objects in a set X , can be modeled as a group action $g : G \times X \rightarrow X$, where the image of g in G and x in X is written as $g \cdot x$. If, for some g , $g \cdot x = y$ then x and y are said to be symmetrical to each other. For each object x , operations g for which $g \cdot x = x$ form a group, the **symmetry group** of the object, a subgroup of G . If the symmetry group of x is the trivial group then x is said to be **asymmetric**, otherwise **symmetric**. A general example is that G is a group of bijections $g : V \rightarrow V$ acting on the set of functions $x : V \rightarrow W$ by $(gx)(v) = x(g^{-1}(v))$ (or a restricted set of such functions that is closed under the group action). Thus a group of bijections of space induces a group action on "objects" in it. The symmetry group of x consists of all g for which $x(v) = x(g(v))$ for all v . G is the symmetry group of the space itself, and of any object that is uniform throughout space. Some subgroups of G may not be the symmetry group of any object. For example, if the group contains for every v and w in V a g such that $g(v) = w$, then only the symmetry groups of constant functions x contain that group. However, the symmetry group of constant functions is G itself.

In a modified version for vector fields, we have $(gx)(v) = h(g, x(g^{-1}(v)))$ where h rotates any vectors and pseudovectors in x , and inverts any vectors (but not pseudovectors) according to rotation and inversion in g , see symmetry in physics. The symmetry group of x consists of all g for which $x(v) = h(g, x(g(v)))$ for all v . In this case the symmetry group of a constant function may be a proper subgroup of G : a constant vector has only rotational symmetry with respect to rotation about an axis if that axis is in the direction of the vector, and only inversion symmetry if it is zero.

For a common notion of symmetry in Euclidean space, G is the Euclidean group $E(n)$, the group of isometries, and V is the Euclidean space. The **rotation group** of an object is the symmetry group if G is restricted to $E^+(n)$, the group

of direct isometries. (For generalizations, see the next subsection.) Objects can be modeled as functions x , of which a value may represent a selection of properties such as color, density, chemical composition, etc. Depending on the selection we consider just symmetries of sets of points (x is just a Boolean function of position v), or, at the other extreme, e.g. symmetry of right and left hand with all their structure.

For a given symmetry group, the properties of part of the object, fully define the whole object. Considering points equivalent which, due to the symmetry, have the same properties, the equivalence classes are the orbits of the group action on the space itself. We need the value of x at one point in every orbit to define the full object. A set of such representatives forms a fundamental domain. The smallest fundamental domain does not have a symmetry; in this sense, one can say that symmetry relies upon asymmetry.

An object with a desired symmetry can be produced by choosing for every orbit a single function value. Starting from a given object x we can e.g.:

- take the values in a fundamental domain (i.e., add copies of the object)
- take for each orbit some kind of average or sum of the values of x at the points of the orbit (ditto, where the copies may overlap)

If it is desired to have no more symmetry than that in the symmetry group, then the object to be copied should be asymmetric.

As pointed out above, some groups of isometries are not the symmetry group of any object, except in the modified model for vector fields. For example, this applies in 1D for the group of all translations. The fundamental domain is only one point, so we can not make it asymmetric, so any "pattern" invariant under translation is also invariant under reflection (these are the uniform "patterns").

In the vector field version continuous translational symmetry does not imply reflectional symmetry: the function value is constant, but if it contains nonzero vectors, there is no reflectional symmetry. If there is also reflectional symmetry, the constant function value contains no nonzero vectors, but it may contain nonzero pseudovectors. A corresponding 3D example is an infinite cylinder with a current perpendicular to the axis; the magnetic field (a pseudovector) is, in the direction of the cylinder, constant, but nonzero. For vectors (in particular the current density) we have symmetry in every plane perpendicular to the cylinder, as well as cylindrical symmetry. This cylindrical symmetry without mirror planes through the axis is also only possible in the vector field version of the symmetry concept. A similar example is a cylinder rotating about its axis, where magnetic field and current density are replaced by angular momentum and velocity, respectively.

A symmetry group is said to act transitively on a repeated feature of an object if, for every pair of occurrences of the feature there is a symmetry operation mapping the first to the second. For example, in 1D, the symmetry group of $\{..., 1, 2, 5, 6, 9, 10, 13, 14, ...\}$ acts transitively on all these points, while $\{..., 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, ...\}$ does *not* act transitively on all points. Equivalently, the first set is only one conjugacy class with respect to isometries, while the second has two classes.

Symmetric functions

A symmetric function is a function which is unchanged by any permutation of its variables. For example, $x + y + z$ and $xy + yz + xz$ are symmetric functions, whereas $x^2 - yz$ is not.

A function may be unchanged by a sub-group of all the permutations of its variables. For example, $ac + 3ab + bc$ is unchanged if a and b are exchanged; its symmetry group is isomorphic to C_2 .

Symmetry in logic

A dyadic relation R is symmetric if and only if, whenever it's true that Rab , it's true that Rba . Thus, "is the same age as" is symmetrical, for if Paul is the same age as Mary, then Mary is the same age as Paul.

Symmetric binary logical connectives are "and" ($\wedge$, and), "or" ($\vee$), "biconditional" (if and only if) ($\leftrightarrow$), NAND ("not-and"), XOR ("not-biconditional"), and NOR ("not-or").

Symmetry in science

Symmetry in physics

Symmetry in physics has been generalized to mean invariance—that is, lack of any visible change—under any kind of transformation, for example arbitrary coordinate transformations. This concept has become one of the most powerful tools of theoretical physics, as it has become evident that practically all laws of nature originate in symmetries. In fact, this role inspired the Nobel laureate PW Anderson to write in his widely read 1972 article *More is Different* that "it is only slightly overstating the case to say that physics is the study of symmetry." See Noether's theorem (which, in greatly simplified form, states that for every continuous mathematical symmetry, there is a corresponding conserved quantity; a conserved current, in Noether's original language); and also, Wigner's classification, which says that the symmetries of the laws of physics determine the properties of the particles found in nature.

Symmetry in physical objects

Classical objects

Although an everyday object may appear exactly the same after a symmetry operation such as a rotation or an exchange of two identical parts has been performed on it, it is readily apparent that such a symmetry is true only as an approximation for any ordinary physical object.

For example, if one rotates a precisely machined aluminum equilateral triangle 120 degrees around its center, a casual observer brought in before and after the rotation will be unable to decide whether or not such a rotation took place. However, the reality is that each corner of a triangle will always appear unique when examined with sufficient precision. An observer armed with sufficiently detailed measuring equipment such as optical or electron microscopes will not be fooled; he will immediately recognize that the object has been rotated by looking for details such as crystals or minor deformities.

Such simple thought experiments show that assertions of symmetry in everyday physical objects are always a matter of approximate similarity rather than of precise mathematical sameness. The most important consequence of this approximate nature of symmetries in everyday physical objects is that such symmetries have minimal or no impacts on the physics of such objects. Consequently, only the deeper symmetries of space and time play a major role in classical physics—that is, the physics of large, everyday objects.

Quantum objects

Remarkably, there exists a realm of physics for which mathematical assertions of simple symmetries in real objects cease to be approximations. That is the domain of quantum physics, which for the most part is the physics of very small, very simple objects such as electrons, protons, light, and atoms.

Unlike everyday objects, objects such as electrons have very limited numbers of configurations, called states, in which they can exist. This means that when symmetry operations such as exchanging the positions of components are applied to them, the resulting new configurations often cannot be distinguished from the originals no matter how diligent an observer is. Consequently, for sufficiently small and simple objects the generic mathematical symmetry assertion $F(x) = x$ ceases to be approximate, and instead becomes an experimentally precise and accurate description of the situation in the real world.

Consequences of quantum symmetry

While it makes sense that symmetries could become exact when applied to very simple objects, the immediate intuition is that such a detail should not affect the physics of such objects in any significant way. This is in part because it is very difficult to view the concept of exact similarity as physically meaningful. Our mental picture of such situations is invariably the same one we use for large objects: We picture objects or configurations that are very, very similar, but for which if we could "look closer" we would still be able to tell the difference.

However, the assumption that exact symmetries in very small objects should not make any difference in their physics was discovered in the early 1900s to be spectacularly incorrect. The situation was succinctly summarized by Richard Feynman in the direct transcripts of his Feynman Lectures on Physics, Volume III, Section 3.4, *Identical particles*. (Unfortunately, the quote was edited out of the printed version of the same lecture.)

"... if there is a physical situation in which it is impossible to tell which way it happened, it *always* interferes; it *never* fails."

The word "interferes" in this context is a quick way of saying that such objects fall under the rules of quantum mechanics, in which they behave more like waves that interfere than like everyday large objects.

In short, when an object becomes so simple that a symmetry assertion of the form $F(x) = x$ becomes an exact statement of experimentally verifiable sameness, x ceases to follow the rules of classical physics and must instead be modeled using the more complex—and often far less intuitive—rules of quantum physics.

This transition also provides an important insight into why the mathematics of symmetry are so deeply intertwined with those of quantum mechanics. When physical systems make the transition from symmetries that are approximate to ones that are exact, the mathematical expressions of those symmetries cease to be approximations and are transformed into precise definitions of the underlying nature of the objects. From that point on, the correlation of such objects to their mathematical descriptions becomes so close that it is difficult to separate the two.

Generalizations of symmetry

If we have a given set of objects with some structure, then it is possible for a symmetry to merely convert only one object into another, instead of acting upon all possible objects simultaneously. This requires a generalization from the concept of symmetry group to that of a groupoid. Indeed, A. Connes in his book 'Non-commutative geometry' writes that Heisenberg discovered quantum mechanics by considering the groupoid of transitions of the hydrogen spectrum.

The notion of groupoid also leads to notions of multiple groupoids, namely sets with many compatible groupoid structures, a structure which trivialises to abelian groups if one restricts to groups. This leads to prospects of 'higher order symmetry' which have been a little explored, as follows.

The automorphisms of a set, or a set with some structure, form a group, which models a homotopy 1-type. The automorphisms of a group G naturally form a crossed module $G \rightarrow \text{Aut}(G)$, and crossed modules give an

algebraic model of homotopy 2-types. At the next stage, automorphisms of a crossed module fit into a structure known as a crossed square, and this structure is known to give an algebraic model of homotopy 3-types. It is not known how this procedure of generalising symmetry may be continued, although crossed n -cubes have been defined and used in algebraic topology, and these structures are only slowly being brought into theoretical physics.^{[7] [8]}

Physicists have come up with other directions of generalization, such as supersymmetry and quantum groups, yet the different options are indistinguishable during various circumstances.

Symmetry in chemistry

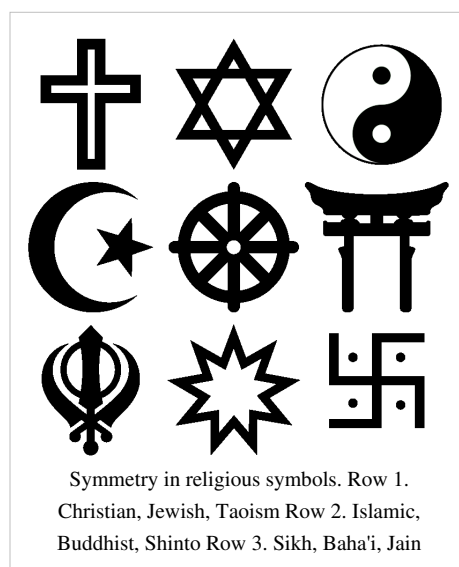
Symmetry is important to chemistry because it explains observations in spectroscopy, quantum chemistry and crystallography. It draws heavily on group theory.

Symmetry in history, religion, and culture

In any human endeavor for which an impressive visual result is part of the desired objective, symmetries play a profound role. The innate appeal of symmetry can be found in our reactions to happening across highly symmetrical natural objects, such as precisely formed crystals or beautifully spiraled seashells. Our first reaction in finding such an object often is to wonder whether we have found an object created by a fellow human, followed quickly by surprise that the symmetries that caught our attention are derived from nature itself. In both reactions we give away our inclination to view symmetries both as beautiful and, in some fashion, informative of the world around us.

Symmetry in religious symbols

The tendency of people to see purpose in symmetry suggests at least one reason why symmetries are often an integral part of the symbols of world religions. Just a few of many examples include the sixfold rotational symmetry of Judaism's Star of David, the twofold point symmetry of Taoism's Taijitu or Yin-Yang, the bilateral symmetry of Christianity's cross and Sikhism's Khanda, or the fourfold point symmetry of Jain's ancient (and peacefully intended) version of the swastika. With its strong prohibitions against the use of representational images, Islam, and in particular the Sunni branch of Islam, has developed intricate use of symmetries.



Symmetry in Social Interactions

People observe the symmetrical nature, often including asymmetrical balance, of social interactions in a variety of contexts. These include assessments of reciprocity, empathy, apology, dialog, respect, justice, and revenge. Symmetrical interactions send the message "we are all the same" while asymmetrical interactions send the message "I am special; better than you." Peer relationships are based on symmetry, power relationships are based on asymmetry.^[9]

Symmetry in architecture

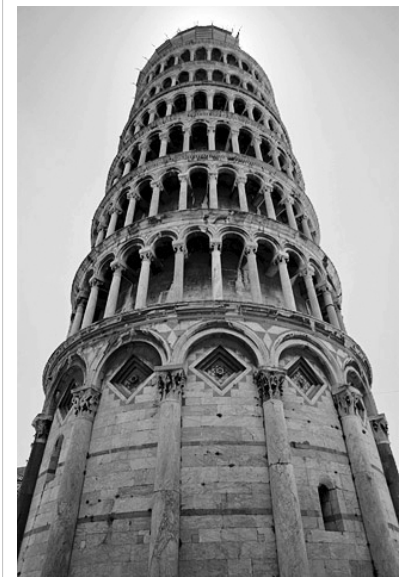
Another human endeavor in which the visual result plays a vital part in the overall result is architecture. Both in ancient times, the ability of a large structure to impress or even intimidate its viewers has often been a major part of its purpose, and the use of symmetry is an inescapable aspect of how to accomplish such goals.

Just a few examples of ancient examples of architectures that made powerful use of symmetry to impress those around them included the Egyptian Pyramids, the Greek Parthenon, the first and second Temple of Jerusalem,

China's Forbidden City, Cambodia's Angkor Wat complex, and the many temples and pyramids of ancient Pre-Columbian civilizations. More recent historical examples of architectures emphasizing symmetries include Gothic architecture cathedrals, and American President Thomas Jefferson's Monticello home. India's unparalleled Taj Mahal is in a category by itself, as it may arguably be one of the most impressive and beautiful uses of symmetry in architecture that the world has ever seen.

An interesting example of a broken symmetry in architecture is the Leaning Tower of Pisa, whose notoriety stems in no small part not for the intended symmetry of its design, but for the violation of that symmetry from the lean that developed while it was still under construction. Modern examples of architectures that make impressive or complex use of various symmetries include Australia's Sydney Opera House and Houston, Texas's simpler Astrodome.

Symmetry finds its ways into architecture at every scale, from the overall external views, through the layout of the individual floor plans, and down to the design of individual building elements such as intricately caved doors, stained glass windows, tile mosaics, friezes, stairwells, stair rails, and balustrades. For sheer complexity and sophistication in the exploitation of symmetry as an architectural element, Islamic buildings such as the Taj Mahal often eclipse those of other cultures and ages, due in part to the general prohibition of Islam against using images of people or animals.^{[10] [11]}



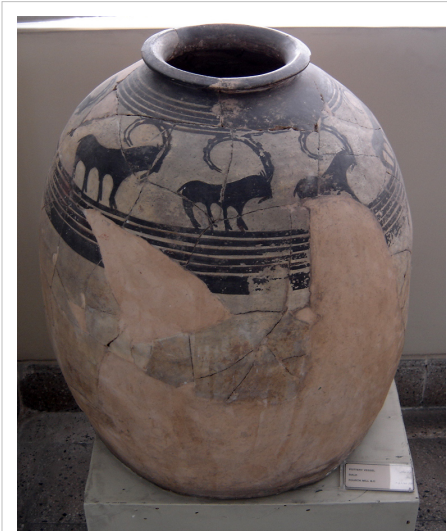
Leaning Tower of Pisa

Symmetry in pottery and metal vessels

Since the earliest uses of pottery wheels to help shape clay vessels, pottery has had a strong relationship to symmetry. As a minimum, pottery created using a wheel necessarily begins with full rotational symmetry in its cross-section, while allowing substantial freedom of shape in the vertical direction. Upon this inherently symmetrical starting point cultures from ancient times have tended to add further patterns that tend to exploit or in many cases reduce the original full rotational symmetry to a point where some specific visual objective is achieved. For example, Persian pottery dating from the fourth millennium B.C. and earlier used symmetric zigzags, squares, cross-hatchings, and repetitions of figures to produce more complex and visually striking overall designs.

Cast metal vessels lacked the inherent rotational symmetry of wheel-made pottery, but otherwise provided a similar opportunity to decorate their surfaces with patterns pleasing to those who used them.

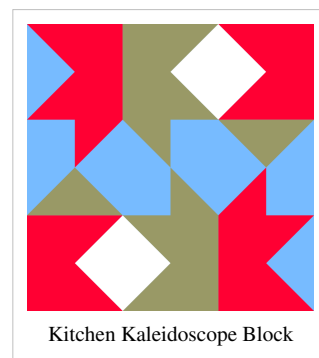
The ancient Chinese, for example, used symmetrical patterns in their bronze castings as early as the 17th century B.C. Bronze vessels exhibited both a bilateral main motif and a repetitive translated border design.^{[12] [13] [14]}



Persian vessel (4th millennium B.C.)

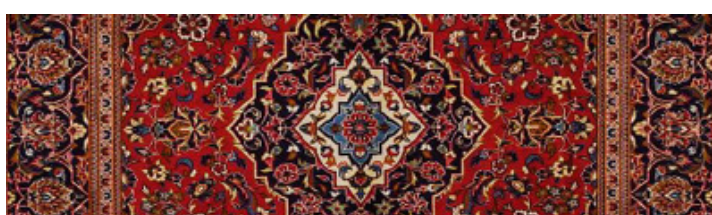
Symmetry in quilts

As quilts are made from square blocks (usually 9, 16, or 25 pieces to a block) with each smaller piece usually consisting of fabric triangles, the craft lends itself readily to the application of symmetry.^[15]



Symmetry in carpets and rugs

A long tradition of the use of symmetry in carpet and rug patterns spans a variety of cultures. American Navajo Indians used bold diagonals and rectangular motifs. Many Oriental rugs have intricate reflected centers and borders that translate a pattern. Not surprisingly, rectangular rugs typically use quadrilateral symmetry—that is, motifs that are reflected across both the horizontal and vertical axes.^{[16] [17]}

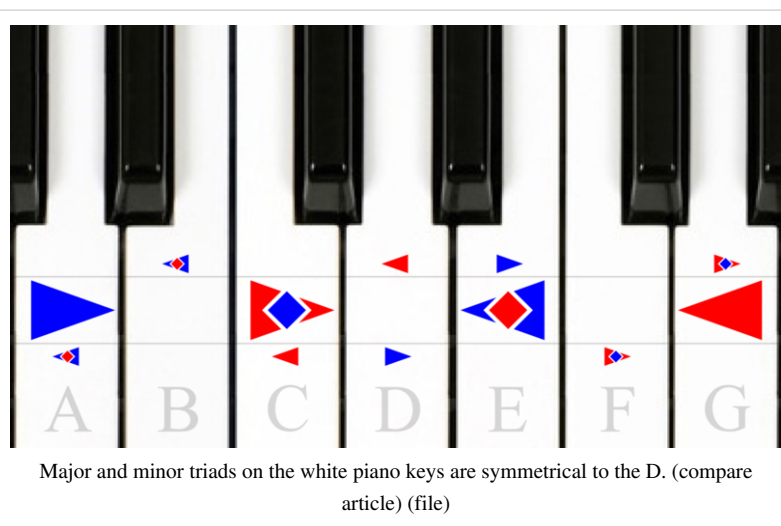


Symmetry in music

Symmetry is of course not restricted to the visual arts. Its role in the history of music touches many aspects of the creation and perception of music.

Musical form

Symmetry has been used as a formal constraint by many composers, such as the arch (swell) form (ABCBA) used by Steve Reich, Béla Bartók, and James Tenney. In classical music, Bach used the symmetry concepts of permutation and invariance.^[18]



Pitch structures

Symmetry is also an important consideration in the formation of scales and chords, traditional or tonal music being made up of non-symmetrical groups of pitches, such as the diatonic scale or the major chord. Symmetrical scales or chords, such as the whole tone scale, augmented chord, or diminished seventh chord (diminished-diminished seventh), are said to lack direction or a sense of forward motion, are ambiguous as to the key or tonal center, and

have a less specific diatonic functionality. However, composers such as Alban Berg, Béla Bartók, and George Perle have used axes of symmetry and/or interval cycles in an analogous way to keys or non-tonal tonal centers.

Perle (1992) explains "C-E, D-F#, [and] Eb-G, are different instances of the same interval...the other kind of identity. .has to do with axes of symmetry. C-E belongs to a family of symmetrically related dyads as follows:"

D D# E F F# G G#
D C# C B A# A G#

Thus in addition to being part of the interval-4 family, C-E is also a part of the sum-4 family (with C equal to 0).

+ 2 3 4 5 6 7 8
2 1 0 11 10 9 8
4 4 4 4 4 4 4

Interval cycles are symmetrical and thus non-diatonic. However, a seven pitch segment of C5 (the cycle of fifths, which are enharmonic with the cycle of fourths) will produce the diatonic major scale. Cyclic tonal progressions in the works of Romantic composers such as Gustav Mahler and Richard Wagner form a link with the cyclic pitch successions in the atonal music of Modernists such as Bartók, Alexander Scriabin, Edgard Varèse, and the Vienna school. At the same time, these progressions signal the end of tonality.

The first extended composition consistently based on symmetrical pitch relations was probably Alban Berg's *Quartet*, Op. 3 (1910). (Perle, 1990)

Equivalency

Tone rows or pitch class sets which are invariant under retrograde are horizontally symmetrical, under inversion vertically. See also Asymmetric rhythm.

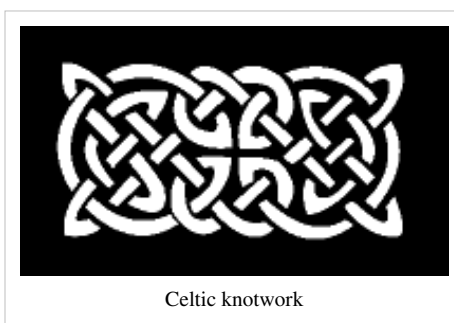
Symmetry in other arts and crafts

The concept of symmetry is applied to the design of objects of all shapes and sizes. Other examples include beadwork, furniture, sand paintings, knotwork, masks, musical instruments, and many other endeavors.

Symmetry in aesthetics

The relationship of symmetry to aesthetics is complex. Certain simple symmetries, and in particular bilateral symmetry, seem to be deeply ingrained in the inherent perception by humans of the likely health or fitness of other living creatures, as can be seen by the simple experiment of distorting one side of the image of an attractive face and asking viewers to rate the attractiveness of the resulting image. Consequently, such symmetries that mimic biology tend to have an innate appeal that in turn drives a powerful tendency to create artifacts with similar symmetry. One only needs to imagine the difficulty in trying to market a highly asymmetrical car or truck to general automotive buyers to understand the power of biologically inspired symmetries such as bilateral symmetry.

Another more subtle appeal of symmetry is that of simplicity, which in turn has an implication of safety, security, and familiarity. A highly symmetrical room, for example, is unavoidably also a room in which anything out of place or potentially threatening can be identified easily and quickly. For example, people who have grown up in houses full of exact right angles and precisely identical artifacts can find their first experience in staying in a room with *no* exact right angles and *no* exactly identical artifacts to be highly disquieting. Symmetry thus can be a source of



Celtic knotwork

comfort not only as an indicator of biological health, but also of a safe and well-understood living environment.

Opposed to this is the tendency for excessive symmetry to be perceived as boring or uninteresting. Humans in particular have a powerful desire to exploit new opportunities or explore new possibilities, and an excessive degree of symmetry can convey a lack of such opportunities. Most people display a preference for figures that have a certain degree of simplicity and symmetry, but enough complexity to make them interesting.^[19]

Yet another possibility is that when symmetries become too complex or too challenging, the human mind has a tendency to "tune them out" and perceive them in yet another fashion: as noise that conveys no useful information.

Finally, perceptions and appreciation of symmetries are also dependent on cultural background. The far greater use of complex geometric symmetries in many Islamic cultures, for example, makes it more likely that people from such cultures will appreciate such art forms (or, conversely, to rebel against them).

As in many human endeavors, the result of the confluence of many such factors is that effective use of symmetry in art and architecture is complex, intuitive, and highly dependent on the skills of the individuals who must weave and combine such factors within their own creative work. Along with texture, color, proportion, and other factors, symmetry is a powerful ingredient in any such synthesis; one only need to examine the Taj Mahal to powerful role that symmetry plays in determining the aesthetic appeal of an object.

Modernist architecture rejects symmetry, stating *only a bad architect relies on symmetry*; instead of symmetrical layout of blocks, masses and structures, Modernist architecture relies on wings and balance of masses. This notion of getting rid of symmetry was first encountered in International style. Some people find asymmetrical layouts of buildings and structures revolutionizing; other find them restless, boring and unnatural.

A few examples of the more explicit use of symmetries in art can be found in the remarkable art of M. C. Escher, the creative design of the mathematical concept of a wallpaper group, and the many applications (both mathematical and real world) of tiling.

Notes

- [1] Penrose, Roger (2007). *Fearful Symmetry*. City: Princeton. ISBN 9780691134826.
- [2] For example, Aristotle ascribed spherical shape to the heavenly bodies, attributing this formally defined geometric measure of symmetry to the natural order and perfection of the cosmos.
- [3] Weyl 1982
- [4] For example, operations such as moving across a regularly patterned tile floor or rotating an eight-sided vase, or complex transformations of equations or in the way music is played.
- [5] See e.g., Mainzer, Klaus (2005). *Symmetry And Complexity: The Spirit and Beauty of Nonlinear Science*. World Scientific. ISBN 9812561927.
- [6] Symmetric objects can be material, such as a person, crystal, quilt, floor tiles, or molecule, or it can be an abstract structure such as a mathematical equation or a series of tones (music).
- [7] n-category cafe (<http://golem.ph.utexas.edu/category/>) - discussion of n -groups
- [8] 'Higher dimensional group theory' (<http://www.bangor.ac.uk/r.brown/hdaweb2.htm>)
- [9] Emotional Competency (<http://www.emotionalcompetency.com/symmetry.htm>) Entry describing Symmetry
- [10] Williams: Symmetry in Architecture (<http://members.tripod.com/vismath/kim/>)
- [11] Aslaksen: Mathematics in Art and Architecture (<http://www.math.nus.edu.sg/aslaksen/teaching/math-art-arch.shtml>)
- [12] Chinavoc: The Art of Chinese Bronzes (<http://www.chinavoc.com/arts/handicraft/bronze.htm>)
- [13] Grant: Iranian Pottery in the Oriental Institute (http://www-oi.uchicago.edu/OI/MUS/VOL/NN_SUM94/NN_Sum94.html)
- [14] The Metropolitan Museum of Art - Islamic Art (<http://www.metmuseum.org/collections/department.asp?dep=14>)
- [15] Quate: Exploring Geometry Through Quilts (<http://its.guilford.k12.nc.us/webquests/quilts/quilts.htm>)
- [16] Mallet: Tribal Oriental Rugs (<http://www.marlamallett.com/default.htm>)
- [17] Dilucchio: Navajo Rugs (<http://navajocentral.org/rugs.htm>)
- [18] see ("Fugue No. 21," pdf (<http://jan.ucc.nau.edu/~tas3/wtc/ii21s.pdf>) or Shockwave (<http://jan.ucc.nau.edu/~tas3/wtc/ii21.html>))
- [19] Arnheim, Rudolf (1969). *Visual Thinking*. University of California Press.

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External links

- Calotta: A World of Symmetry (<http://www.teachersnetwork.org/teachnet/westchester/symmetry.htm>)
 - Dutch: Symmetry Around a Point in the Plane (<http://www.uwgb.edu/dutchs/SYMMETRY/2DPTGRP.HTM>)
 - Chapman: Aesthetics of Symmetry (<http://home.earthlink.net/~jdc24/symmetry.htm>)
 - ISIS Symmetry (<http://www.mi.sanu.ac.rs/~jablans/isis0.htm>)
 - *International Symmetry Association - ISA* (http://www.symmetry.hu/isa_articles.html)
 - Institute Symmetrion (http://www.symmetry.hu/aus_symmetrion.html)
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Representations

<i>Representations</i>	
	
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Representations is an interdisciplinary journal in the humanities and social sciences published by the University of California Press, in Berkeley, California. Published quarterly since 1983, it is the founding publication of the New Historicism movement of the 1980s. It covers topics including literary, historical, and cultural studies.

Beginning

Representations originated as the intellectual project of a group of UC Berkeley professors in the early 1980s, and its first issue was published in February 1983. The founding editorial board was chaired by Stephen Greenblatt^[3] and Svetlana Alpers.

Scope

Representations publishes articles that employ a theoretically inflected, empirically grounded, and interdisciplinary treatment of phenomena. The stated aim of the journal is "to transform and enrich the understanding of cultures" by examining "the symbolic dimension of social practice and the social dimension of artistic practice."

The journal frequently publishes thematic special issues that articulate the terms of new and recurring debates—for example, the 2007 issue on the legacies of American Orientalism^[4], the 2006 issue on cross-cultural mimesis^[5], and the 2005 issue on political and intellectual redress.^[6]

In both its thematic and general issues *Representations* seeks to encourage innovative research among scholars who explore the way artifacts, institutions, and modes of thought both reflect and give a heightened account of the social, cultural, and historical circumstances in which they arise.

Topics of concern

- The Body, Gender, and Sexuality
- Culture and Law
- Empire, Imperialism, and The New World
- History and Memory
- Music
- Narrative and Poetics
- National Identities
- Philosophy and Religion
- Politics and Aesthetics
- Race and Ethnicity
- Science Studies
- Society, Class, and Power
- Visual Culture

Anthologized texts from the journal

The UC Press Representations Books series has collected and reprinted many essays originally published in the journal, including:

- *The Making of the Modern Body: Sexuality and Society in the Nineteenth Century*, edited by Catherine Gallagher and Thomas Laqueur
- *Representing the English Renaissance*, edited by Stephen Greenblatt
- *Misogyny, Misandry, Misanthropy*, edited by R. Howard Bloch and Frances Ferguson
- *Law and the Order of Culture*, edited by Robert Post
- *The New American Studies: Essays from Representations*, edited by Philip Fisher
- *New World Encounters*, edited by Stephen Greenblatt
- *Future Libraries*, edited by R. Howard Bloch and Carla Hesse
- *The Fate of "Culture": Geertz and Beyond*, edited by Sherry B. Ortner

Notes

- [1] <http://www.worldcat.org/issn/0734-6018>
- [2] <http://www.representations.org>
- [3] Representations (http://www.representations.org/vision_board.php)
- [4] <http://caliber.ucpress.net/toc/rep/99/1> Forms of Asia
- [5] <http://caliber.ucpress.net/toc/rep/94/1> cf.
- [6] University of California Press - Error (<http://caliber.ucpress.net/toc/rep/92/1>)

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- Randolph Starn, "Making Representations," *Chronicle of the University of California* 6 (Spring 2004): 160-67 (<http://cshe.berkeley.edu/publications/publications.php?id=20>).
- H. Aram Veesser, ed., *The New Historicism Reader* (<http://www.routledge.com/books/The-New-Historicism-Reader-isbn9780415907828>), New York: Routledge, 1994.

External links

- Official website (<http://www.representations.org>)

Representations of the symmetric group

In mathematics, the **representation theory of the symmetric group** is a particular case of the representation theory of finite groups, for which a concrete and detailed theory can be obtained. This has a large area of potential applications, from symmetric function theory to problems of quantum mechanics for a number of identical particles.

The symmetric group S_n has order $n!$. Its conjugacy classes are labeled by partitions of n . Therefore according to the representation theory of a finite group, the number of inequivalent irreducible representations, over the complex numbers, is equal to the number of partitions of n . Unlike the general situation for finite groups, there is in fact a natural way to parametrize irreducible representation by the same set that parametrizes conjugacy classes, namely by partitions of n or equivalently Young diagrams of size n .

Each such irreducible representation can in fact be realized over the integers (every permutation acting by a matrix with integer coefficients); it can be explicitly constructed by computing the Young symmetrizers acting on a space generated by the Young tableaux of shape given by the Young diagram.

Over other fields the situation can become much more complicated. If the field K has characteristic equal to zero or greater than n then by Maschke's theorem the group algebra KS_n is semisimple. In these cases the irreducible representations defined over the integers give the complete set of irreducible representations (after reduction modulo the characteristic if necessary).

However, the irreducible representations of the symmetric group are not known in arbitrary characteristic. In this context it is more usual to use the language of modules rather than representations. The representation obtained from an irreducible representation defined over the integers by reducing modulo the characteristic will not in general be irreducible. The modules so constructed are called *Specht modules*, and every irreducible does arise inside some such module. There are now fewer irreducibles, and although they can be classified they are very poorly understood. For example, even their dimensions are not known in general.

The determination of the irreducible modules for the symmetric group over an arbitrary field is widely regarded as one of the most important open problems in representation theory.

Low dimensional representations

The lowest dimensional representations of the symmetric groups can be described explicitly, as done in (Burnside 1955, p. 468). This work was extended to the smallest k degrees (explicitly for $k=4$, and $k=7$) in (Rasala 1977), and over arbitrary fields in (James 1983). The smallest two degrees in characteristic zero are described here:

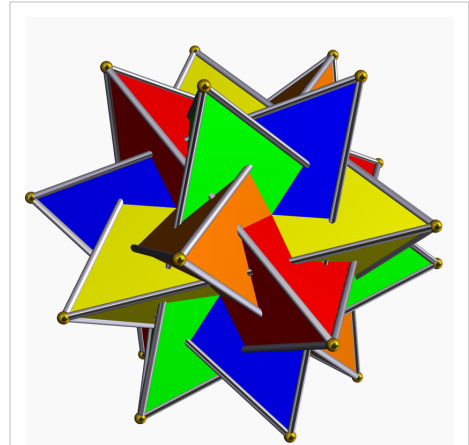
Every symmetric group has a one-dimensional representation called the **trivial representation**, where every element acts as the one by one identity matrix. For $n \geq 2$, there is another irreducible representation of degree 1, called the **sign representation** or **alternating character**, which takes a permutation to the one by one matrix with entry ± 1 based on the sign of the permutation. These are the only one-dimensional representations of the symmetric groups, as one-dimensional representations are abelian, and the abelianization of the symmetric group is C_2 , the cyclic group of order 2.

For all n , there is an n -dimensional representation of the symmetric group of order n , called the **permutation representation**, which consists of permuting n coordinates. This has the trivial subrepresentation consisting of vectors whose coordinates are all equal. The orthogonal complement consists of those vectors whose coordinates sum to zero, and when $n \geq 2$, the representation on this subspace is an $n-1$ dimensional irreducible representation. Another $n-1$ dimensional irreducible representation is found by tensoring with the sign representation.

For $n \geq 7$, these are the lowest-dimensional irreducible representations of S_n – all other irreducible representations have dimension at least n . However for $n = 4$, the surjection from S_4 to S_3 allows S_4 to inherit a two-dimensional irreducible representation. For $n = 6$, the exceptional transitive embedding of S_5 into S_6 produces another pair of five-dimensional irreducible representations.

Alternating group

The representation theory of the alternating groups is similar, though the sign representation disappears. For $n \geq 7$, the lowest dimensional irreducible representations are the trivial representation in dimension one, and the $(n - 1)$ -dimensional representation from the other summand of the permutation representation, with all other irreducible representations having higher dimension, but there are exceptions for smaller n .



The compound of five tetrahedra, on which A_5 acts, giving a 3-dimensional representation.

The alternating groups for $n \geq 5$ have only one one-dimensional irreducible representation, the trivial representation. For $n = 3, 4$ there are two additional one-dimensional irreducible representations, corresponding to maps to the cyclic group of order 3: $A_3 \cong C_3$ and $A_4 \rightarrow A_4/V \cong C_3$.

- For $n \geq 7$, there is just one irreducible representation of degree $n - 1$, and this is the smallest degree of a non-trivial irreducible representation.
- For $n = 3$ the obvious analogue of the $(n - 1)$ -dimensional representation is reducible – the permutation representation coincides with the regular representation, and thus breaks up into the three one-dimensional representations, as $A_3 \cong C_3$ is abelian; see the discrete Fourier transform for representation theory of cyclic groups.
- For $n = 4$, there is just one $n - 1$ irreducible representation, but there are the exceptional irreducible representations of dimension 1.
- For $n = 5$, there are two dual irreducible representations of dimension 3, corresponding to its action as icosahedral symmetry.
- For $n = 6$, there is an extra irreducible representation of dimension 5 corresponding to the exceptional transitive embedding of A_5 in A_6 .

Notes

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Representations of a finite group

In mathematics, representation theory is a technique for analyzing abstract groups in terms of groups of linear transformations. See the article on group representations for an introduction. This article discusses the representation theory of groups that have a finite number of elements.

Basic definitions

All the linear representations in this article are finite dimensional and assumed to be complex unless otherwise stated. A **representation of G** is a group homomorphism $\rho: G \rightarrow \text{GL}(n, \mathbb{C})$ from G to the general linear group $\text{GL}(n, \mathbb{C})$. Thus to specify a representation, we just assign a square matrix to each element of the group, in such a way that the matrices behave in the same way as the group elements when multiplied together.

We say that ρ is a **real representation of G** if the matrices are real. In other words if $\rho(G) \subset \text{GL}(n, \mathbb{R})$.

Other formulations

A representation $\rho: G \rightarrow \text{GL}(n, \mathbb{C})$ defines a group action of G on the vector space $\mathbb{C}^n$. Moreover this action completely determines ρ . Hence to specify a representation it is enough to specify how it acts on its representing vector space.

Alternatively, the action of a group G on a complex vector space V induces a left action of group algebra $\mathbb{C}[G]$ on the vector space V , and vice-versa. Hence representations are equivalent to left $\mathbb{C}[G]$ -modules.

The group algebra $\mathbb{C}[G]$ is a $|G|$ -dimensional algebra over the complex numbers, on which G acts. (See Peter–Weyl for the case of compact groups.) In fact $\mathbb{C}[G]$ is a representation for $G \times G$. More specifically, if g_1 and g_2 are elements of G and h is an element of $\mathbb{C}[G]$ corresponding to the element h of G ,

$$(g_1 \cdot g_2)[h] = g_1 h g_2^{-1}.$$

$\mathbb{C}[G]$ can also be considered as a representation of G in three different ways:

- Conjugation: $g[h] = g h g^{-1}$
- As a left action: $g[h] = g h$ (a regular representation)

- As a right action: $g[h] = h g^{-1}$ (also);
- these are all to be 'found' inside the $G \times G$ action.

Example

For many groups it is entirely natural to represent the group through matrices. Consider for example the dihedral group D_4 of symmetries of a square. This is generated by the two reflection matrices

$$m = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Here m is a reflection that maps (x,y) to $(-x,y)$, while n maps (x,y) to (y,x) . Multiplying these matrices together creates a set of 8 matrices that form the group. As discussed above, we can either think of the representation in terms of the matrices, or in terms of the action on the two-dimensional vector space (x,y) .

This representation is *faithful* - that is, there is a one-to-one correspondence between the matrices and the elements of the group. It is also *irreducible*, because there is no subspace of (x,y) that is invariant under the action of the group.

Discrete Fourier transform

If G is a finite cyclic group, then its representation theory is called the discrete Fourier transform; this example is central to digital signal processing.

All irreducible representations are 1-dimensional (characters), and correspond to sending a generator of G to a root of unity, not necessarily primitive (the trivial representation sends a generator to 1, for instance).

A function on G is called the *time domain* representation of the function, while the corresponding expression in terms of characters is called the *frequency domain* representation of the function: changing from the time domain description to the frequency domain description is called the *discrete Fourier transform*, and the opposite direction is called the *inverse discrete Fourier transform*.

The character table, which in this case is the matrix of the transform, is the DFT matrix, which is, up to normalization factor, the Vandermonde matrix for the n th roots of unity; the order of rows and columns depends on a choice of generator and primitive root of unity.

The group of characters is isomorphic to G itself, but not naturally so, and is known as the dual group, $\widehat{G}$, in the language of Pontryagin duality, and the original group G can be recovered as the double dual.

Abelian groups

More generally, any finite abelian group is a direct sum of finite cyclic groups (by the fundamental theorem of finitely generated abelian groups, though there is no canonical decomposition), and thus the representation theory of finite abelian groups is completely described by that of finite cyclic groups, that is, by the discrete Fourier transform.

If an abelian group is expressed as a direct product, and the dual group likewise decomposed, and the elements of each sorted in lexicographic order, then the character table of the product group is the Kronecker product (tensor product) of the character tables for the two component groups, which is just a statement that the value of a product homomorphism on a product group is the product of the values: $(\rho \times \sigma)(g, h) = \rho(g) \cdot \sigma(h)$.

Morphisms between representations

Given two representations $\rho: G \rightarrow \text{GL}(n, \mathbb{C})$ and $\tau: G \rightarrow \text{GL}(m, \mathbb{C})$ a morphism between ρ and τ is a linear map $T: \mathbb{C}^n \rightarrow \mathbb{C}^m$ so that for all g in G we have the following commuting relation: $T \circ \rho(g) = \tau(g) \circ T$.

According to Schur's lemma, a non-zero morphism between two irreducible complex representations is invertible, and moreover, is given in matrix form as a scalar multiple of the identity matrix.

This result holds as the complex numbers are algebraically closed. For a counterexample over the real numbers, consider the two dimensional irreducible real representation of the cyclic group $C_4 = \langle x \rangle$ given by:

$$\rho: x \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ defines an automorphism of ρ , which is clearly not a scalar multiple of the identity matrix.

Subrepresentations and irreducible representations

As noted earlier, a representation ρ defines an action on a vector space $\mathbb{C}^n$. It may turn out that $\mathbb{C}^n$ has an invariant subspace $V \subset \mathbb{C}^n$. The action of G is given by complex matrices and this in turn defines a new representation $\sigma: G \rightarrow \text{GL}(V)$. We call σ a subrepresentation of ρ . A representation without subrepresentations is called irreducible.

Constructing new representations from old

There are number of ways to combine representations to obtain new representations. Each of these methods involves the application of a construction from linear algebra to representation theory.

- Given two representations ρ_1, ρ_2 we may construct their direct sum $\rho_1 \oplus \rho_2$ by $(\rho_1 \oplus \rho_2)(g)(v, w) = (\rho_1(g)v, \rho_2(g)w)$.
- The tensor representation of ρ_1, ρ_2 is defined by $(\rho_1 \otimes \rho_2)(v \otimes w) = \rho_1(v) \otimes \rho_2(w)$.
- Let $\rho: G \rightarrow \text{GL}(n, \mathbb{C})$ be a representation. Then ρ induces a representation ρ^* on the dual of the vector space $\text{Hom}(\mathbb{C}^n, \mathbb{C})$. Let $f: \mathbb{C}^n \rightarrow \mathbb{C}$ be a linear functional. The representation ρ^* is then defined by the rule $\rho^*(g)(f) = f(\rho(g)^{-1})$. The representation ρ^* is called either the dual representation or the contragredient representation of ρ .
- Furthermore, if a representation ρ has a subrepresentation σ then the quotient of the representing vector spaces for ρ and σ has a well defined action of G on it. We call the resulting representation the quotient representation of ρ by σ .

Young tableau

For the symmetric groups, a graphical method exists to determine their finite representations that associates with each representation a Young tableau (also known as a Young diagram). The direct product of two representations may easily be decomposed into a direct sum of irreducible representation by a set of rules for the "direct product" of two Young diagrams. Each diagram also contains information about the dimension of the representation to which it corresponds. Young tableaux provide a far cleaner way of working with representations than the algebraic methods that underlie their use.

Applying Schur's lemma

Lemma. If $f: A \otimes B \rightarrow C$ is a morphism of representations, then the corresponding linear transformation obtained by dualizing B is: $f': A \rightarrow C \otimes B^*$ is also a morphism of representations. Similarly, if $g: A \rightarrow B \otimes C$ is a morphism of representations, dualizing it will give another morphism of representations $g': A \otimes C^* \rightarrow B$.

If ρ is an n -dimensional irreducible representation of G with the underlying vector space V , then we can define a $G \times G$ morphism of representations, for all g in G and x in V

$$\begin{aligned} f: \mathbf{C}[G] \otimes (1_G \otimes V) &\rightarrow (V \otimes 1_G) \\ f:(g \otimes x) &= \rho(g)[x] \end{aligned}$$

where 1_G is the trivial representation of G . This defines a $G \times G$ morphism of representations.

Now we use the above lemma and obtain the $G \times G$ morphism of representations

$$f': \bar{V} \otimes V \rightarrow \overline{\mathbf{C}[G]}.$$

The dual representation of $\mathbf{C}[G]$ as a $G \times G$ -representation is equivalent to $\mathbf{C}[G]$. An isomorphism is given if we define the contraction $\langle g, h \rangle = \delta_{gh}$. So, we end up with a $G \times G$ -morphism of representations

$$f'': \bar{V} \otimes V \rightarrow \mathbf{C}[G].$$

Then

$$f''(x \otimes y) = \sum_{g \in G} \langle x, \rho(g)[y] \rangle g$$

for all x in $\bar{V}$ and y in V .

By Schur's lemma, the image of f'' is a $G \times G$ irreducible representation, which is therefore $n \times n$ dimensional, which also happens to be a subrepresentation of $\mathbf{C}[G]$ (f'' is nonzero).

This is n direct sum equivalent copies V . Note that if ρ_1 and ρ_2 are equivalent G -irreducible representations, the respective images of the intertwining matrices would give rise to the *same* $G \times G$ -irreducible representation of $\mathbf{C}[G]$.

Here, we use the fact that if f is a function over G , then

$$\sum_{g \in G} f(g) h g k^{-1} = \sum_{g \in G} f(h^{-1} g k) g$$

We convert $\mathbf{C}[G]$ into a Hilbert space by introducing the norm where $\langle g, h \rangle$ is 1 if g is h and zero otherwise. This is different from the 'contraction' given a couple of paragraphs back, in that this form is sesquilinear. This makes $\mathbf{C}[G]$ a unitary representation of $G \times G$. In particular, we now have the concepts of orthogonal complement and orthogonality of subrepresentations.

In particular, if $\mathbf{C}[G]$ contains two inequivalent irreducible $G \times G$ subrepresentations, then both subrepresentations are orthogonal to each other. To see this, note that for every subspace of a Hilbert space, there exists a unique linear transformation from the Hilbert space to itself which maps points on the subspace to itself while mapping points on its orthogonal complement to zero. This is called the projection map. The projection map associated with the first irreducible representation is an intertwiner. Restricted to the second irreducible representation, it gives an intertwiner from the second irreducible representation to the first. Using Schur's lemma, this must be zero.

Now suppose $A \otimes B$ is a $G \times G$ -irreducible representation of $\mathbf{C}[G]$.

Note. The complex irreducible representations of $G \times H$ are always a direct product of a complex irreducible representation of G and a complex irreducible representation of H . This is not the case for real irreducible representations. As an example, there is a 2 dimensional real irreducible representation of the group $C_3 \times C_3$ which transforms nontrivially under both copies of C_3 but **cannot** be expressed as the direct product of two irreducible representations of C_3 .

This representation is also a G -representation (n_A direct sum copies of B where n_A is the dimension of A). If Y is an element of this representation (and hence also of $\mathbf{C}[G]$) and X an element of its dual representation (which is a subrepresentation of the dual representation of $\mathbf{C}[G]$), then

$$\begin{aligned} f''(X \otimes Y) &= \sum_{g \in G} \langle X, \rho(g)[Y] \rangle g = \sum_{g \in G} \langle X, Y g^{-1} \rangle g \\ &= \sum_{g \in G} \langle X, g^{-1} \rangle g Y = \sum_{g \in G} \langle X, g^{-1} \rangle (g, e)[Y] \end{aligned}$$

where e is the identity of G . Though the f'' defined a couple of paragraphs back is only defined for G -irreducible representations, and though $A \otimes B$ is not a G -irreducible representation in general, we claim this argument could be made correct since $A \otimes B$ is simply the direct sum of copies of B s, and we have shown that each copy all maps to the same $G \times G$ -irreducible subrepresentation of $\mathbf{C}[G]$, we have just showed that $\bar{B} \otimes B$ as an irreducible $G \times G$ -subrepresentation of $\mathbf{C}[G]$ is contained in $A \otimes B$ as another irreducible $G \times G$ -subrepresentation of $\mathbf{C}[G]$. Using Schur's lemma again, this means both irreducible representations are the same.

Putting all of this together,

Theorem. $\mathbf{C}[G] \cong \sum_V \bar{V} \otimes_G V$ where the sum is taken over the inequivalent G -irreducible representations V .

Corollary. If there are p inequivalent G -irreducible representations, V_i , each of dimension n_i , then $|G| = n_1^2 + \dots + n_p^2$.

Character theory

Main article: Character theory

There is a mapping from G to the complex numbers for each representation called the character given by the trace of the linear transformation upon the representation generated by the element of G in question

$$\chi_\rho(g) = \text{Tr}[\rho(g)].$$

All elements of G belonging to the same conjugacy class have the same character: in other words χ_ρ is a class function on G . This follows from

$$\text{Tr}[\rho(ghg^{-1})] = \text{Tr}[\rho(g)\rho(h)\rho(g)^{-1}] = \text{Tr}[\rho(h)]$$

by the cyclic property of the trace of a matrix.

What are the characters of $\mathbf{C}[G]$? Using the property that gh^{-1} is only the same as g if $h = e$, $\chi_{\mathbf{C}[G]}(g)$ is $|G|$ if $g=e$ and 0 otherwise.

The character of a direct sum of representations is simply the sum of their individual characters.

Putting all of this together,

$$\sum_{i=1}^p n_i \chi_{V_i}(g) = |G| \delta_{ge}$$

with the Kronecker delta on the right hand side.

Repeat this, working with characters of $G \times G$ instead of characters, of G which I'll call Δ . Then, $\Delta_{\mathbf{C}[G]}(g, h)$ is the number of elements k in G satisfying $g k h^{-1} = k$. This is equal to

$$\sum_{i=1}^p \chi_{V_i}(g) \chi_{V_i}(h) = \sum_{i=1}^p \chi_{V_i}(g^{-1}) \chi_{V_i}(h) = \sum_{i=1}^p \chi_{V_i}(g)^* \chi_{V_i}(h)$$

where $*$ denotes complex conjugation. After all, $\mathbf{C}[G]$ is a unitary representation and any subrepresentation of a finite unitary representation is another unitary representation; and all irreducible representations are (equivalent to) a subrepresentation of $\mathbf{C}[G]$.

Consider

$$\sum_{h \in G} \Delta_{C[G]}((g, hkh^{-1})).$$

This is $|G|$ times the number of elements which commute with g ; which is $|G|^2$ divided by the size of the conjugacy class of g , if g and k belong to the same conjugacy class, but zero otherwise. Therefore, for each conjugacy class C_i of size m_i , the characters are the same for each element of the conjugacy class and so we can just call $\chi_p(C_i)$ by an abuse of notation). Then,

$$\frac{|G|}{|C_i|} \delta_{ij} = \sum_{k=1}^p \chi_{V_k}(C_i)^* \chi_{V_k}(C_j).$$

Note that

$$\sum_{g \in G} \rho(g)$$

is a self-intertwiner (or *invariant*). This linear transformation, when applied to $\mathbf{C}[G]$ (as a representation of the *second* copy of $G \times G$), would give as its image the 1-dimensional subrepresentation generated by

$$\sum_{g \in G} g,$$

which is obviously the trivial representation.

Since we know $\mathbf{C}[G]$ contains all irreducible representations up to equivalence and using Schur's lemma, we conclude that

$$\sum_{g \in G} \rho(g)$$

for irreducible representations is zero if it's not the trivial irreducible representation; and it's of course $|G|$ if the irreducible representation is trivial.

Given two irreducible representations V_i and V_j , we can construct a G -representation

$$\bar{V}_i \otimes V_j,$$

this time *not* as a $G \times G$ representation but an ordinary G -representation. See direct product of representations. Then,

$$\chi_{\bar{V}_i \otimes V_j}(g) = \chi_{\bar{V}_i}(g) \chi_{V_j}(g) = \chi_{V_i}(g)^* \chi_{V_j}(g).$$

It can be shown that any irreducible representation can be turned into a unitary irreducible representation. So, the direct product of two irreducible representations can also be turned into a unitary representations and now, we have the neat orthogonality property allowing us to decompose the direct product into a direct sum of irreducible representations (we're also using the property that for finite dimensional representations, if you keep taking proper subrepresentations, you'll hit an irreducible representation eventually. There's no infinite strictly decreasing sequence of positive integers). See Maschke's theorem.

If $i \neq j$, then this decomposition does not contain the trivial representation (Otherwise, we'd have a nonzero intertwiner from V_j to V_i contradicting Schur's lemma). If $i = j$, then it contains exactly one copy of the trivial representation (Schur's lemma states that if A and B are two intertwiners from V_i to itself, since they're both multiples of the identity, A and B are linearly dependent). Therefore,

$$\sum_{g \in G} \chi_{V_i}(g)^* \chi_{V_j}(g) = \sum_k |C_k| \chi_{V_i}(C_k)^* \chi_{V_j}(C_k) = |G| \delta_{ij}.$$

Applying a result of linear algebra to both orthogonality relations ($|C_i|$ is always positive), we find that the number of conjugacy classes is greater than or equal to the number of inequivalent irreducible representations; and also at the same time less than or equal to. The conclusion, then, is that the number of conjugacy classes of G is the same as the number of inequivalent irreducible representations of G .

Corollary. If two representations have the same characters, then they are equivalent.

Proof. Characters can be thought of as elements of a q -dimensional vector space where q is the number of conjugacy classes. Using the orthogonality relations derived above, we find that the q characters for the q inequivalent irreducible representations forms a basis set. Also, according to Maschke's theorem, both representations can be expressed as the direct sum of irreducible representations. Since the character of the direct sum of representations is the sum of their characters, from linear algebra, we see they are equivalent.

We know that any irreducible representation can be turned into a unitary representation. It turns out the Hilbert space norm is unique up to multiplication by a positive number. To see this, note that the conjugate representation of the irreducible representation is equivalent to the dual irreducible representation with the Hilbert space norm acting as the intertwiner. Using Schur's lemma, all possible Hilbert space norms can only be a multiple of each other.

Let ρ be an irreducible representation of a finite group G on a vector space V of (finite) dimension n with character χ . It is a fact that $\chi(g) = n$ if and only if $\rho(g) = \text{id}$ (see for instance Exercise 6.7 from Serre's book below). A consequence of this is that if χ is a *non-trivial* irreducible character of G such that $\chi(g) = \chi(1)$ for some $g \neq 1$ then G contains a proper non-trivial normal subgroup (the normal subgroup is the kernel of ρ). Conversely, if G contains a proper non-trivial normal subgroup N , then the composition of the natural surjective group homomorphism $G \rightarrow G/N$ with the regular representation of G/N produces a representation π of G which has kernel N . Taking χ to be the character of some *non-trivial* subrepresentation of π , we have a character satisfying the hypothesis in the direct statement above. Altogether, whether or not G is simple can be determined immediately by looking at the character table of G .

History

The general features of the representation theory of a finite group G , over the complex numbers, were discovered by Ferdinand Georg Frobenius in the years before 1900. Later the modular representation theory of Richard Brauer was developed.

Generalizations

The Peter–Weyl theorem extends many results about representations of finite groups to representations of compact groups.

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The standard graduate level reference for representations of groups in general, particularly Lie groups.

- James, Gordon; and Liebeck, Martin (1993). *Representations and Characters of Finite Groups*. Cambridge: Cambridge University Press. ISBN 0-521-44590-6.

A beautiful and readable introduction; designed for self study.

- Serre, Jean-Pierre (1977). *Linear Representations of Finite Groups*. Springer-Verlag. ISBN 0-387-90190-6.

A very well-written introduction to stated topic: concise *and* extremely readable.

External links

- *character of a finite group* (<http://planetmath.org/encyclopedia/Character2.html>) at PlanetMath.

Representations of finite groups of Lie type

In mathematics, **Deligne–Lusztig theory** is a way of constructing linear representations of finite groups of Lie type using ℓ -adic cohomology with compact support, introduced by Deligne & Lusztig (1976).

Lusztig (1984) used these representations to find all representations of all finite simple groups of Lie type.

Motivation

Suppose that G is a reductive group defined over a finite field, with Frobenius map F .

Macdonald conjectured that there should be a map from *general position* characters of F -stable maximal tori to irreducible representations of G^F (the fixed points of F). For general linear groups this was already known by the work of Green (1955). This was the main result proved by Deligne and Lusztig; they found a virtual representation for all characters of an F -stable maximal torus, which is irreducible (up to sign) when the character is in general position.

When the maximal torus is split, these representations were well known and are given by parabolic induction of characters of the torus (extend the character to a Borel subgroup, then induce it up to G). The representations of parabolic induction can be constructed using functions on a space, which can be thought of as elements of a suitable zeroth cohomology group. Deligne and Lusztig's construction is a generalization of parabolic induction to non-split tori using higher cohomology groups. (Parabolic induction can also be done with tori of G replaced by Levi subgroups of G , and there is a generalization of Deligne–Lusztig theory to this case too.)

Drinfel'd proved that the discrete series representations of $\mathrm{SL}_2(\mathbf{F}_q)$ can be found in the ℓ -adic cohomology groups

$$H_c^1(X, \mathbf{Q}_\ell)$$

of the affine curve X defined by

$$xy^q - yx^q = 1.$$

The construction of Deligne and Lusztig is a generalization of this fundamental example to other groups. The affine curve X is generalized to a T^F bundle over a "Deligne–Lusztig variety" where T is a maximal torus of G , and instead of using just the first cohomology group they use an alternating sum of ℓ -adic cohomology groups with compact support to construct virtual representations.

The Deligne–Lusztig construction is formally similar to Weyl's construction of the representations of a compact group from the characters of a maximal torus. The case of compact groups is easier partly because there is only one conjugacy class of maximal tori. The Borel–Weil–Bott construction of representations of algebraic groups using coherent sheaf cohomology is also similar.

For real semisimple groups there is an analogue of the construction of Deligne and Lusztig, using Zuckerman functors to construct representations.

Deligne–Lusztig varieties

The construction of Deligne–Lusztig characters uses a family of auxiliary algebraic varieties X_T called Deligne–Lusztig varieties, constructed from a reductive linear algebraic group G defined over a finite field $\mathbf{F}_q$.

If B is a Borel subgroup of G and T a maximal torus of B then we write

$$W_{T,B}$$

for the Weyl group (normalizer mod centralizer)

$$N_G(T)/T$$

of G with respect to T , together with the simple roots corresponding to B . If B_1 is another Borel subgroup with maximal torus T_1 then there is a canonical isomorphism from T to T_1 that identifies the two Weyl groups. So we can identify all these Weyl groups, and call it 'the' Weyl group W of G . Similarly there is a canonical isomorphism between any two maximal tori with given choice of positive roots, so we can identify all these and call it 'the' maximal torus T of G .

By the Bruhat decomposition

$$G = BWB,$$

the subgroup B_1 can be written as the conjugate of B by bw for some $b \in B$ and $w \in W$ (identified with $W_{T,B}$) where w is uniquely determined. In this case we say that B and B_1 are in **relative position** w .

Suppose that w is in the Weyl group of G , and write X for the smooth projective variety of all Borel subgroups of G . The **Deligne–Lusztig variety** $X(w)$ consists of all Borel subgroups B of G such that B and $F(B)$ are in relative position w . In other words it is the inverse image of the G -homogeneous space of pairs of Borel subgroups in relative position w , under the Lang isogeny with formula

$$g.F(g)^{-1}.$$

For example, if $w=1$ then $X(w)$ is 0-dimensional and its points are the rational Borel subgroups of G .

We let $T(w)$ be the torus T , with the rational structure for which the Frobenius is wF . The G^F conjugacy classes of F -stable maximal tori of G can be identified with the F -conjugacy classes of W , where we say $w \in W$ is F -conjugate to elements of the form $v w F(v)^{-1}$ for $v \in W$. If the group G is split, so that F acts trivially on W , this is the same as ordinary conjugacy, but in general for non-split groups G , F may act on W via a non-trivial diagram automorphism. The F -stable conjugacy classes can be identified with elements of the non-abelian galois cohomology group of torsors

$$H^1(F, W).$$

Fix a maximal torus T of G and a Borel subgroup B containing it, both invariant under the Frobenius map F , and write U for the unipotent radical of B . If we choose a representative w' of the normalizer $N(T)$ representing w , then we define $X'(w')$ to be the elements of G/U with $F(u) = u w'$. This is acted on freely by $T(F)$, and the quotient is isomorphic to $X(T)$. So for each character θ of $T(w)^F$ we get a corresponding local system F_θ on $X(w)$. The Deligne–Lusztig virtual representation

$$R^\theta(w)$$

of G^F is defined by the alternating sum

$$R^\theta(w) = \sum_i (-1)^i H_c^i(X(w), F_\theta)$$

of l -adic compactly supported cohomology groups of $X(w)$ with coefficients in the l -adic local system F_θ .

If T is a maximal F -invariant torus of G contained in a Borel subgroup B such that B and FB are in relative position w then $R^\theta(w)$ is also denoted by R_{TCB}^θ , or by R_T^θ as up to isomorphism it does not depend on the choice of B .

Properties of Deligne–Lusztig characters

- The character of R_T^θ does not depend on the choice of prime $l \neq p$, and if $\theta=1$ its values are rational integers.
- Every irreducible character of G^F occurs in at least one character $R^\theta(w)$.
- The inner product of R_T^θ and $R_{T'}^{\theta'}$ is equal to the number of elements of $W(T, T')^F$ taking θ to θ' . The set $W(T, T')$ is the set of elements of G taking T to T' under conjugation, modulo the group T^F which acts on it in the obvious way (so if $T=T'$ it is the Weyl group). In particular the inner product is 0 if w and w' are not F -conjugate. If θ is in general position then R_T^θ has norm 1 and is therefore an irreducible character up to sign. So this verifies Macdonald's conjecture.
- The representation R_T^θ contains the trivial representation if and only if $\theta=1$ (in which case the trivial representation occurs exactly once).
- The representation R_T^θ has dimension

$$\dim(R_T^\theta) = \frac{\pm |G^F|}{|T^F| |U^F|}$$

where U^F is a Sylow p -subgroup of G^F , of order the largest power of p dividing $|G^F|$.

- The restriction of the character R_T^θ to unipotent elements u does not depend on θ and is called a **Green function**, denoted by $Q_{T,G}(u)$ (the Green function is defined to be 0 on elements that are not unipotent). The character formula gives the character of R_T^θ in terms of Green functions of subgroups as follows:

$$R_T^\theta(x) = \frac{1}{|G_s^F|} \sum_{g \in G^F, g^{-1}sg \in T} Q_{gTg^{-1}, G_s}(u) \theta(g^{-1}sg)$$

where $x=su$ is the Jordan–Chevalley decomposition of x as the product of commuting semisimple and unipotent elements s and u , and G_s is the identity component of the centralizer of s in G . In particular the character value vanishes unless the semisimple part of x is conjugate under G^F to something in the torus T .

- The Deligne–Lusztig variety is usually affine, in particular whenever the characteristic p is larger than the Coxeter number h of the Weyl group. If it is affine and the character θ is in general position (so that the Deligne–Lusztig character is irreducible up to sign) then only one of the cohomology groups $H^i(X(w), F_\theta)$ is non-zero (the one with i equal to the length of w) so this cohomology group gives a model for the irreducible representation. In general it is possible for more than one cohomology group to be non-zero, for example when θ is 1.

Lusztig's classification of irreducible characters

Lusztig classified all the irreducible characters of G^F by decomposing such a character into a semisimple character and a unipotent character (of another group) and separately classifying the semisimple and unipotent characters.

The dual group

The representations of G^F are classified using conjugacy classes of the **dual group** of G . A reductive group over a finite field determines a root datum (with choice of Weyl chamber) together with an action of the Frobenius element on it. The dual group G^* of a reductive algebraic group G defined over a finite field is the one with dual root datum (and adjoint Frobenius action). This is similar to the Langlands dual group (or L-group), except here the dual group is defined over a finite field rather than over the complex numbers. The dual group has the same root system, except that root systems of type B and C get exchanged.

The local Langlands conjectures state (very roughly) that representations of an algebraic group over a local field should be closely related to conjugacy classes in the Langlands dual group. Lusztig's classification of representations of reductive groups over finite fields can be thought of as a verification of an analogue of this conjecture for finite fields (though Langlands never stated his conjecture for this case).

Jordan decomposition

In this section G will be a reductive group with connected center.

An irreducible character is called **unipotent** if it occurs in some $R^1_{T'}$, and is called **semisimple** if its average value on regular unipotent elements is non-zero (in which case the average value is 1 or -1). If p is a good prime for G (meaning that it does not divide any of the coefficients of roots expressed as linear combinations of simple roots) then an irreducible character is semisimple if and only if its order is not divisible by p .

An arbitrary irreducible character has a "Jordan decomposition": to it one can associate a semisimple character (corresponding to some semisimple element s of the dual group), and a unipotent representation of the centralizer of s . The dimension of the irreducible character is the product of the dimensions of its semisimple and unipotent components.

This (more or less) reduces the classification of irreducible characters to the problem of finding the semisimple and the unipotent characters.

Geometric conjugacy

Two pairs (T, θ) , (T', θ') of a maximal torus T of G fixed by F and a character θ of T^F are called **geometrically conjugate** if they are conjugate under some element of $G(k)$, where k is the algebraic closure of $\mathbf{F}$. If an irreducible representation occurs in both R_T^θ and $R_{T'}^{\theta'}$ then (T, θ) , (T', θ') need not be conjugate under G^F , but are always geometrically conjugate. For example if $\theta = \theta' = 1$ and T and T' are not conjugate, then the identity representation occurs in both Deligne-Lusztig characters, and the corresponding pairs $(T, 1)$, $(T', 1)$ are geometrically conjugate but not conjugate.

The geometric conjugacy classes of pairs (T, θ) are parameterized by geometric conjugacy classes of semisimple elements s of the group G^{*F} of elements of the dual group G^* fixed by F . Two elements of G^{*F} are called geometrically conjugate if they are conjugate over the algebraic closure of the finite field; if the center of G is connected this is equivalent to conjugacy in G^{*F} . The number of geometric conjugacy classes of pairs (T, θ) is $|Z^{0F}|q^l$ where Z^0 is the identity component of the center Z of G and l is the semisimple rank of G .

Classification of semisimple characters

In this subsection G will be a reductive group with connected center Z . (The case when the center is not connected has some extra complications.)

The semisimple characters of G correspond to geometric conjugacy classes of pairs (T, θ) (where T is a maximal torus invariant under F and θ is a character of T^F); in fact among the irreducible characters occurring in the Deligne-Lusztig characters of a geometric conjugacy class there is exactly one semisimple character. If the center of G is connected there are $|Z^F|q^l$ semisimple characters. If κ is a geometric conjugacy class of pairs (T, θ) then the character of the corresponding semisimple representation is given up to sign by

$$\sum_{(T, \theta) \in \kappa, \text{ mod } G^F} \frac{R_T^\theta}{(R_T^\theta, R_T^\theta)}$$

and its dimension is the p' part of the index of the centralizer of the element s of the dual group corresponding to it.

The semisimple characters are (up to sign) exactly the duals of the regular characters, under a certain duality operation on generalized characters introduced by Curtis and Kawanaka. An irreducible character is called **regular** if it occurs in the **Gelfand-Graev representation** G^F , which is the representation induced from a certain "non-degenerate" 1-dimensional character of the Sylow p -subgroup. It is reducible, and any irreducible character of G^F occurs at most once in it. If κ is a geometric conjugacy class of pairs (T, θ) then the character of the corresponding regular representation is given by

$$\sum_{(T,\theta) \in \kappa, \bmod G^F} \frac{\epsilon_G \epsilon_T R_T^\theta}{(R_T^\theta, R_T^\theta)}$$

and its dimension is the p' part of the index of the centralizer of the element s of the dual group corresponding to it times the p -part of the order of the centralizer.

Classification of unipotent characters

These can be found from the cuspidal unipotent characters: those that cannot be obtained from decomposition of parabolically induced characters of smaller rank groups. The unipotent cuspidal characters were listed by Lusztig using rather complicated arguments. The number of them depends only on the type of the group and not on the underlying field; for example, there are none for groups of type A_n , at most 1 for all classical groups, and 13 for groups of type E_8 . The unipotent characters can be found by decomposing the characters induced from the cuspidal ones, using results of Howlett and Lehrer. The number of unipotent characters depends only on the root system of the group and not on the field (or the center). The dimension of the unipotent characters can be given by universal polynomials in the order of the ground field depending only on the root system; for example the Steinberg representation has dimension q^r , where r is the number of positive roots of the root system.

Lusztig discovered that the unipotent characters of a group G^F (with irreducible Weyl group) fall into families of size 4^n ($n \geq 0$), 8, 21, or 39. The characters of each family are indexed by conjugacy classes of pairs (x, σ) where x is in one of the groups $\mathbf{Z}/2\mathbf{Z}^n$, S_3 , S_4 , S_5 respectively, and σ is a representation of its centralizer. (The family of size 39 only occurs for groups of type E_8 , and the family of size 21 only occurs for groups of type F_4 .) The families are in turn indexed by special representations of the Weyl group, or equivalently by 2-sided cells of the Weyl group. For example, the group $E_8(\mathbf{F}_q)$ has 46 families of unipotent characters corresponding to the 46 special representations of the Weyl group of E_8 . There are 23 families with 1 character, 18 families with 4 characters, 4 families with 8 characters, and one family with 39 characters (which includes the 13 cuspidal unipotent characters).

Examples

Suppose that q is an odd prime power, and G is the algebraic group SL_2 . We describe the Deligne-Lusztig representations of the group $SL_2(\mathbf{F}_q)$. (The representation theory of these groups was well known long before Deligne-Lusztig theory.)

The irreducible representations are:

- The trivial representations of dimension 1.
- The Steinberg representation of dimension q
- The $(q-3)/2$ irreducible principal series representations of dimension $q+1$, together with 2 representations of dimension $(q+1)/2$ coming from a reducible principal series representation.
- The $(q-1)/2$ irreducible discrete series representations of dimension $q-1$, together with 2 representations of dimension $(q-1)/2$ coming from a reducible discrete series representation.

There are two classes of tori associated to the two elements (or conjugacy classes) of the Weyl group, denoted by $T(1)$ (cyclic of order $q-1$) and $T(w)$ (cyclic of order $q+1$). The non-trivial element of the Weyl group acts on the characters of these tori by changing each character to its inverse. So the Weyl group fixes a character if and only if it has order 1 or 2. By the orthogonality formula, $R^\theta(w)$ is (up to sign) irreducible if θ does not have order 1 or 2, and a sum of two irreducible representations if it has order 1 or 2.

The Deligne-Lusztig variety $X(1)$ for the split torus is 0-dimensional with $q+1$ points, and can be identified with the points of 1-dimensional projective space defined over $\mathbf{F}_q$. The representations $R^\theta(1)$ are given as follows:

- 1+Steinberg if $\theta=1$
- The sum of the 2 representations of dimension $(q+1)/2$ if θ has order 2.
- An irreducible principal series representation if θ has order greater than 2.

The Deligne-Lusztig variety $X(w)$ for the non-split torus is 1-dimensional, and can be identified with the complement of $X(1)$ in 1-dimensional projective space. So it is the set of points $(x:y)$ of projective space not fixed by the Frobenius map $(x:y) \rightarrow (x^q:y^q)$, in other words the points with

$$xy^q - yx^q \neq 0$$

Drinfeld's variety of points (x,y) of affine space with

$$xy^q - yx^q = 1$$

maps to $X(w)$ in the obvious way, and is acted on freely by the group of $q+1$ th roots λ of 1 (which can be identified with the elements of the non-split torus that are defined over $\mathbf{F}_q$), with λ taking (x,y) to $(\lambda x, \lambda y)$. The Deligne Lusztig variety is the quotient of Drinfeld's variety by this group action. The representations $-R^\theta(w)$ are given as follows:

- Steinberg -1 if $\theta=1$
- The sum of the 2 representations of dimension $(q-1)/2$ if θ has order 2.
- An irreducible discrete series representation if θ has order greater than 2.

The unipotent representations are the trivial representation and the Steinberg representation, and the semisimple representations are all the representations other than the Steinberg representation. (In this case the semisimple representations do not correspond exactly to geometric conjugacy classes of the dual group, as the center of G is not connected.)

Intersection cohomology and character sheaves

Lusztig (1985) replaced the ℓ -adic cohomology used to define the Deligne-Lusztig representations with intersection ℓ -adic cohomology, and introduced certain perverse sheaves called **character sheaves**. The representations defined using intersection cohomology are related to those defined using ordinary cohomology by Kazhdan-Lusztig polynomials. The F -invariant irreducible character sheaves are closely related to the irreducible characters of the group G^F .

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Representations of Lie groups

In mathematics and theoretical physics, the idea of a **representation of a Lie group** plays an important role in the study of continuous symmetry. A great deal is known about such representations, a basic tool in their study being the use of the corresponding 'infinitesimal' representations of Lie algebras. The physics literature sometimes passes over the distinction between Lie group representations and Lie algebra representations.

Representations on a complex finite-dimensional vector space

Let us first discuss representations acting on finite-dimensional complex vector spaces. A representation of a Lie group G on a finite-dimensional complex vector space V is a smooth group homomorphism $\Psi: G \rightarrow \text{Aut}(V)$ from G to the automorphism group of V .

For n -dimensional V , the automorphism group of V is identified with a subset of complex square-matrices of order n . The automorphism group of V is given the structure of a smooth manifold using this identification. The condition that Ψ is smooth, in the definition above, means that Ψ is a smooth map from the smooth manifold G to the smooth manifold $\text{Aut}(V)$.

If a basis for the complex vector space V is chosen, the representation can be expressed as a homomorphism into $\text{GL}(n, \mathbb{C})$. This is known as a *matrix representation*.

Representations on a finite-dimensional vector space over an arbitrary field

A representation of a Lie group G on a vector space V (over a field K) is a smooth (i.e. respecting the differential structure) group homomorphism $G \rightarrow \text{Aut}(V)$ from G to the automorphism group of V . If a basis for the vector space V is chosen, the representation can be expressed as a homomorphism into $\text{GL}(n, K)$. This is known as a *matrix representation*. Two representations of G on vector spaces V , W are *equivalent* if they have the same matrix representations with respect to some choices of bases for V and W .

On the Lie algebra level, there is a corresponding linear mapping from the Lie algebra of G to $\text{End}(V)$ preserving the Lie bracket $[\cdot, \cdot]$. See representation of Lie algebras for the Lie algebra theory.

If the homomorphism is in fact a monomorphism, the representation is said to be *faithful*.

A unitary representation is defined in the same way, except that G maps to unitary matrices; the Lie algebra will then map to skew-hermitian matrices.

If G is a compact Lie group, every finite-dimensional representation is equivalent to a unitary one.

Representations on Hilbert spaces

A representation of a Lie group G on a complex Hilbert space V is a group homomorphism $\Psi: G \rightarrow B(V)$ from G to $B(V)$, the group of bounded linear operators of V which have a bounded inverse, such that the map $G \times V \rightarrow V$ given by $(g, v) \rightarrow \Psi(g)v$ is continuous.

This definition can handle representations on **infinite-dimensional** Hilbert spaces. Such representations can be found in e.g. quantum mechanics, but also in Fourier analysis as shown in the following example.

Let $G = \mathbf{R}$, and let the complex Hilbert space V be $L^2(\mathbf{R})$. We define the representation $\Psi: \mathbf{R} \rightarrow B(L^2(\mathbf{R}))$ by $\Psi(r)\{f(x)\} \rightarrow f(r^{-1}x)$.

See also Wigner's classification for representations of the Poincaré group.

Classification

If G is a semisimple group, its finite-dimensional representations can be decomposed as direct sums of irreducible representations. The irreducibles are indexed by highest weight; the allowable (*dominant*) highest weights satisfy a suitable positivity condition. In particular, there exists a set of *fundamental weights*, indexed by the vertices of the Dynkin diagram of G , such that dominant weights are simply non-negative integer linear combinations of the fundamental weights. The characters of the irreducible representations are given by the Weyl character formula.

If G is a commutative Lie group, then its irreducible representations are simply the continuous characters of G : see Pontryagin duality for this case.

A quotient representation is a quotient module of the group ring.

Formulaic examples

Let $\mathbf{F}_q$ be a finite field of order q and characteristic p . Let G be a finite group of Lie type, that is, G is the $\mathbf{F}_q$ -rational points of a connected reductive group G defined over $\mathbf{F}_q$. For example, if n is a positive integer $GL(n, \mathbf{F}_q)$ and $SL(n, \mathbf{F}_q)$ are finite groups of Lie type. Let $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, where I_n is the $n \times n$ identity matrix. Let

$$Sp_{2n}(\mathbf{F}_q) = \{g \in GL_{2n}(\mathbf{F}_q) \mid {}^t g J g = J\}.$$

Then $Sp(2, \mathbf{F}_q)$ is a symplectic group of rank n and is a finite group of Lie type. For $G = GL(n, \mathbf{F}_q)$ or $SL(n, \mathbf{F}_q)$ (and some other examples), the *standard Borel subgroup* B of G is the subgroup of G consisting of the upper triangular elements in G . A *standard parabolic subgroup* of G is a subgroup of G which contains the standard Borel subgroup B . If P is a standard parabolic subgroup of $GL(n, \mathbf{F}_q)$, then there exists a partition $(n_1, \dots, n_r)$ of n (a set of positive integers n_j such that $n_1 + \dots + n_r = n$) such that $P = P_{(n_1, \dots, n_r)} = M \times N$, where

$M \simeq GL_{n_1}(\mathbf{F}_q) \times \dots \times GL_{n_r}(\mathbf{F}_q)$ has the form

$$M = \left\{ \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_r \end{pmatrix} \mid A_j \in GL_{n_j}(\mathbf{F}_q), 1 \leq j \leq r \right\},$$

and

$$N = \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{n_r} \end{pmatrix} \right\},$$

where $*$ denotes arbitrary entries in $\mathbb{F}_q$.

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Representations of Lie algebras

In the mathematical field of representation theory, a **Lie algebra representation** or **representation of a Lie algebra** is a way of writing a Lie algebra as a set of matrices (or endomorphisms of a vector space) in such a way that the Lie bracket is given by the commutator.

The notion is closely related to that of a representation of a Lie group. Roughly speaking, the representations of Lie algebras are the differentiated form of representations of Lie groups, while the representations of the universal cover of a Lie group are the integrated form of the representations of its Lie algebra.

Formal definition

A **representation** of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism

$$\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$$

from $\mathfrak{g}$ to the Lie algebra of endomorphisms on a vector space V (with the commutator as the Lie bracket), sending an element x of $\mathfrak{g}$ to an element ρ_x of $\mathfrak{gl}(V)$.

Explicitly, this means that

$$\rho_{[x,y]} = [\rho_x, \rho_y] = \rho_x \rho_y - \rho_y \rho_x$$

for all x, y in $\mathfrak{g}$. The vector space V , together with the representation ρ , is called a **$\mathfrak{g}$ -module**. (Many authors abuse terminology and refer to V itself as the representation).

One can equivalently define a $\mathfrak{g}$ -module as a vector space V together with a bilinear map $\mathfrak{g} \times V \rightarrow V$ such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

for all x, y in $\mathfrak{g}$ and v in V . This is related to the previous definition by setting $x \cdot v = \rho_x(v)$.

Infinitesimal Lie group representations

If $\phi : G \rightarrow H$ is a homomorphism of Lie groups, and $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively, then the induced map $\phi_* : \mathfrak{g} \rightarrow \mathfrak{h}$ on tangent spaces is a Lie algebra homomorphism. In particular, a representation of Lie groups

$$\phi : G \rightarrow \mathrm{GL}(V)$$

determines a Lie algebra homomorphism

$$\phi_* : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$$

from $\mathfrak{g}$ to the Lie algebra of the general linear group $\mathrm{GL}(V)$, i.e. the endomorphism algebra of V .

A partial converse to this statement says that every representation of a finite-dimensional (real or complex) Lie algebra lifts to a unique representation of the associated simply connected Lie group, so that representations of simply-connected Lie groups are in one-to-one correspondence with representations of their Lie algebras.

Properties

Representations of a Lie algebra are in one-to-one correspondence with algebra representations of the associated universal enveloping algebra. This follows from the universal property of that construction.

If the Lie algebra is semisimple, then all reducible representations are decomposable. Otherwise, that's not true in general.

If we have two representations, with V_1 and V_2 as their underlying vector spaces and $[\cdot]_1$ and $[\cdot]_2$ as the representations, then the product of both representations would have $V_1 \otimes V_2$ as the underlying vector space and

$$x[v_1 \otimes v_2] = x[v_1] \otimes v_2 + v_1 \otimes x[v_2].$$

If L is a real Lie algebra and $\rho : L \times V \rightarrow V$ is a complex representation of it, we can construct another representation of L called its dual representation as follows.

Let V^* be the dual vector space of V . In other words, V^* is the set of all linear maps from V to $\mathbb{C}$ with addition defined over it in the usual linear way, but scalar multiplication defined over it such that $(z\omega)[X] = \bar{z}\omega[X]$ for any z in $\mathbb{C}$, ω in V^* and X in V . This is usually rewritten as a contraction with a sesquilinear form $\langle \cdot, \cdot \rangle$. i.e. $\langle \omega, X \rangle$ is defined to be $\omega[X]$.

We define $\bar{\rho}$ as follows:

$$\langle \bar{\rho}(A)[\omega], X \rangle + \langle \omega, \rho A[X] \rangle = 0,$$

for any A in L , ω in V^* and X in V . This defines $\bar{\rho}$ uniquely.

Classification

Finite-dimensional representations of semisimple Lie algebras

Similarly to how semisimple Lie algebras can be classified, the finite-dimensional representations of semisimple Lie algebras can be classified. This is a classical theory, widely regarded as beautiful, and a standard reference is (Fulton & Harris 1992).

Briefly, finite-dimensional representations of a semisimple Lie algebra are completely reducible, so it suffices to classify irreducible (simple) representations. Semisimple Lie algebras are classified in terms of the weights of the adjoint representation, the so-called root system; in a similar manner all finite-dimensional irreducible representations can be understood in terms of weights; see weight (representation theory) for details.

Representation on an algebra

If we have a Lie superalgebra L , then a representation of L on an algebra is a (not necessarily associative) $\mathbf{Z}_2$ graded algebra A which is a representation of L as a $\mathbf{Z}_2$ graded vector space and in addition, the elements of L acts as derivations/antiderivations.

More specifically, if H is a pure element of L and x and y are pure elements of A ,

$$H[xy] = (H[x])y + (-1)^{xH}x(H[y])$$

Also, if A is unital, then

$$H[1] = 0$$

Now, for the case of a **representation of a Lie algebra**, we simply drop all the gradings and the (-1) to the some power factors.

A Lie (super)algebra is an algebra and it has an adjoint representation of itself. This is a representation on an algebra: the (anti)derivation property is the superJacobi identity.

If a vector space is both an associative algebra and a Lie algebra and the adjoint representation of the Lie algebra on itself is a representation on an algebra (i.e., acts by derivations on the associative algebra structure), then it is a Poisson algebra. The analogous observation for Lie superalgebras gives the notion of a Poisson superalgebra.

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Representations of the Poincaré group

In mathematics, **the representation theory of the Poincaré group** is an example of the representation theory of a Lie group that is neither a compact group nor a semisimple group. It is fundamental in theoretical physics.

In a physical theory having Minkowski space as the underlying spacetime, the space of physical states is typically a representation of the Poincaré group. (More generally, it may be a projective representation, which amounts to a representation of the double cover of the group.)

In a classical field theory, the physical states are sections of a Poincaré-equivariant vector bundle over Minkowski space. The equivariance condition means that the group acts on the total space of the vector bundle, and the projection to Minkowski space is an equivariant map. Therefore the Poincaré group also acts on the space of sections. Representations arising in this way (and their subquotients) are called covariant field representations, and are not usually unitary.

For a discussion of such unitary representations, see Wigner's classification.

In quantum mechanics, the state of the system is determined by the Schrodinger equation, which is only invariant under Galilean transformations. Quantum field theory is the relativistic extension of quantum mechanics, where relativistic (Lorentz/Poincaré invariant) wave equations are solved, "quantized", and act on a Hilbert space composed of Fock states; eigenstates of the theory's Hamiltonian which are states with a definite number of particles with individual 4-momentum. There are no finite unitary representations of the full Lorentz (and thus Poincaré) transformations due to the nature of Lorentz boosts (rotations in Minkowski space along a space and time axis.

Representation theory of the Lorentz group

The Lorentz group of theoretical physics has a variety of representations, corresponding to particles with integer and half-integer spins in quantum field theory. These representations are normally constructed out of spinors.

The group may also be represented in terms of a set of functions defined on the Riemann sphere. these are the Riemann P-functions, which are expressible as hypergeometric functions. An important special case is the subgroup $SO(3)$, where these reduce to the spherical harmonics, and find practical application in the theory of atomic spectra.

The Lorentz group has no unitary representation of finite dimension, except for the trivial representation (where every group element is represented by 1).

Finding representations

According to general representation theory of Lie groups, one first looks for the representations of the complexification of the Lie algebra of the Lorentz group. A convenient basis for the Lie algebra of the Lorentz group is given by the three generators of rotations $J^k = \epsilon^{ijk} L_{ij}$ and the three generators of boosts $K^i = L_{it}$ where i, j , and k run over the three spatial coordinates and ϵ is the Levi-Civita symbol for a three dimensional spatial slice of Minkowski space. Note that the three generators of rotations transform like components of a pseudovector $\mathbf{J}$ and the three generators of boosts transform like components of a vector $\mathbf{K}$ under the adjoint action of the spatial rotation subgroup.

This motivates the following construction: first complexify, and then change basis to the components of $\mathbf{A} = (\mathbf{J} + i \mathbf{K})/2$ and $\mathbf{B} = (\mathbf{J} - i \mathbf{K})/2$. In this basis, one checks that the components of $\mathbf{A}$ and $\mathbf{B}$ satisfy separately the commutation relations of the Lie algebra $\mathfrak{sl}_2$ and moreover that they commute with each other. In other words, one has the isomorphism

$$\mathfrak{so}(3, 1) \otimes \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}).$$

The utility of this isomorphism comes from the fact that $\mathfrak{sl}_2$ is the complexification of the rotation algebra, and so its irreducible representations correspond to the well-known representations of the spatial rotation group; for each j in $\frac{1}{2}\mathbb{Z}$, one has the $(2j+1)$ -dimensional spin- j representation spanned by the spherical harmonics with j as the highest weight. Thus the finite dimensional irreducible representations of the Lorentz group are simply given by an ordered pair of half-integers (m, n) which fix representations of the subalgebra spanned by the components of $\mathbf{A}$ and $\mathbf{B}$ respectively.

Properties of the (m, n) irrep

Since the angular momentum operator is given by $\mathbf{J} = \mathbf{A} + \mathbf{B}$, the highest weight of the rotation subrepresentation will be $m+n$. So for example, the $(1/2, 1/2)$ representation has spin 1. The (m, n) representation is $(2m+1)(2n+1)$ -dimensional.

Common representations

- $(0, 0)$ the Lorentz scalar representation. This representation is carried by relativistic scalar field theories.
- $(1/2, 0)$ is the left-handed Weyl spinor and $(0, 1/2)$ is the right-handed Weyl spinor representation.
- $(1/2, 0) \oplus (0, 1/2)$ is the bispinor representation (see also Dirac spinor).
- $(1/2, 1/2)$ is the four-vector representation. The electromagnetic vector potential lives in this rep. It is a 1-form field.
- $(1, 0)$ is the self-dual 2-form field representation and $(0, 1)$ is the anti-self-dual 2-form field representation.
- $(1, 0) \oplus (0, 1)$ is the representation of a parity invariant 2-form field. The electromagnetic field tensor transforms under this representation.

- $(1, 1/2) \oplus (1/2, 1)$ is the Rarita-Schwinger field representation.
- $(1, 1)$ is the spin-2 representation of the traceless metric tensor.

Full Lorentz group

The (m, n) representation is irreducible under the restricted Lorentz group (the identity component of the Lorentz group). When one considers the full Lorentz group, which is generated by the restricted Lorentz group along with time and parity reversal, not only is this not an irreducible representation, it is not a representation at all, unless $m=n$. The reason is that this representation is formed in terms of the sum of a vector and a pseudovector, and a parity reversal changes the sign of one, but not the other. The upshot is that a vector in the (m, n) representation is carried into the (n, m) representation by a parity reversal. Thus $(m, n) \oplus (n, m)$ gives an irrep of the full Lorentz group. When constructing theories such as QED which is invariant under parity reversal, Dirac spinors may be used, while theories that do not, such as the electroweak force, must be formulated in terms of Weyl spinors.

Infinite dimensional unitary representations

History

The Lorentz group $SO(3, 1)$ and its double cover $SL(2, \mathbb{C})$ also have infinite dimensional unitary representations, first studied independently by Bargmann (1947), Gelfand & Naimark (1947) and Harish-Chandra (1947) (at the instigation of Paul Dirac). The Plancherel formula for these groups was first obtained by Gelfand and Naimark through involved calculations. The treatment was subsequently considerably simplified by Harish-Chandra (1951) and Gelfand & Graev (1953), based on an analogue for $SL(2, \mathbb{C})$ of the integration formula of Hermann Weyl for compact Lie groups. Elementary accounts of this approach can be found in Rühl (1970) and Knapp (2001).

The theory of spherical functions for the Lorentz group, required for harmonic analysis on the 3-dimensional hyperboloid in Minkowski space, or equivalently 3-dimensional hyperbolic space, is considerably easier than the general theory. It only involves representations from the spherical principal series and can be treated directly, because in radial coordinates the Laplacian on the hyperboloid is equivalent to the Laplacian on $\mathbf{R}$. This theory is discussed in Takahashi (1963), Helgason (1968), Helgason (2000) and the posthumous text of Jorgenson & Lang (2008).

Principal series

The **principal series**, or **unitary principal series**, are the unitary representations induced from the one-dimensional representations of the lower triangular subgroup B of $G = SL(2, \mathbb{C})$. Since the one-dimensional representations of B correspond to the representations of the diagonal matrices, with non-zero complex entries z and z^{-1} , and thus have the form

$$\chi_{\nu, k}(re^{i\theta}) = r^{i\nu} e^{ik\theta},$$

for k an integer and ν real. The representations are irreducible; the only repetitions occur when k is replaced by $-k$. By definition the representations are realised on L^2 sections of line bundles on $G/B = \mathbb{S}^2$, the Riemann sphere. When $k = 0$, these representations constitute the so-called **spherical principal series**.

The restriction of a principal series to the maximal compact subgroup $K = SU(2)$ of G can also be realised as an induced representation of K using the identification $G/B = K/T$, where $T = B \cap K$ is the maximal torus in K consisting of diagonal matrices with $|z|=1$. It is the representation induced from the 1-dimensional representation z^k on T , and is independent of ν . By Frobenius reciprocity, on K they decompose as a direct sum of the irreducible representations of K with dimensions $|k| + 2m + 1$ with m a non-negative integer.

Using the identification between the Riemann sphere minus a point and $\mathbb{C}$, the principal series can be defined directly on $L^2(\mathbb{C})$ by the formula

$$\pi_{\nu,k} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} f(z) = |cz + d|^{-2-i\nu} \left(\frac{cz + d}{|cz + d|} \right)^{-k} f \left(\frac{az + b}{cz + d} \right).$$

Irreducibility can be checked in a variety of ways:

- The representation is already irreducible on B . This can be seen directly, but is also a special case of general results on irreducibility of induced representations due to François Bruhat and George Mackey, relying on the Bruhat decomposition $G = B \cup B s B$ where s is the Weyl group element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.^[1]
- The action of the Lie algebra $\mathfrak{g}$ of G can be computed on the algebraic direct sum of the irreducible subspaces of K can be computed explicitly and it can be verified directly that the lowest dimensional subspace generates this direct sum as a $\mathfrak{g}$ -module.^{[2] [3]}

Complementary series

For $0 < t < 2$, the **complementary series** is defined on L^2 functions f on $\mathbb{C}$ for the inner product

$$(f, g) = \int \int \frac{f(z) \overline{g(w)} dz dw}{|z - w|^{2-t}}.$$

with the action given by

$$\pi_t \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} f(z) = |cz + d|^{-2-t} f \left(\frac{az + b}{cz + d} \right).$$

The complementary series are irreducible and inequivalent. As a representation of K , each is isomorphic to the Hilbert space direct sum of all the odd dimensional irreducible representations of $K = \text{SU}(2)$. Irreducibility can be proved by analysing the action of $\mathfrak{g}$ on the algebraic sum of these subspaces^{[2] [3]} or directly without using the Lie algebra.^{[4] [5]}

Plancherel theorem

The only irreducible unitary representations of $\text{SL}(2, \mathbb{C})$ are the principal series, the complementary series and the trivial representation. Since $-I$ acts $(-1)^k$ on the principal series and trivially on the remainder, these will give all the irreducible unitary representations of the Lorentz group, provided k is taken to be even.

To decompose the left regular representation of G on $L^2(G)$, only the principal series are required. This immediately yields the decomposition on the subrepresentations $L^2(G/\pm I)$, the left regular representation of the Lorentz group, and $L^2(G/K)$, the regular representation on 3-dimensional hyperbolic space. (The former only involves principal series representations with k even and the latter only those with $k = 0$.)

The left and right regular representation λ and ρ are defined on $L^2(G)$ by

$$\lambda(g)f(x) = f(g^{-1}x), \quad \rho(g)f(x) = f(xg).$$

Now if f is an element of $C_c(G)$, the operator $\pi_{\nu,k}(f)$ defined by

$$\pi_{\nu,k}(f) = \int_G f(g) \pi(g) dg$$

is Hilbert-Schmidt. We define a Hilbert space H by

$$H = \bigoplus_{k \geq 0} HS(L^2(\mathbb{C})) \otimes L^2(\mathbb{R}, c_k(\nu^2 + k^2)^{1/2} d\nu),$$

where

$$c_0 = 1/4\pi^{3/2}, \quad c_k = 1/(2\pi)^{3/2} \quad (k \neq 0)$$

and $HS(L^2(\mathbb{C}))$ denotes the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{C})$.^[6] Then the map U defined on $C_c(G)$ by

$$U(f)(\nu, k) = \pi_{\nu,k}(f)$$

extends to a unitary of $L^2(G)$ onto H .

The map U satisfies

$$U(\lambda(x)\rho(y)f)(\nu, k) = \pi_{\nu, k}(x)^{-1}\pi_{\nu, k}(f)\pi_{\nu, k}(y).$$

If f_1, f_2 are in $C_c(G)$ then

$$(f_1, f_2) = \sum_{k \geq 0} c_k^2 \int_{-\infty}^{\infty} \text{Tr}(\pi_{\nu, k}(f_1)\pi_{\nu, k}(f_2)^*)(\nu^2 + k^2) d\nu.$$

Thus if $f = f_1 \star f_2^*$ denotes the convolution of f_1 and f_2^* , where $f_2^*(g) = \overline{f_2(g^{-1})}$, then

$$f(1) = \sum_{k \geq 0} c_k^2 \int_{-\infty}^{\infty} \text{Tr}(\pi_{\nu, k}(f))(\nu^2 + k^2) d\nu.$$

The last two displayed formulas are usually referred to as the **Plancherel formula** and the **Fourier inversion formula** respectively. The Plancherel formula extends to all f_i in $L_2(G)$. By a theorem of Jacques Dixmier and Paul Malliavin, every function f in $C_c^\infty(G)$ is a finite sum of convolutions of similar functions, the inversion formula holds for such f . It can be extended to much wider classes of functions satisfying mild differentiability conditions.^[7]

See also

- Poincaré group
- Wigner's classification

Notes

[1] Knapp 2001, Chapter II

[2] Harish-Chandra 1947

[3] Taylor 1986

[4] Gelfand & Naimark 1947

[5] Takahashi 1963, p. 343

[6] Note that for a Hilbert space H , $\text{HS}(H)$ may be identified canonically with the Hilbert space tensor product of H and its conjugate space.

[7] Knapp 2001

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Double group

In crystallography, the **space group** (or **crystallographic group**, or **Fedorov group**) of a crystal is a description of the symmetry of the crystal, and can have one of 230 types. In mathematics space groups are also studied in dimensions other than 3 where they are sometimes called **Bieberbach groups**, and are discrete cocompact groups of isometries of an oriented Euclidean space.

A definitive source regarding 3-dimensional space groups is the *International Tables for Crystallography* (Hahn (2002)).

History

The space groups in 3 dimensions were first enumerated by Fyodorov (1891), and shortly afterwards were independently enumerated by Barlow (1894) and Schönflies (1891). These first enumerations all contained several minor mistakes, and the correct list of 230 space groups was found during correspondence between Fyodorov and Schönflies.

Space groups in 2 dimensions are the 17 wallpaper groups which have been known for several centuries.

Elements of a space group

The space groups in three dimensions are made from combinations of the 32 crystallographic point groups with the 14 Bravais lattices which belong to one of 7 lattice systems. This results in a space group being some combination of the translational symmetry of a unit cell including lattice centering, the point group symmetry operations of reflection, rotation and improper rotation (also called rotoinversion), and the screw axis and glide plane symmetry operations. The combination of all these symmetry operations results in a total of 230 unique space groups describing all possible crystal symmetries.

Elements fixing a point

The elements of the space group fixing a point of space are rotations, reflections, the identity element, and improper rotations.

Translations

The translations form a normal abelian subgroup of rank 3, called the Bravais lattice. There are 14 possible types of Bravais lattice. The quotient of the space group by the Bravais lattice is a finite group which is one of the 32 possible point groups.

Glide planes

A glide plane is a reflection in a plane, followed by a translation parallel with that plane. This is noted by a , b or c , depending on which axis the glide is along. There is also the n glide, which is a glide along the half of a diagonal of a face, and the d glide, which is a fourth of the way along either a face or space diagonal of the unit cell. The latter is called the diamond glide plane as it features in the diamond structure.

Screw axes

A screw axis is a rotation about an axis, followed by a translation along the direction of the axis. These are noted by a number, n , to describe the degree of rotation, where the number is how many operations must be applied to complete a full rotation (e.g., 3 would mean a rotation one third of the way around the axis each time). The degree of translation is then added as a subscript showing how far along the axis the translation is, as a portion of the parallel lattice vector. So, 2_1 is a twofold rotation followed by a translation of $1/2$ of the lattice vector.

Notation for space groups

There are at least eight methods of naming space groups. Some of these methods can assign several different names to the same space group, so altogether there are many thousands of different names.

- **Number.** The International Union of Crystallography publishes tables of all space group types, and assigns each a unique number from 1 to 230. The numbering is arbitrary, except that groups with the same crystal system or point group are given consecutive numbers.
- **International symbol or Hermann-Mauguin notation.** The Hermann-Mauguin (or international) notation describes the lattice and some generators for the group. It has a shortened form called the **international short symbol**, which is the one most commonly used in crystallography, and usually consists of a set of four symbols. The first describes the centering of the Bravais lattice (P , A , B , C , I , R or F). The next three describe the most prominent symmetry operation visible when projected along one of the high symmetry directions of the crystal. These symbols are the same as used in point groups, with the addition of glide planes and screw axis, described above. By way of example, the space group of quartz is $P3_121$, showing that it exhibits primitive centering of the motif (i.e., once per unit cell), with a threefold screw axis and a twofold rotation axis. Note that it does not explicitly contain the crystal system, although this is unique to each space group (in the case of $P3_121$, it is trigonal).

In the international short symbol the first symbol (3_1 in this example) denotes the symmetry along the major axis (c-axis in trigonal cases), the second (2 in this case) along axes of secondary importance (a and b) and the third symbol the symmetry in another direction. In the trigonal case there also exists a space group $P3_112$. In this space group the twofold axes are not along the a and b-axes but in a direction rotated by 30° .

The international symbols and international short symbols for some of the space groups were changed slightly between 1935 and 2002, so several space groups have 4 different international symbols in use.

- **Hall notation.** Space group notation with an explicit origin. Rotation, translation and axis-direction symbols are clearly separated and inversion centers are explicitly defined. The construction and format of the notation make it particularly suited to computer generation of symmetry information. For example, group number 3 has three Hall symbols: $P\ 2y\ (P\ 1\ 2\ 1)$, $P\ 2\ (P\ 1\ 1\ 2)$, $P\ 2x\ (P\ 2\ 1\ 1)$.
- **Schönflies notation.** The space groups with given point group are numbered by 1, 2, 3, ... (in the same order as their international number) and this number is added as a superscript to the Schönflies symbol for the point group. For example, groups numbers 3 to 5 whose point group is C_2 have Schönflies symbols C_2^1 , C_2^2 , C_2^3 .
- **Shubnikov symbol**
- **2D:Orbifold notation** and **3D:Fibrifold notation.** As the name suggests, the orbifold notation describes the orbifold, given by the quotient of Euclidean space by the space group, rather than generators of the space group. It was introduced by Conway and Thurston, and is not used much outside mathematics. Some of the space groups have several different fibrifolds associated to them, so have several different fibrifold symbols.

Classification systems for space groups

There are (at least) 10 different ways to classify space groups into classes. The relations between some of these are described in the following table. Each classification system is a refinement of the ones below it.

(Crystallographic) space group types (230 in three dimensions). Two space groups, considered as subgroups of the group of affine transformations of space, have the same space group type if they are conjugate by an orientation-preserving affine transformation. In three dimensions, for 11 of the affine space groups, there is no orientation-preserving map from the group to its mirror image, so if one distinguishes groups from their mirror images these each split into two cases. So there are $54+11=65$ space group types that preserve orientation.	
Affine space group types (219 in three dimensions). Two space groups, considered as subgroups of the group of affine transformations of space, have the same affine space group type if they are conjugate under an affine transformation. The affine space group type is determined by the underlying abstract group of the space group. In three dimensions there are 54 affine space group types that preserve orientation.	
Arithmetic crystal classes (73 in three dimensions). These are determined by the point group together with the action of the point group on the subgroup of translations. In other words the arithmetic crystal classes correspond to conjugacy classes of finite subgroup of the general linear group $GL_n(\mathbf{Z})$ over the integers. A space group is called symmorphic (or split) if there is a point such that all symmetries are the product of asymmetry fixing this point and a translation. Equivalently, a space group is symmorphic if it is a semidirect product of its point group with its translation subgroup. There are 73 symmorphic space groups, with exactly one in each arithmetic crystal class. There are also 157 nonsymmorphic space group types with varying numbers in the arithmetic crystal classes.	
(geometric) Crystal classes (32 in three dimensions). The crystal class of a space group is determined by its point group: the quotient by the subgroup of translations, acting on the lattice. Two space groups are in the same crystal class if and only if their point groups, which are subgroups of $GL_2(\mathbf{Z})$, are conjugate in the larger group $GL_2(\mathbf{Q})$.	Bravais flocks (14 in three dimensions). These are determined by the underlying Bravais lattice type. These correspond to conjugacy classes of lattice point groups in $GL_2(\mathbf{Z})$, where the lattice point group is the group of symmetries of the underlying lattice that fix a point of the lattice, and contains the point group.
Crystal systems. (7 in three dimensions) Crystal systems are an ad hoc modification of the lattice systems to make them compatible with the classification according to point groups. They differ from crystal families in that the hexagonal crystal family is split into two subsets, called the trigonal and hexagonal crystal systems. The trigonal crystal system is larger than the rhombohedral lattice system, the hexagonal crystal system is smaller than the hexagonal lattice system, and the remaining crystal systems and lattice systems are the same.	Lattice systems (7 in three dimensions). The lattice system of a space group is determined by the conjugacy class of the lattice point group (a subgroup of $GL_2(\mathbf{Z})$) in the larger group $GL_2(\mathbf{Q})$. In three dimensions the lattice point group can have one of the 7 different orders 2, 4, 8, 12, 16, 24, or 48. The hexagonal crystal family is split into two subsets, called the rhombohedral and hexagonal lattice systems.
Crystal families (6 in three dimensions). The point group of a space group does not quite determine its lattice system, because occasionally two space groups with the same point group may be in different lattice systems. Crystal families are formed from lattice systems by merging the two lattice systems whenever this happens, so that the crystal family of a space group is determined by either its lattice system or its point group. In 3 dimensions the only two lattice families that get merged in this way are the hexagonal and rhombohedral lattice systems, which are combined into the hexagonal crystal family. The 6 crystal families in 3 dimensions are called triclinic, monoclinic, orthorhombic, tetragonal, hexagonal, and cubic. Crystal families are commonly used in popular books on crystals, where they are sometimes called crystal systems.	

Conway, Delgado Friedrichs, and Huson et al. (2001) gave another classification of the space groups, according to the fibrifold structures on the corresponding orbifold. They divided the 219 affine space groups into reducible and irreducible groups. The reducible groups fall into 17 classes corresponding to the 17 wallpaper groups, and the remaining 35 irreducible groups are the same as the cubic groups and are classified separately.

Space groups in other dimensions

Bieberbach's theorems

In n dimensions, an affine space group, or Bieberbach group, is a discrete subgroup of isometries of n -dimensional Euclidean space with a compact fundamental domain. Bieberbach (1911, 1912) proved that the subgroup of translations of any such group contains n linearly independent translations, and is a free abelian subgroup of finite index, and is also the unique maximal normal abelian subgroup. He also showed that in any dimension n there are only a finite number of possibilities for the isomorphism class of the underlying group of a space group, and moreover the action of the group on Euclidean space is unique up to conjugation by affine transformations. This answers part of Hilbert's 18th problem. Zassenhaus (1948) showed that conversely any group that is the extension of $\mathbf{Z}^n$ by a finite group acting faithfully is an affine space group. Combining these results shows that classifying space groups in n dimensions up to conjugation by affine transformations is essentially the same as classifying isomorphism classes for groups that are extensions of $\mathbf{Z}^n$ by a finite group acting faithfully.

It is essential in Bieberbach's theorems to assume that the group acts as isometries; the theorems do not generalize to discrete cocompact groups of affine transformations of Euclidean space. A counter-example is given by the 3-dimensional Heisenberg group of the integers acting by translations on the Heisenberg group of the reals, identified with 3-dimensional Euclidean space. This is a discrete cocompact group of affine transformations of space, but does not contain a subgroup $\mathbf{Z}^3$.

Classification in small dimensions

This table give the number of space group types in small dimensions.

Dimension	Lattice types (sequence A004030 ^[1] in OEIS)	point groups (sequence A004028 ^[2] in OEIS)	Crystallographic space group types (sequence A006227 ^[3] in OEIS)	Affine space group types (sequence A004029 ^[4] in OEIS)	Classification
0	1	1	1	1	Trivial group
1	1	2	2	2	One is the group of integers and the other is the infinite dihedral group; see symmetry groups in one dimension
2	5	10	17	17	these 2D space groups are also called wallpaper groups or plane groups .
3	14	32	230	219	In 3D there are 230 crystallographic space group types, which reduces to 219 affine space group types because of some types being different from their mirror image; these are said to differ by "enantiomorphous character" (e.g. $P3_112$ and $P3_212$). Usually "space group" refers to 3D. They were enumerated independently by Barlow (1894), Fedorov (1891) and Schönflies (1891).
4	64	227	4895	4783	The 4895 4-dimensional groups were enumerated by Harold Brown, Rolf Bülow, and Joachim Neubüser et al. (1978).

5	189	955		222018	Plesken & Schulz (2000) enumerated the ones of dimension 5
6		7104	28934974	28927922	Plesken & Schulz (2000) enumerated the ones of dimension 6

Double groups and time reversal

In addition to crystallographic space groups there are also magnetic space groups or double groups. These symmetries contain an element known as time reversal. They are of importance in magnetic structures that contain ordered unpaired spins, i.e. ferro-, ferri- or antiferromagnetic structures as studied by neutron diffraction. The time reversal element flips a magnetic spin while leaving all other structure the same and it can be combined with a number of other symmetry elements. Including time reversal there are 1651 magnetic space groups in 3D (Kim 1999, p.428).

Table of space groups in 3 dimensions

Crystal system	Point group		#	Space groups (international short symbol)
	Hermann-Mauguin	Schönflies		
Triclinic (2)	1	C_1	1	P1
	1	C_i	2	P1
Monoclinic (13)	2	C_2	3-5	P2, $P2_1$, C2
	m	C_s	6-9	Pm, Pc, Cm, Cc
	2/m	C_{2h}	10-15	P2/m, $P2_1/m$, C2/m, P2/c, $P2_1/c$, C2/c
Orthorhombic (59)	222	D_2	16-24	P222, $P222_1$, $P2_12_12$, $P2_12_12_1$, C222 $_1$, C222, F222, I222, $I2_12_12_1$
	mm2	C_{2v}	25-46	Pmm2, Pmc2 $_1$, Pcc2, Pma2, Pca2 $_1$, Pnc2, Pmn2 $_1$, Pba2, Pna2 $_1$, Pnn2, Cmm2, Cmc2 $_1$, Ccc2, Amm2, Aem2, Ama2, Aea2, Fmm2, Fdd2, Imm2, Iba2, Ima2
	mmm	D_{2h}	47-74	Pmmm, Pnnn, Pccm, Pban, Pmma, Pnna, Pmna, Pcca, Pbam, Pccn, Pbcm, Pnnm, Pmnm, Pbcn, Pbca, Pnma, Cmcm, Cmce, Cmmm, Cccm, Cmme, Ccce, Fmmm, Fddd, Immm, Ibam, Ibca, Imma
Tetragonal (68)	4	C_4	75-80	P4, $P4_1$, $P4_2$, $P4_3$, I4, $I4_1$
	4	S_4	81-82	P4, I4
	4/m	C_{4h}	83-88	P4/m, $P4_2/m$, P4/n, $P4_2/n$, I4/m, $I4_1/a$
	422	D_4	89-98	P422, $P42_12$, $P4_122$, $P4_12_12$, $P4_222$, $P4_22_12$, $P4_322$, $P4_32_12$, I422, $I4_122$
	4mm	C_{4v}	99-110	P4mm, P4bm, $P4_2cm$, $P4_2nm$, P4cc, P4nc, $P4_2mc$, $P4_2bc$, I4mm, I4cm, $I4_1md$, $I4_1cd$
	42m	D_{2d}	111-122	P42m, P42c, $P42_1m$, $P42_1c$, P4m2, P4c2, P4b2, P4n2, I4m2, I4c2, I42m, I42d
	4/mmm	D_{4h}	123-142	P4/mmm, P4/mcc, P4/nbm, P4/nnc, P4/mbm, P4/mnc, P4/nmm, P4/ncc, $P4_2/mmc$, $P4_2/mcm$, $P4_2/nbc$, $P4_2/nnm$, $P4_2/mbc$, $P4_2/mnm$, $P4_2/nmc$, $P4_2/ncm$, I4/mmm, $I4/mcm$, $I4_1/amd$, $I4_1/acd$
Trigonal (25)	3	C_3	143-146	P3, $P3_1$, $P3_2$, R3
	3	S_6	147-148	P3, R3
	32	D_3	149-155	P312, P321, $P3_112$, $P3_121$, $P3_212$, $P3_221$, R32
	3m	C_{3v}	156-161	P3m1, P31m, P3c1, P31c, R3m, R3c
	3m	D_{3d}	162-167	P31m, P31c, P3m1, P3c1, R3m, R3c,

Hexagonal (27)	6	C_6	168-173	$P6, P6_1, P6_5, P6_2, P6_4, P6_3$
	6	C_{3h}	174	$P6$
	6/m	C_{6h}	175-176	$P6/m, P6_3/m$
	622	D_6	177-182	$P622, P6_122, P6_522, P6_222, P6_422, P6_322$
	6mm	C_{6v}	183-186	$P6mm, P6cc, P6_3cm, P6_3mc$
	6m2	D_{3h}	187-190	$P6m2, P6c2, P62m, P62c$
	6/mmm	D_{6h}	191-194	$P6/mmm, P6/mcc, P6_3/mcm, P6_3/mmc$
Cubic (36)	23	T	195-199	$P23, F23, I23, P2_13, I2_13$
	m3	T_h	200-206	$Pm3, Pn3, Fm3, Fd3, Im3, Pa3, Ia3$
	432	O	207-214	$P432, P4_232, F432, F4_132, I432, P4_332, P4_132, I4_132$
	43m	T_d	215-220	$P43m, F43m, I43m, P43n, F43c, I43d$
	m3m	O_h	221-230	$Pm3m, Pn3n, Pm3n, Pn3m, Fm3m, Fm3c, Fd3m, Fd3c, Im3m, Ia3d$

Note. An e plane is a double glide plane, one having glides in two different directions. They are found in five space groups, all in the orthorhombic system and with a centered lattice. The use of the symbol e became official with Hahn (2002).

The lattice system can be found as follows. If the crystal system is not trigonal then the lattice system is of the same type. If the crystal system is trigonal, then the lattice system is hexagonal unless the space group is one of the seven in the rhombohedral lattice system consisting of the 7 trigonal space groups in the table above whose name begins with R. (The term rhombohedral system is also sometimes used as an alternative name for the whole trigonal system.) The hexagonal lattice system is larger than the hexagonal crystal system, and consists of the hexagonal crystal system together with the 18 groups of the trigonal crystal system other than the seven whose names begin with R.

The Bravais lattice of the space group is determined by the lattice system together with the initial letter of its name, which for the non-rhombohedral groups is P, I, F, or C, standing for the principal, body centered, face centered, or C-face centered lattices.

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External links

- International Union of Crystallography ^[10]
- Point Groups and Bravais Lattices ^[11]
- Bilbao Crystallographic Server ^[12]
- Space Group Info (old) ^[13]
- Space Group Info (new) ^[14]
- Crystal Lattice Structures: Index by Space Group ^[15]
- Full list of 230 crystallographic space groups ^[16]
- Interactive 3D visualization of all 230 crystallographic space groups ^[17]
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Noether's Theorem

Noether's (first) theorem states that any differentiable symmetry of the action of a physical system has a corresponding conservation law. The theorem was proved by German mathematician Emmy Noether in 1915 and published in 1918.^[1] The action of a physical system is the integral over time of a Lagrangian function (which may or may not be an integral over space of a Lagrangian density function), from which the system's behavior can be determined by the principle of least action.

Noether's theorem has become a fundamental tool of modern theoretical physics and the calculus of variations. A generalization of the seminal formulations on constants of motion in Lagrangian and Hamiltonian mechanics (1788 and 1833, respectively), it does not apply to systems that cannot be modeled with a Lagrangian; for example, dissipative systems with continuous symmetries need not have a corresponding conservation law.

For illustration, if a physical system behaves the same regardless of how it is oriented in space, its Lagrangian is rotationally symmetric; from this symmetry, Noether's theorem shows the angular momentum of the system must be conserved. The physical system itself need not be symmetric; a jagged asteroid tumbling in space conserves angular momentum despite its asymmetry – it is the laws of motion that are symmetric. As another example, if a physical experiment has the same outcome regardless of place or time (having the same outcome, say, somewhere in Asia on a Tuesday or in America on a Wednesday), then its Lagrangian is symmetric under continuous translations in space and time; by Noether's theorem, these symmetries account for the conservation laws of linear momentum and energy within this system, respectively. (These examples are just for illustration; in the first one, Noether's theorem added nothing new – the results were known to follow from Lagrange's equations and from Hamilton's equations.)

Noether's theorem is important, both because of the insight it gives into conservation laws, and also as a practical calculational tool. It allows researchers to determine the conserved quantities from the observed symmetries of a physical system. Conversely, it allows researchers to consider whole classes of hypothetical Lagrangians to describe a physical system. For illustration, suppose that a new field is discovered that conserves a quantity X . Using Noether's theorem, the types of Lagrangians that conserve X because of a continuous symmetry can be determined, and then their fitness judged by other criteria.

There are numerous different versions of Noether's theorem, with varying degrees of generality. The original version only applied to ordinary differential equations (particles) and not partial differential equations (fields). The original versions also assume that the Lagrangian only depends upon the first derivative, while later versions generalize the theorem to Lagrangians depending on the n^{th} derivative. There is also a quantum version of this theorem, known as the Ward–Takahashi identity. Generalizations of Noether's theorem to superspaces also exist.

Informal statement of the theorem

All fine technical points aside, Noether's theorem can be stated informally

If a system has a continuous symmetry property, then there are corresponding quantities whose values are conserved in time.^[2]

A more sophisticated version of the theorem states that:

To every differentiable symmetry generated by local actions, there corresponds a conserved current.

The word "symmetry" in the above statement refers more precisely to the covariance of the form that a physical law takes with respect to a one-dimensional Lie group of transformations satisfying certain technical criteria. The conservation law of a physical quantity is usually expressed as a continuity equation.

The formal proof of the theorem uses only the condition of invariance to derive an expression for a current associated with a conserved physical quantity. The conserved quantity is called the *Noether charge* and the flow carrying that 'charge' is called the *Noether current*. The Noether current is defined up to a solenoidal vector field.

Historical context

A conservation law states that some quantity X describing a system remains constant throughout its motion; expressed mathematically, the rate of change of X (its derivative with respect to time) is zero:

$$\frac{dX}{dt} = 0.$$

Such quantities are said to be conserved; they are often called constants of motion, although motion *per se* need not be involved, just evolution in time. For example, if the energy of a system is conserved, its energy is constant at all times, which imposes a constraint on the system's motion and may help to solve for it. Aside from the insight that such constants of motion give into the nature of a system, they are a useful calculational tool; for example, an approximate solution can be corrected by finding the nearest state that satisfies the necessary conservation laws.

The earliest constants of motion discovered were momentum and energy, which were proposed in the 17th century by René Descartes and Gottfried Leibniz on the basis of collision experiments, and refined by subsequent researchers. Isaac Newton was the first to enunciate the conservation of momentum in its modern form, and showed that it was a consequence of Newton's third law; interestingly, conservation of momentum still holds even in situations when Newton's third law is incorrect. Modern physics has revealed that the conservation laws of momentum and energy are only approximately true, but their modern refinements – the conservation of four-momentum in special relativity and the zero covariant divergence of the stress-energy tensor in general relativity – are rigorously true within the limits of those theories. The conservation of angular momentum, a generalization to rotating rigid bodies, likewise holds in modern physics. Another important conserved quantity, discovered in studies of the celestial mechanics of astronomical bodies, was the Laplace–Runge–Lenz vector.

In the late 18th and early 19th centuries, physicists developed more systematic methods for discovering conserved quantities. A major advance came in 1788 with the development of Lagrangian mechanics, which is related to the principle of least action. In this approach, the state of the system can be described by any type of generalized coordinates $\mathbf{q}$; the laws of motion need not be expressed in a Cartesian coordinate system, as was customary in Newtonian mechanics. The action is defined as the time integral I of a function known as the Lagrangian L

$$I = \int L(\mathbf{q}, \dot{\mathbf{q}}, t) dt$$

where the dot over $\mathbf{q}$ signifies the rate of change of the coordinates $\mathbf{q}$

$$\dot{\mathbf{q}} = \frac{d\mathbf{q}}{dt}.$$

Hamilton's principle states that the physical path $\mathbf{q}(t)$ – the one truly taken by the system – is a path for which infinitesimal variations in that path cause no change in I , at least up to first order. This principle results in the Euler–Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) = \frac{\partial L}{\partial \mathbf{q}}.$$

Thus, if one of the coordinates, say q_k , does not appear in the Lagrangian, the right-hand side of the equation is zero, and the left-hand side shows that

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = \frac{dp_k}{dt} = 0$$

where the conserved momentum p_k is defined as the left-hand quantity in parentheses. The absence of the coordinate q_k from the Lagrangian implies that the Lagrangian is unaffected by changes or transformations of q_k ; the Lagrangian is invariant, and is said to exhibit a kind of symmetry. This is the seed idea from which Noether's theorem was born.

Several alternative methods for finding conserved quantities were developed in the 19th century, especially by William Rowan Hamilton. For example, he developed a theory of canonical transformations that allowed researchers

to change coordinates so that coordinates disappeared from the Lagrangian, resulting in conserved quantities. Another approach and perhaps the most efficient for finding conserved quantities is the Hamilton–Jacobi equation.

Mathematical expression

The essence of Noether's theorem is the following: Imagine that the action I defined above is invariant under small perturbations (warpings) of the time variable t and the generalized coordinates $\mathbf{q}$; (in a notation commonly used by physicists) we write

$$t \rightarrow t' = t + \delta t$$

$$\mathbf{q} \rightarrow \mathbf{q}' = \mathbf{q} + \delta \mathbf{q}$$

where the perturbations δt and $\delta \mathbf{q}$ are both small but variable. For generality, assume that there might be several such symmetry transformations of the action, say, N ; we may use an index $r = 1, 2, 3, \dots, N$ to keep track of them. Then a generic perturbation can be written as a linear sum of the individual types of perturbations

$$\delta t = \sum_r \epsilon_r T_r$$

$$\delta \mathbf{q} = \sum_r \epsilon_r \mathbf{Q}_r.$$

Using these definitions, Emmy Noether showed that the N quantities

$$\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} - L \right) T_r - \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \mathbf{Q}_r$$

are conserved, i.e., are constants of motion; this is a simple version of Noether's theorem.

Examples

For illustration, consider a Lagrangian that does not depend on time, i.e., that is invariant (symmetric) under changes $t \rightarrow t + \delta t$, without any change in the coordinates $\mathbf{q}$. In this case, $N = 1$, $T = 1$ and $\mathbf{Q} = 0$; the corresponding conserved quantity is the total energy H ^[3]

$$H = \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} - L.$$

Similarly, consider a Lagrangian that does not depend on a coordinate q_k , i.e., that is invariant (symmetric) under changes $q_k \rightarrow q_k + \delta q_k$. In that case, $N = 1$, $T = 0$, and $\mathbf{Q}_k = 1$; the conserved quantity is the corresponding momentum p_k ^[4]

$$p_k = \frac{\partial L}{\partial \dot{q}_k}.$$

In special and general relativity, these apparently separate conservation laws are aspects of a single conservation law, that of the stress-energy tensor,^[5] that is derived in the next section.

The conservation of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is slightly more complicated to derive, but analogous to its linear momentum counterpart.^[6] It is assumed that the symmetry of the Lagrangian is rotational, i.e., that the Lagrangian does not depend on the absolute orientation of the physical system in space. For concreteness, assume that the Lagrangian does not change under small rotations of an angle $\delta\theta$ about an axis $\mathbf{n}$; such a rotation transforms the Cartesian coordinates by the equation

$$\mathbf{r} \rightarrow \mathbf{r} + \delta\theta \mathbf{n} \times \mathbf{r}.$$

Since time is not being transformed, T equals zero. Taking $\delta\theta$ as the ϵ parameter and the Cartesian coordinates $\mathbf{r}$ as the generalized coordinates $\mathbf{q}$, the corresponding $\mathbf{Q}$ variables are given by

$$\mathbf{Q} = \mathbf{n} \times \mathbf{r}.$$

Then Noether's theorem states that the following quantity is conserved

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \mathbf{Q}_r = \mathbf{p} \cdot (\mathbf{n} \times \mathbf{r}) = \mathbf{n} \cdot (\mathbf{r} \times \mathbf{p}) = \mathbf{n} \cdot \mathbf{L}.$$

In other words, the component of the angular momentum $\mathbf{L}$ along the $\mathbf{n}$ axis is conserved. If $\mathbf{n}$ is arbitrary, i.e., if the system is insensitive to any rotation, then every component of $\mathbf{L}$ is conserved; in short, angular momentum is conserved.

Field-theory version

Although useful in its own right, the version of her theorem just given was a special case of the general version she derived in 1915. To give the flavor of the general theorem, a version of the Noether theorem for continuous fields in four-dimensional space-time is now given. Since field theory problems are more common in modern physics than mechanics problems, this field-theory version is the most commonly used version of Noether's theorem.

Let there be a set of differentiable fields ϕ_k defined over all space and time; for example, the temperature $T(\mathbf{x}, t)$ would be representative of such a field, being a number defined at every place and time. The principle of least action can be applied to such fields, but the action is now an integral over space and time

$$I = \int L(\phi, \partial_\mu \phi, x^\mu) d^4x$$

(the theorem can actually be further generalized to the case where the Lagrangian depends on up to the n^{th} derivative using jet bundles)

Let the action be invariant under certain transformations of the space-time coordinates x^μ and the fields ϕ_k

$$\begin{aligned} x^\mu &\rightarrow x^\mu + \delta x^\mu \\ \phi &\rightarrow \phi + \delta \phi \end{aligned}$$

where the transformations can be indexed by $r = 1, 2, 3, \dots, N$

$$\begin{aligned} \delta x^\mu &= \epsilon_r X_r^\mu \\ \delta \phi &= \epsilon_r \Psi_r \end{aligned}$$

For such systems, Noether's theorem states that there are N conserved current densities

$$j_r^\nu = - \left(\frac{\partial L}{\partial \phi_{,\nu}} \right) \cdot \Psi_r + \sum_\sigma \left[\left(\frac{\partial L}{\partial \phi_{,\nu}} \right) \cdot \phi_{,\sigma} - L \delta_\sigma^\nu \right] X_r^\sigma$$

In such cases, the conservation law is expressed in a four-dimensional way

$$\sum_\nu \frac{\partial j_r^\nu}{\partial x^\nu} = 0$$

which expresses the idea that the amount of a conserved quantity within a sphere cannot change unless some of it flows out of the sphere. For example, electric charge is conserved; the amount of charge within a sphere cannot change unless some of the charge leaves the sphere.

For illustration, consider a physical system of fields that behaves the same under translations in time and space, as considered above; in other words, the fields do not depend on the absolute position in space and time. In that case, $N = 4$, one for each dimension of space and time. Since only the positions in space-time are being warped, not the fields, the Ψ are all zero and the X_μ^ν equal the Kronecker delta δ_μ^ν , where we have used μ instead of r for the index. In that case, Noether's theorem corresponds to the conservation law for the stress-energy tensor $T_\mu^{\nu[5]}$

$$T_\mu^\nu = \sum_\sigma \left[\left(\frac{\partial L}{\partial \phi_{,\nu}} \right) \cdot \phi_{,\sigma} - L \delta_\sigma^\nu \right] \delta_\mu^\sigma = \left(\frac{\partial L}{\partial \phi_{,\nu}} \right) \cdot \phi_{,\mu} - L \delta_\mu^\nu$$

The conservation of electric charge can be derived by considering transformations of the fields themselves.^[7] In quantum mechanics, the probability amplitude $\psi(\mathbf{x})$ of finding a particle at a point $\mathbf{x}$ is a complex field, because it ascribes a complex number to every point in space and time. The probability amplitude itself is physically

unmeasurable; only the probability $p = |\psi|^2$ is directly measurable. Therefore, the system is invariant under transformations of the ψ field and its complex conjugate field ψ^* that leave $|\psi|^2$ unchanged, such as

$$\psi \rightarrow e^{i\theta} \psi, \quad \psi^* \rightarrow e^{-i\theta} \psi^*$$

In the limit when θ becomes infinitesimally small ($\delta\theta$), it may be taken as the ϵ , and the Ψ are equal to $i\psi$ and $-i\psi^*$, respectively. A specific example is the Klein–Gordon equation, the relativistically correct version of the Schrödinger equation for spinless particles, which has the Lagrangian density

$$L = \psi_{,\nu} \psi^*_{,\mu} \eta^{\nu\mu} + m^2 \psi \psi^*.$$

In this case, Noether's theorem states that the conserved current equals

$$j^\nu = i \left(\frac{\partial \psi}{\partial x^\mu} \psi^* - \frac{\partial \psi^*}{\partial x^\mu} \psi \right) \eta^{\nu\mu}$$

which, when multiplied by the charge, equals the electric current density. This transformation was first noted by Hermann Weyl and is one of the fundamental gauge symmetries of modern physics.

Derivations

One independent variable

Consider the simplest case, a system with one independent variable, time. Suppose the dependent variables $\mathbf{q}$ are such that the action integral

$$I = \int_{t_1}^{t_2} L[\mathbf{q}[t], \dot{\mathbf{q}}[t], t] dt$$

is invariant under brief infinitesimal variations in the dependent variables. In other words, they satisfy the Euler–Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} [t] = \frac{\partial L}{\partial \mathbf{q}} [t].$$

And suppose that the integral is invariant under a continuous symmetry. Mathematically such a symmetry is represented as a flow, ϕ , which acts on the variables as follows

$$t \rightarrow t' = t + \epsilon T$$

$$\mathbf{q}[t] \rightarrow \mathbf{q}'[t'] = \phi[\mathbf{q}[t], \epsilon] = \phi[\mathbf{q}[t' - \epsilon T], \epsilon]$$

where ϵ is a real variable indicating the amount of flow and T is a real constant (which could be zero) indicating how much the flow shifts time.

$$\dot{\mathbf{q}}[t] \rightarrow \dot{\mathbf{q}}'[t'] = \frac{d}{dt} \phi[\mathbf{q}[t], \epsilon] = \frac{\partial \phi}{\partial \mathbf{q}} [\mathbf{q}[t' - \epsilon T], \epsilon] \dot{\mathbf{q}}[t' - \epsilon T].$$

The action integral flows to

$$I'[\epsilon] = \int_{t_1 + \epsilon T}^{t_2 + \epsilon T} L[\mathbf{q}'[t'], \dot{\mathbf{q}}'[t'], t'] dt'$$

$$= \int_{t_1 + \epsilon T}^{t_2 + \epsilon T} L[\phi[\mathbf{q}[t' - \epsilon T], \epsilon], \frac{\partial \phi}{\partial \mathbf{q}} [\mathbf{q}[t' - \epsilon T], \epsilon] \dot{\mathbf{q}}[t' - \epsilon T], t'] dt'$$

which may be regarded as a function of ϵ . Calculating the derivative at $\epsilon = 0$ and using the symmetry, we get

$$0 = \frac{dI'}{d\epsilon} [0] = L[\mathbf{q}[t_2], \dot{\mathbf{q}}[t_2], t_2] T - L[\mathbf{q}[t_1], \dot{\mathbf{q}}[t_1], t_1] T$$

$$+ \int_{t_1}^{t_2} \frac{\partial L}{\partial \mathbf{q}} \left(-\frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}} T + \frac{\partial \phi}{\partial \epsilon} \right) + \frac{\partial L}{\partial \dot{\mathbf{q}}} \left(-\frac{\partial^2 \phi}{(\partial \mathbf{q})^2} \dot{\mathbf{q}}^2 T + \frac{\partial^2 \phi}{\partial \epsilon \partial \mathbf{q}} \dot{\mathbf{q}} - \frac{\partial \phi}{\partial \mathbf{q}} \ddot{\mathbf{q}} T \right) dt.$$

Notice that the Euler–Lagrange equations imply

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}} T \right) &= \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}} T + \frac{\partial L}{\partial \dot{\mathbf{q}}} \left(\frac{d}{dt} \frac{\partial \phi}{\partial \mathbf{q}} \right) \dot{\mathbf{q}} T + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \ddot{\mathbf{q}} T \\ &= \frac{\partial L}{\partial \mathbf{q}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}} T + \frac{\partial L}{\partial \dot{\mathbf{q}}} \left(\frac{\partial^2 \phi}{(\partial \mathbf{q})^2} \dot{\mathbf{q}} \right) \dot{\mathbf{q}} T + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \ddot{\mathbf{q}} T. \end{aligned}$$

Substituting this into the previous equation, one gets

$$\begin{aligned} 0 &= \frac{dI'}{d\epsilon} [0] = L[\mathbf{q}[t_2], \dot{\mathbf{q}}[t_2], t_2] T - L[\mathbf{q}[t_1], \dot{\mathbf{q}}[t_1], t_1] T - \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}}[t_2] T + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}}[t_1] T \\ &\quad + \int_{t_1}^{t_2} \frac{\partial L}{\partial \mathbf{q}} \frac{\partial \phi}{\partial \epsilon} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial^2 \phi}{\partial \epsilon \partial \mathbf{q}} \dot{\mathbf{q}} dt. \end{aligned}$$

Again using the Euler–Lagrange equations we get

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \epsilon} \right) = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \frac{\partial \phi}{\partial \epsilon} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial^2 \phi}{\partial \epsilon \partial \mathbf{q}} \dot{\mathbf{q}} = \frac{\partial L}{\partial \mathbf{q}} \frac{\partial \phi}{\partial \epsilon} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial^2 \phi}{\partial \epsilon \partial \mathbf{q}} \dot{\mathbf{q}}.$$

Substituting this into the previous equation, one gets

$$\begin{aligned} 0 &= L[\mathbf{q}[t_2], \dot{\mathbf{q}}[t_2], t_2] T - L[\mathbf{q}[t_1], \dot{\mathbf{q}}[t_1], t_1] T - \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}}[t_2] T + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}}[t_1] T \\ &\quad + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \epsilon} [t_2] - \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \epsilon} [t_1]. \end{aligned}$$

From which one can see that

$$\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}} - L \right) T - \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \epsilon}$$

is a constant of the motion, i.e. a conserved quantity. Since $\phi[\mathbf{q}, 0] = \mathbf{q}$, we get $\frac{\partial \phi}{\partial \mathbf{q}} = 1$ and so the conserved quantity simplifies to

$$\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \dot{\mathbf{q}} - L \right) T - \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \epsilon}.$$

To avoid excessive complication of the formulas, this derivation assumed that the flow does not change as time passes. The same result can be obtained in the more general case.

Field-theoretic derivation

Noether's theorem may also be derived for tensor fields φ^A where the index A ranges over the various components of the various tensor fields. These field quantities are functions defined over a four-dimensional space whose points are labeled by coordinates x^μ where the index μ ranges over time ($\mu=0$) and three spatial dimensions ($\mu=1,2,3$). These four coordinates are the independent variables; and the values of the fields at each event are the dependent variables. Under an infinitesimal transformation, the variation in the coordinates is written

$$x^\mu \rightarrow \xi^\mu = x^\mu + \delta x^\mu$$

whereas the transformation of the field variables is expressed as

$$\phi^A \rightarrow \alpha^A(\xi^\mu) = \phi^A(x^\mu) + \delta \phi^A(x^\mu).$$

By this definition, the field variations $\delta \varphi^A$ result from two factors: intrinsic changes in the field themselves and changes in coordinates, since the transformed field α^A depends on the transformed coordinates ξ^μ . To isolate the intrinsic changes, the field variation at a single point x^μ may be defined

$$\alpha^A(x^\mu) = \phi^A(x^\mu) + \bar{\delta} \phi^A(x^\mu).$$

If the coordinates are changed, the boundary of the region of space-time over which the Lagrangian is being integrated also changes; the original boundary and its transformed version are denoted as Ω and Ω' , respectively.

Noether's theorem begins with the assumption that a specific transformation of the coordinates and field variables does not change the action, which is defined as the integral of the Lagrangian density over the given region of spacetime. Expressed mathematically, this assumption may be written as

$$\int_{\Omega'} L(\alpha^A, \alpha^A_{,\nu}, \xi^\mu) d^4\xi - \int_{\Omega} L(\phi^A, \phi^A_{,\nu}, x^\mu) d^4x = 0$$

where the comma subscript indicates a partial derivative with respect to the coordinate(s) that follows the comma, e.g.

$$\phi^A_{,\sigma} = \frac{\partial \phi^A}{\partial x^\sigma}.$$

Since ξ is a dummy variable of integration, and since the change in the boundary Ω is infinitesimal by assumption, the two integrals may be combined using the four-dimensional version of the divergence theorem into the following form

$$\int_{\Omega} \left\{ \left[L(\alpha^A, \alpha^A_{,\nu}, x^\mu) - L(\phi^A, \phi^A_{,\nu}, x^\mu) \right] + \frac{\partial}{\partial x^\sigma} \left[L(\phi^A, \phi^A_{,\nu}, x^\mu) \delta x^\sigma \right] \right\} d^4x = 0.$$

The difference in Lagrangians can be written to first-order in the infinitesimal variations as

$$\left[L(\alpha^A, \alpha^A_{,\nu}, x^\mu) - L(\phi^A, \phi^A_{,\nu}, x^\mu) \right] = \frac{\partial L}{\partial \phi^A} \bar{\delta} \phi^A + \frac{\partial L}{\partial \phi^A_{,\sigma}} \bar{\delta} \phi^A_{,\sigma}.$$

However, because the variations are defined at the same point as described above, the variation and the derivative can be done in reverse order; they commute

$$\bar{\delta} \phi^A_{,\sigma} = \bar{\delta} \frac{\partial \phi^A}{\partial x^\sigma} = \frac{\partial}{\partial x^\sigma} (\bar{\delta} \phi^A).$$

Using the Euler–Lagrange field equations

$$\frac{\partial}{\partial x^\sigma} \left(\frac{\partial L}{\partial \phi^A_{,\sigma}} \right) = \frac{\partial L}{\partial \phi^A}$$

the difference in Lagrangians can be written neatly as

$$\left[L(\alpha^A, \alpha^A_{,\nu}, x^\mu) - L(\phi^A, \phi^A_{,\nu}, x^\mu) \right] = \frac{\partial}{\partial x^\sigma} \left(\frac{\partial L}{\partial \phi^A_{,\sigma}} \right) \bar{\delta} \phi^A + \frac{\partial L}{\partial \phi^A_{,\sigma}} \bar{\delta} \phi^A_{,\sigma} = \frac{\partial}{\partial x^\sigma} \left(\frac{\partial L}{\partial \phi^A_{,\sigma}} \bar{\delta} \phi^A \right).$$

Thus, the change in the action can be written as

$$\int_{\Omega} \frac{\partial}{\partial x^\sigma} \left\{ \frac{\partial L}{\partial \phi^A_{,\sigma}} \bar{\delta} \phi^A + L(\phi^A, \phi^A_{,\nu}, x^\mu) \delta x^\sigma \right\} d^4x = 0.$$

Since this holds for any region Ω , the integrand must be zero

$$\frac{\partial}{\partial x^\sigma} \left\{ \frac{\partial L}{\partial \phi^A_{,\sigma}} \bar{\delta} \phi^A + L(\phi^A, \phi^A_{,\nu}, x^\mu) \delta x^\sigma \right\} = 0.$$

For any combination of the various symmetry transformations, the perturbation can be written

$$\begin{aligned} \delta x^\mu &= \epsilon X^\mu \\ \delta \phi^A &= \epsilon \Psi^A = \bar{\delta} \phi^A + \epsilon \mathcal{L}_X \phi^A \end{aligned}$$

where $\mathcal{L}_X \phi^A$ is the Lie derivative of ϕ^A in the X^μ direction. When ϕ^A is a scalar or $X^\mu_{,\nu} = 0$,

$$\mathcal{L}_X \phi^A = \frac{\partial \phi^A}{\partial x^\mu} X^\mu.$$

These equations imply that the field variation taken at one point equals

$$\bar{\delta}\phi^A = \epsilon\Psi^A - \epsilon\mathcal{L}_X\phi^A.$$

Differentiating the above divergence with respect to ϵ at $\epsilon=0$ and changing the sign yields the conservation law

$$\frac{\partial}{\partial x^\sigma} j^\sigma = 0$$

where the conserved current equals

$$j^\sigma = \left[\frac{\partial L}{\partial \phi^A_{,\sigma}} \mathcal{L}_X \phi^A - L X^\sigma \right] - \left(\frac{\partial L}{\partial \phi^A_{,\sigma}} \right) \Psi^A.$$

Manifold/fiber bundle derivation

Suppose we have an n -dimensional oriented Riemannian manifold, M and a target manifold T . Let $\mathcal{C}$ be the configuration space of smooth functions from M to T . (More generally, we can have smooth sections of a fiber bundle over M .)

Examples of this M in physics include:

- In classical mechanics, in the Hamiltonian formulation, M is the one-dimensional manifold $\mathbf{R}$, representing time and the target space is the cotangent bundle of space of generalized positions.
- In field theory, M is the spacetime manifold and the target space is the set of values the fields can take at any given point. For example, if there are m real-valued scalar fields, $\phi_1, \dots, \phi_m$, then the target manifold is $\mathbf{R}^m$. If the field is a real vector field, then the target manifold is isomorphic to $\mathbf{R}^3$.

Now suppose there is a functional

$$\mathcal{S} : \mathcal{C} \rightarrow \mathbb{R},$$

called the action. (Note that it takes values into $\mathbb{R}$, rather than $\mathbb{C}$; this is for physical reasons, and doesn't really matter for this proof.)

To get to the usual version of Noether's theorem, we need additional restrictions on the action. We assume $\mathcal{S}[\phi]$ is the integral over M of a function

$$\mathcal{L}(\phi, \partial_\mu \phi, x)$$

called the Lagrangian density, depending on ϕ , its derivative and the position. In other words, for ϕ in $\mathcal{C}$

$$\mathcal{S}[\phi] = \int_M \mathcal{L}[\phi(x), \partial_\mu \phi(x), x] d^n x.$$

Suppose we are given boundary conditions, ie., a specification of the value of ϕ at the boundary if M is compact, or some limit on ϕ as x approaches ∞ . Then the subspace of $\mathcal{C}$ consisting of functions ϕ such that all functional derivatives of $\mathcal{S}$ at ϕ are zero, that is:

$$\frac{\delta \mathcal{S}[\phi]}{\delta \phi(x)} \approx 0$$

and that ϕ satisfies the given boundary conditions, is the subspace of on shell solutions. (See principle of stationary action)

Now, suppose we have an infinitesimal transformation on $\mathcal{C}$, generated by a functional derivation, Q such that

$$Q \left[\int_N \mathcal{L} d^n x \right] \approx \int_{\partial N} f^\mu[\phi(x), \partial\phi, \partial\partial\phi, \dots] ds_\mu$$

for all compact submanifolds N or in other words,

$$Q[\mathcal{L}(x)] \approx \partial_\mu f^\mu(x)$$

for all x , where we set

$$\mathcal{L}(x) = \mathcal{L}[\phi(x), \partial_\mu \phi(x), x].$$

If this holds on shell and off shell, we say Q generates an off-shell symmetry. If this only holds on shell, we say Q generates an on-shell symmetry. Then, we say Q is a generator of a one parameter symmetry Lie group.

Now, for any N , because of the Euler–Lagrange theorem, on shell (and only on-shell), we have

$$\begin{aligned} Q \left[\int_N \mathcal{L} d^n x \right] &= \int_N \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right] Q[\phi] d^n x + \int_{\partial N} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} Q[\phi] ds_\mu \\ &\approx \int_{\partial N} f^\mu ds_\mu. \end{aligned}$$

Since this is true for any N , we have

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} Q[\phi] - f^\mu \right] \approx 0.$$

But this is the continuity equation for the current J^μ defined by:^[8]

$$J^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} Q[\phi] - f^\mu,$$

which is called the **Noether current** associated with the symmetry. The continuity equation tells us that if we integrate this current over a space-like slice, we get a conserved quantity called the Noether charge (provided, of course, if M is noncompact, the currents fall off sufficiently fast at infinity).

Comments

Noether's theorem is really a reflection of the relation between the boundary conditions and the variational principle. Assuming no boundary terms in the action, Noether's theorem implies that

$$\int_{\partial N} J^\mu ds_\mu \approx 0.$$

Noether's theorem is an on shell theorem. The quantum analog of Noether's theorem are the Ward–Takahashi identities.

Generalization to Lie algebras

Suppose say we have two symmetry derivations Q_1 and Q_2 . Then, $[Q_1, Q_2]$ is also a symmetry derivation. Let's see this explicitly. Let's say

$$Q_1[\mathcal{L}] \approx \partial_\mu f_1^\mu$$

and

$$Q_2[\mathcal{L}] \approx \partial_\mu f_2^\mu$$

Then,

$$[Q_1, Q_2][\mathcal{L}] = Q_1[Q_2[\mathcal{L}]] - Q_2[Q_1[\mathcal{L}]] \approx \partial_\mu f_{12}^\mu$$

where $f_{12}^\mu = Q_1[f_2^\mu] - Q_2[f_1^\mu]$. So,

$$j_{12}^\mu = \left(\frac{\partial}{\partial (\partial_\mu \phi)} \mathcal{L} \right) (Q_1[Q_2[\phi]] - Q_2[Q_1[\phi]]) - f_{12}^\mu.$$

This shows we can extend Noether's theorem to larger Lie algebras in a natural way.

Generalization of the proof

This applies to *any* local symmetry derivation Q satisfying $QS \approx 0$, and also to more general local functional differentiable actions, including ones where the Lagrangian depends on higher derivatives of the fields. Let ϵ be any arbitrary smooth function of the spacetime (or time) manifold such that the closure of its support is disjoint from the boundary. ϵ is a test function. Then, because of the variational principle (which does *not* apply to the boundary, by the way), the derivation distribution q generated by $q[\epsilon][\Phi(x)] = \epsilon(x)Q[\Phi(x)]$ satisfies $q[\epsilon][S] \approx 0$ for any ϵ , or more compactly, $q(x)[S] \approx 0$ for all x not on the boundary (but remember that $q(x)$ is a shorthand for a derivation *distribution*, not a derivation parametrized by x in general). This is the generalization of Noether's theorem. To see how the generalization related to the version given above, assume that the action is the spacetime integral of a Lagrangian that only depends on ϕ and its first derivatives. Also, assume

$$Q[\mathcal{L}] \approx \partial_\mu f^\mu$$

Then,

$$\begin{aligned} q[\epsilon][S] &= \int q[\epsilon][\mathcal{L}] d^n x \\ &= \int \left\{ \left(\frac{\partial}{\partial \phi} \mathcal{L} \right) \epsilon Q[\phi] + \left[\frac{\partial}{\partial (\partial_\mu \phi)} \mathcal{L} \right] \partial_\mu (\epsilon Q[\phi]) \right\} d^n x \\ &= \int \left\{ \epsilon Q[\mathcal{L}] + \partial_\mu \epsilon \left[\frac{\partial}{\partial (\partial_\mu \phi)} \mathcal{L} \right] Q[\phi] \right\} d^n x \\ &\approx \int \epsilon \partial_\mu \left\{ f^\mu - \left[\frac{\partial}{\partial (\partial_\mu \phi)} \mathcal{L} \right] Q[\phi] \right\} d^n x \end{aligned}$$

for all ϵ .

More generally, if the Lagrangian depends on higher derivatives, then

$$\partial_\mu \left[f^\mu - \left[\frac{\partial}{\partial (\partial_\mu \phi)} \mathcal{L} \right] Q[\phi] - 2 \left[\frac{\partial}{\partial (\partial_\mu \partial_\nu \phi)} \mathcal{L} \right] \partial_\nu Q[\phi] + \partial_\nu \left[\left[\frac{\partial}{\partial (\partial_\mu \partial_\nu \phi)} \mathcal{L} \right] Q[\phi] \right] - \dots \right] \approx 0.$$

Examples

Example 1: Conservation of energy

Looking at the specific case of a Newtonian particle of mass m , coordinate x , moving under the influence of a potential V , coordinatized by time t . The action, S , is:

$$\begin{aligned} S[x] &= \int L[x(t), \dot{x}(t)] dt \\ &= \int \left(\frac{m}{2} \sum_{i=1}^3 \dot{x}_i^2 - V(x(t)) \right) dt. \end{aligned}$$

Consider the generator of time translations $Q = \partial/\partial t$. In other words, $Q[x(t)] = \dot{x}(t)$. Note that x has an explicit dependence on time, whilst V does not; consequently:

$$Q[L] = m \sum_i \dot{x}_i \ddot{x}_i - \sum_i \frac{\partial V(x)}{\partial x_i} \dot{x}_i = \frac{d}{dt} \left[\frac{m}{2} \sum_i \dot{x}_i^2 - V(x) \right]$$

so we can set

$$f = \frac{m}{2} \sum_i \dot{x}_i^2 - V(x).$$

Then,

$$\begin{aligned}
j &= \sum_{i=1}^3 \frac{\partial L}{\partial \dot{x}_i} Q[x_i] - f \\
&= m \sum_i \dot{x}_i^2 - \left[\frac{m}{2} \sum_i \dot{x}_i^2 - V(x) \right] \\
&= \frac{m}{2} \sum_i \dot{x}_i^2 + V(x).
\end{aligned}$$

The right hand side is the energy and Noether's theorem states that $\dot{j} = 0$ (i.e. the principle of conservation of energy is a consequence of invariance under time translations).

More generally, if the Lagrangian does not depend explicitly on time, the quantity

$$\sum_{i=1}^3 \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i - L$$

(called the Hamiltonian) is conserved.

Example 2: Conservation of center of momentum

Still considering 1-dimensional time, let

$$\begin{aligned}
\mathcal{S}[\vec{x}] &= \int \mathcal{L}[\vec{x}(t), \dot{\vec{x}}(t)] dt \\
&= \int \left[\sum_{\alpha=1}^N \frac{m_{\alpha}}{2} (\dot{\vec{x}}_{\alpha})^2 - \sum_{\alpha < \beta} V_{\alpha\beta}(\vec{x}_{\beta} - \vec{x}_{\alpha}) \right] dt
\end{aligned}$$

i.e. N Newtonian particles where the potential only depends pairwise upon the relative displacement.

For $\vec{Q}$, let's consider the generator of Galilean transformations (i.e. a change in the frame of reference). In other words,

$$Q_i[x_{\alpha}^j(t)] = t \delta_i^j.$$

Note that

$$\begin{aligned}
Q_i[\mathcal{L}] &= \sum_{\alpha} m_{\alpha} \dot{x}_{\alpha}^i - \sum_{\alpha < \beta} \partial_i V_{\alpha\beta}(\vec{x}_{\beta} - \vec{x}_{\alpha})(t - t) \\
&= \sum_{\alpha} m_{\alpha} \dot{x}_{\alpha}^i.
\end{aligned}$$

This has the form of $\frac{d}{dt} \sum_{\alpha} m_{\alpha} x_{\alpha}^i$ so we can set

$$\vec{f} = \sum_{\alpha} m_{\alpha} \vec{x}_{\alpha}.$$

Then,

$$\begin{aligned}
\vec{j} &= \sum_{\alpha} \left(\frac{\partial}{\partial \dot{\vec{x}}_{\alpha}} \mathcal{L} \right) \cdot \vec{Q}[\vec{x}_{\alpha}] - \vec{f} \\
&= \sum_{\alpha} (m_{\alpha} \dot{\vec{x}}_{\alpha} t - m_{\alpha} \vec{x}_{\alpha}) \\
&= \vec{P} t - M \vec{x}_{CM}
\end{aligned}$$

where $\vec{P}$ is the total momentum, M is the total mass and $\vec{x}_{CM}$ is the center of mass. Noether's theorem states:

$$\dot{j} = 0 \Rightarrow \vec{P} - M \dot{\vec{x}}_{CM} = 0.$$

Example 3: Conformal transformation

Both examples 1 and 2 are over a 1-dimensional manifold (time). An example involving spacetime is a conformal transformation of a massless real scalar field with a quartic potential in $(3 + 1)$ -Minkowski spacetime.

$$\begin{aligned}\mathcal{S}[\phi] &= \int \mathcal{L}[\phi(x), \partial_\mu \phi(x)] d^4x \\ &= \int \left(\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \lambda \phi^4 \right) d^4x\end{aligned}$$

For Q , consider the generator of a spacetime rescaling. In other words,

$$Q[\phi(x)] = x^\mu \partial_\mu \phi(x) + \phi(x).$$

The second term on the right hand side is due to the "conformal weight" of ϕ . Note that

$$Q[\mathcal{L}] = \partial^\mu \phi (\partial_\mu \phi + x^\nu \partial_\mu \partial_\nu \phi + \partial_\mu \phi) - 4\lambda \phi^3 (x^\mu \partial_\mu \phi + \phi).$$

This has the form of

$$\partial_\mu \left[\frac{1}{2} x^\mu \partial^\nu \phi \partial_\nu \phi - \lambda x^\mu \phi^4 \right] = \partial_\mu (x^\mu \mathcal{L})$$

(where we have performed a change of dummy indices) so set

$$f^\mu = x^\mu \mathcal{L}.$$

Then,

$$\begin{aligned}j^\mu &= \left[\frac{\partial}{\partial(\partial_\mu \phi)} \mathcal{L} \right] Q[\phi] - f^\mu \\ &= \partial^\mu \phi (x^\nu \partial_\nu \phi + \phi) - x^\mu \left(\frac{1}{2} \partial^\nu \phi \partial_\nu \phi - \lambda \phi^4 \right).\end{aligned}$$

Noether's theorem states that $\partial_\mu j^\mu = 0$ (as one may explicitly check by substituting the Euler–Lagrange equations into the left hand side).

(Aside: If one tries to find the Ward–Takahashi analog of this equation, one runs into a problem because of anomalies.)

Applications

Application of Noether's theorem allows physicists to gain powerful insights into any general theory in physics, by just analyzing the various transformations that would make the form of the laws involved invariant. For example:

- the invariance of physical systems with respect to spatial translation (in other words, that the laws of physics do not vary with locations in space) gives the law of conservation of linear momentum;
- invariance with respect to rotation gives the law of conservation of angular momentum;
- invariance with respect to time translation gives the well-known law of conservation of energy

In quantum field theory, the analog to Noether's theorem, the Ward–Takahashi identity, yields further conservation laws, such as the conservation of electric charge from the invariance with respect to a change in the phase factor of the complex field of the charged particle and the associated gauge of the electric potential and vector potential.

The Noether charge is also used in calculating the entropy of stationary black holes.^[9]

Notes

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- [9] Calculating the entropy of stationary black holes (<http://arxiv.org/abs/gr-qc/9503052>)

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External links

- English translation of Noether's paper. (<http://arxiv.org/pdf/physics/0503066>)
- John Baez (2002) " Noether's Theorem in a Nutshell. (<http://math.ucr.edu/home/baez/noether.html>)"
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Goldstone theorem

In particle and condensed matter physics, **Goldstone bosons** (also known as **Nambu-Goldstone bosons**) are bosons that appear necessarily in models exhibiting spontaneous breakdown of continuous symmetries (cf. spontaneously broken symmetry). They were discovered by Nambu in the context of the BCS superconductivity mechanism,^[1] and subsequently elucidated by Goldstone,^[2] and systematically generalized in the context of quantum field theory.^[3]

These spinless bosons correspond to the spontaneously broken internal symmetry generators, and are characterized by the quantum numbers of these. They transform **nonlinearly** (shift) under the action of these generators, and can thus be excited out of the asymmetric vacuum by these generators. Thus, they can be thought of as the excitations of the field in the broken symmetry directions in group space—and are massless if the spontaneously broken symmetry is *not also broken explicitly*. If, instead, the symmetry is not exact, i.e., if it is *explicitly broken as well as spontaneously broken*, then the Nambu-Goldstone bosons are not massless, though they typically remain relatively light; they are then called **pseudo-Goldstone bosons** or **pseudo-Nambu-Goldstone bosons** (abbreviated *PNGBs*).

Goldstone's theorem

Goldstone's theorem examines a generic continuous symmetry which is spontaneously broken; i.e., its currents are conserved, but the ground state (vacuum) is not invariant under the action of the corresponding charges. Then, necessarily, new massless (or light, if the symmetry is not exact) scalar particles appear in the spectrum of possible excitations. There is one scalar particle—called a Nambu-Goldstone boson—for each generator of the symmetry that is broken; i.e., that does not preserve the ground state. The Nambu-Goldstone mode is a long-wavelength fluctuation of the corresponding order parameter.

There is an arguable loophole in the theorem. If one reads the theorem carefully, it only states that there exist non-vacuum states with arbitrarily small energies. Take for example a chiral $N = 1$ super QCD model with a nonzero squark VEV which is conformal in the IR. The chiral symmetry is a global symmetry which is (partially) spontaneously broken. Some of the "Goldstone bosons" associated with this spontaneous symmetry breaking are charged under the unbroken gauge group and hence, these composite bosons have a continuous mass spectrum with arbitrarily small masses but yet there is no Goldstone boson with exactly zero mass. In other words, the Goldstone bosons are infraparticles.

In theories with gauge symmetry, the Goldstone bosons are "eaten" by the gauge bosons. The latter become massive and their new, longitudinal polarization is provided by the Goldstone boson.

A simple example

Consider a complex scalar field ϕ (phi), with the constraint that $\phi^* \phi = k^2$. One way to get a constraint of this sort is by including a potential

$$\lambda^2(\phi^* \phi - k^2)^2,$$

and taking the limit as $\lambda \rightarrow \infty$ (this is called the Abelian nonlinear σ -model).

The constraint, and the action, below, are invariant under a $U(1)$ phase transformation, $\delta\phi = i\varepsilon\phi$. The field can be redefined to give a real scalar field (i.e., a spin-zero particle) θ without any constraint by

$$\phi = ke^{i\theta}$$

where θ is the Nambu-Goldstone boson (actually $k\theta$ is), and the $U(1)$ symmetry transformation effects a *shift* on θ , namely $\delta\theta = \varepsilon$, but does not preserve the ground state $|0\rangle$, (i.e. the above infinitesimal transformation *does not annihilate it*—the hallmark of invariance), as evident in the charge of the current below. The vacuum is degenerate and noninvariant under the spontaneously broken symmetry.

The corresponding Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu \phi^*)\partial_\mu \phi + m^2 \phi^* \phi = -\frac{1}{2}(-ike^{-i\theta}\partial^\mu \theta)(ike^{i\theta}\partial_\mu \theta) + m^2 k^2 = -\frac{k^2}{2}(\partial^\mu \theta)(\partial_\mu \theta) + m^2 k^2,$$

and the symmetry-induced conserved $U(1)$ current is $J_\mu = -k^2 \partial_\mu \theta$. The charge resulting from this current, Q , shifts θ and the ground state to a new, degenerate, ground state. Thus, a vacuum with $\langle \theta \rangle = 0$ will shift to a *different* vacuum with $\langle \theta \rangle = -\varepsilon$. The current connects the original vacuum with the Nambu-Goldstone state, $\langle 0 | J_0(0) | \theta \rangle \neq 0$.

Note that the constant term $m^2 k^2$ in the Lagrangian density has no physical significance, and the other term in it is simply the kinetic term for a massless scalar.

In general, in a theory with several scalar fields, φ_j , the Nambu-Goldstone mode φ_g is massless, and parameterises the curve of possible (degenerate) vacuum states. Its hallmark under the broken symmetry transformation is **nonvanishing vacuum expectation** $\langle \delta \varphi_g \rangle$, an order parameter, for vanishing $\langle \varphi_g \rangle = 0$, at some ground state $|0\rangle$ chosen at the minimum of the potential, $\langle \partial V / \partial \varphi_j \rangle = 0$.

This nonvanishing vacuum expectation of the transformation increment, $\langle \delta \varphi_g \rangle$, specifies the relevant (Goldstone) *null eigenvector of the mass matrix*, $\langle \partial^2 V / \partial \varphi_i \partial \varphi_j \rangle \langle \delta \varphi_j \rangle = 0$, and hence the corresponding zero-mass eigenvalue.

Goldstone's Argument

The principle behind Goldstone's argument is that the ground state is not unique. Normally, by current conservation, the charge operator for any symmetry current is time-independent,

$$\frac{d}{dt}Q = \frac{d}{dt} \int_x J^0(x) = 0.$$

Acting with the charge operator on the vacuum either *annihilates the vacuum*, if that is symmetric; else, if *not*, as is the case in spontaneous symmetry breaking, it produces a zero-frequency state out of it, through its shift transformation feature illustrated above. (Actually, here, the charge itself is ill-defined; but its better behaved commutators with fields, so, then, the transformation shifts, are still time-invariant, $d\langle \delta \varphi_g \rangle / dt = 0$.)

So, if the vacuum is not invariant under the symmetry, action of the charge operator produces a state which is different from the vacuum chosen, but which has zero frequency. This is a long-wavelength oscillation of a field which is nearly stationary: there are states with zero frequency, so that the theory cannot have a mass gap.

This argument is clarified by taking the limit carefully. If an approximate charge operator is applied to the vacuum,

$$\frac{d}{dt}Q_A = \frac{d}{dt} \int_x e^{\frac{-x^2}{2A^2}} J^0(x) = - \int_x e^{\frac{-x^2}{2A^2}} \nabla \cdot J = \int_x \nabla \left(e^{\frac{-x^2}{2A^2}} \right) \cdot J,$$

a state with approximately vanishing time derivative is produced,

$$\left\| \frac{d}{dt}Q_A |0\rangle \right\| \approx \frac{1}{A} \|Q_A |0\rangle\|.$$

Assuming a mass gap m_0 , the frequency of any state like the above, which is orthogonal to the vacuum, is at least m_0 ,

$$\left\| \frac{d}{dt}|\psi\rangle \right\| = \|H|\psi\rangle\| \geq m_0 \| |\psi\rangle \|.$$

Letting A become large leads to a contradiction.

This argument fails, however, when the symmetry is gauged, because then the symmetry generator is only performing a gauge transformation. A gauge transformed state is the same exact state, so that acting with a symmetry generator does not get one out of the vacuum.

Nonrelativistic theories

A version of Goldstone's theorem also applies to nonrelativistic theories (and also relativistic theories with spontaneously broken spacetime symmetries, such as Lorentz symmetry or conformal symmetry, rotational, or translational invariance).

It essentially states that, for each spontaneously broken symmetry, there corresponds some quasiparticle with no energy gap—the nonrelativistic version of the mass gap. (Note that the energy here is really $H - \mu N - \vec{\alpha} \cdot \vec{P}$ and not H .) However, two *different* spontaneously broken generators may now give rise to the *same* Nambu-Goldstone boson. For example, in a superfluid, both the U(1) particle number symmetry and Galilean symmetry are spontaneously broken. However, the phonon is the Goldstone boson for both.

In general, the phonon is effectively the Nambu-Goldstone boson for spontaneously broken Galilean/Lorentz symmetry. However, in contrast to the case of internal symmetry breaking, when spacetime symmetries are broken, the order parameter *need not* be a scalar field, but may be a tensor field, and the corresponding independent massless modes may now be *fewer* than the number of spontaneously broken generators, because the Goldstone modes may now be linearly dependent among themselves: e.g., the Goldstone modes for some generators might be expressed as gradients of Goldstone modes for other broken generators.

Nambu-Goldstone fermions

Spontaneously broken global fermionic symmetries, which occur in some supersymmetric models, lead to Nambu-Goldstone fermions, or *goldstinos*.^{[4] [5]} These have spin $\frac{1}{2}$, instead of 0, and carry all quantum numbers of the respective supersymmetry generators broken spontaneously.

(Vestigial bosonic superpartners of these goldstinos under supersymmetries, called *sgoldstinos*, might sometimes also appear. Strictly speaking, however, spontaneous symmetry, and supersymmetry, breaking smashes up ("reduces") multiplet and supermultiplet structures, into the characteristic nonlinear realizations of broken supersymmetry, so that, in a technical sense, goldstinos are superpartners of *all* particles in the theory, of *any spin*.)

Goldstone bosons in nature

- In fluids, the phonon is longitudinal and it is the Goldstone boson of the spontaneously broken Galilean symmetry. In solids, the situation is more complicated; the Goldstone bosons are the longitudinal and transverse phonons and they happen to be the Goldstone bosons of spontaneously broken Galilean, translational, and rotational symmetry with no simple one-to-one correspondence between the Goldstone modes and the broken symmetries.
- In magnets, the original rotational symmetry (present in the absence of an external magnetic field) is spontaneously broken such that the magnetization points into a specific direction. The Goldstone bosons then are the *magnons*, i.e., spin waves in which the local magnetization direction oscillates.
- The *pions* are the pseudo-Goldstone bosons that result from the spontaneous breakdown of the chiral-flavor symmetries of QCD effected by quark condensation due to the strong interaction. These symmetries are further explicitly broken by the masses of the quarks, so that the pions are not massless, but their mass is *significantly smaller* than typical hadron masses.
- The longitudinal polarization components of the W and Z bosons correspond to the Goldstone bosons of the spontaneously broken part of the electroweak symmetry $SU(2) \otimes U(1)$. Because this symmetry is gauged, the three would-be Goldstone bosons are "eaten" by the three gauge bosons corresponding to the three broken generators; this gives these three gauge bosons a mass, and the associated necessary third polarization degree of freedom. This is described in the Standard Model through the Higgs mechanism.

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Stone-von Neumann theorem

In mathematics and in theoretical physics, the **Stone–von Neumann theorem** is any one of a number of different formulations of the uniqueness of the canonical commutation relations between position and momentum operators. The name is for Marshall Stone and von Neumann (1931).

Trying to represent the commutation relations

In quantum mechanics, physical observables are represented mathematically by linear operators on Hilbert spaces. For a single particle moving on the real line $\mathbf{R}$, there are two important observables: position and momentum. In the quantum-mechanical description of such a particle, the position operator Q and momentum operator P are respectively given by

$$\begin{aligned} [Q\psi](x) &= x\psi(x) \\ [P\psi](x) &= \frac{\hbar}{i}\psi'(x) \end{aligned}$$

on the domain V of infinitely differentiable functions of compact support on $\mathbf{R}$. We assume $\hbar$ is a fixed *non-zero* real number — in quantum theory $\hbar$ is (up to a factor of 2π) Planck's constant, which is not dimensionless; it takes a small numerical value in terms of units in the macroscopic world. The operators P , Q satisfy the commutation relation

$$QP - PQ = -\frac{\hbar}{i}\mathbf{1}.$$

Already in his classic volume, Hermann Weyl observed that this commutation law was impossible for linear operators P , Q acting on finite dimensional spaces (as is clear by applying the trace of a matrix), unless $\hbar$ vanishes. A little analysis shows that in fact any two self-adjoint operators satisfying the above commutation relation cannot be both bounded.

Uniqueness of representation

One would like to classify representations of the canonical commutation relation by two self-adjoint operators acting on separable Hilbert spaces, up to unitary equivalence. By Stone's theorem, there is a one-to-one correspondence between self-adjoint operators and (strongly continuous) one parameter unitary groups. Let Q and P be two self-adjoint operators satisfying the canonical commutation relation, and e^{itQ} and e^{isP} be the corresponding unitary groups given by functional calculus. A formal computation with power series shows that

$$e^{itQ}e^{isP} - e^{ist}e^{isP}e^{itQ} = 0.$$

Conversely, given two one parameter unitary groups $U(t)$ and $V(s)$ satisfying the relation

$$U(t)V(s) = e^{ist}V(s)U(t) \quad \forall s, t, \quad (*)$$

formally differentiating at 0 shows that the two infinitesimal generators satisfy the canonical commutation relation. These formal calculations can be made rigorous. Therefore there is a one-to-one correspondence between representations of the canonical commutation relation and two one parameter unitary groups $U(t)$ and $V(s)$ satisfying (*). This formulation of the canonical commutation relations for one parameter unitary groups is called the **Weyl form** of the CCR.

The problem now thus becomes classifying two jointly irreducible one parameter unitary groups $U(t)$ and $V(s)$ satisfying the Weyl relation on separable Hilbert spaces. The answer is the content of the **Stone-von Neumann theorem**: all such pairs of one parameter unitary groups are unitarily equivalent. In other words, for any two such $U(t)$ and $V(s)$ acting jointly irreducibly on a Hilbert space H , there is a unitary operator

$$W : L^2(\mathbb{R}) \rightarrow H$$

so that

$$W^*U(t)W = e^{isQ} \quad \text{and} \quad W^*V(t)W = e^{isP},$$

where P and Q are the position and momentum operators from above.

Historically this result was significant because it was a key step in proving that Heisenberg's matrix mechanics which presents quantum mechanical observables and dynamics in terms of infinite matrices, is unitarily equivalent to Schrödinger's wave mechanical formulation (see Schrödinger picture).

Representation theory formulation

In terms of representation theory, the Stone–von Neumann theorem classifies certain unitary representations of the Heisenberg group. This is discussed in more detail in the Heisenberg group section, below.

Informally stated, with certain technical assumptions, every representation of the Heisenberg group H_{2n+1} is equivalent to the position operators and momentum operators on $\mathbf{R}^n$. Alternatively, that they are all equivalent to the Weyl algebra (or CCR algebra) on a symplectic space of dimension $2n$.

More formally, there is a unique (up to scale) non-trivial central strongly continuous unitary representation.

This was later generalized by Mackey theory – and was the motivation for the introduction of the Heisenberg group in quantum physics.

In detail:

- The continuous Heisenberg group is a central extension of the abelian Lie group $\mathbf{R}^{2n}$ by a copy of $\mathbf{R}$,
- the corresponding Heisenberg algebra is a central extension of the abelian Lie algebra $\mathbf{R}^{2n}$ (with trivial bracket) by a copy of $\mathbf{R}$,
- the discrete Heisenberg group is a central extension of the free abelian group $\mathbf{Z}^{2n}$ by a copy of $\mathbf{Z}$, and
- the discrete Heisenberg group module p is a central extension of the free abelian p -group $(\mathbf{Z}/p\mathbf{Z})^{2n}$ by a copy of $\mathbf{Z}/p\mathbf{Z}$.

These are thus all semidirect product, and hence relatively easily understood. In all cases, if one has a representation $H \rightarrow A$ where the center maps to zero, then one simply has a representation of the corresponding abelian group or algebra, which is Fourier theory.

If the center does not map to zero, one has a more interesting theory, particularly if one restricts oneself to *central* representations.

Concretely, by a central representation one means a representation such that the center of the Heisenberg group maps into the center of the algebra: for example, if one is studying matrix representations or representations by operators on a Hilbert space, then the center of the matrix algebra or the operator algebra is the scalar matrices. Thus the representation of the center of the Heisenberg group is determined by a scale value, called the **quantization** value (in physics terms, Planck's constant), and if this goes to zero, one gets a representation of the abelian group (in physics terms, this is the classical limit).

More formally, the group algebra of the Heisenberg group $K[H]$ has center $K[\mathbf{R}]$, so rather than simply thinking of the group algebra as an algebra over the field of scalars K , one may think of it as an algebra over the commutative algebra $K[\mathbf{R}]$. As the center of a matrix algebra or operator algebra is the scalar matrices, a $K[\mathbf{R}]$ -structure on the matrix algebra is a choice of scalar matrix – a choice of scale. Given such a choice of scale, a central representation of the Heisenberg group is a map of $K[\mathbf{R}]$ -algebras $K[H] \rightarrow A$, which is the formal way of saying that it sends the center to a chosen scale. Then the Stone–von Neumann theorem is that, given a quantization value, every strongly continuous unitary representation is unitarily equivalent to the standard representation as position and momentum.

Reformulation via Fourier transform

Let G be a locally compact abelian group and $\hat{G}$ be the Pontryagin dual of G . The Fourier-Plancherel transform defined by

$$f \mapsto \hat{f}(\gamma) = \int_G \overline{\gamma(t)} f(t) d\mu(t)$$

extends to a C^* -isomorphism from the group C^* -algebra $C^*(G)$ of G and $C_0(\hat{G})$, i.e. the spectrum of $C^*(G)$ is precisely $\hat{G}$. When G is the real line $\mathbf{R}$, this is Stone's theorem characterizing one parameter unitary groups. The theorem of Stone-von Neumann can also be restated using similar language.

The group G acts on the C^* -algebra $C_0(G)$ by right translation ϱ : for s in G and f in $C_0(G)$,

$$(s \cdot f)(t) = f(t + s).$$

Under the isomorphism given above, this action becomes the natural action of G on $C^*(\hat{G})$:

$$(\widehat{s \cdot f})(\gamma) = \gamma(s) \hat{f}(\gamma).$$

So a covariant representation corresponding to the C^* -crossed product

$$C^*(\hat{G}) \rtimes_{\hat{\rho}} G$$

is a unitary representation $U(s)$ of G and $V(\gamma)$ of $\hat{G}$ such that

$$U(s)V(\gamma)U^*(s) = \gamma(s)V(\gamma).$$

It is a general fact that covariant representations are in one-to-one correspondence with $*$ -representation of the corresponding crossed product. On the other hand, all irreducible representations of

$$C_0(G) \rtimes_{\rho} G$$

are unitarily equivalent to the $K(L^2(G))$, the compact operators on $L^2(G)$. Therefore all pairs $\{U(s), V(\gamma)\}$ are unitarily equivalent. Specializing to the case where $G = \mathbf{R}$ yields the Stone-von Neumann theorem.

The Heisenberg group

The commutation relations for P, Q look very similar to the commutation relations that define the Lie algebra of general Heisenberg group H_n for n a positive integer. This is the Lie group of $(n+2) \times (n+2)$ square matrices of the form

$$M(a, b, c) = \begin{bmatrix} 1 & a & c \\ 0 & 1_n & b \\ 0 & 0 & 1 \end{bmatrix}.$$

In fact, using the Heisenberg group, we can formulate a far-reaching generalization of the Stone von Neumann theorem. Note that the center of H_n consists of matrices $M(0, 0, c)$.

Theorem. For each non-zero real number h there is an irreducible representation U_h acting on the Hilbert space $L^2(\mathbf{R}^n)$ by

$$[U_h(M(a, b, c))]\psi(x) = e^{i(b \cdot x + hc)} \psi(x + ha).$$

All these representations are unitarily inequivalent and any irreducible representation which is not trivial on the center of H_n is unitarily equivalent to exactly one of these.

Note that U_h is a unitary operator because it is the composition of two operators which are easily seen to be unitary: the translation to the *left* by $h \cdot a$ and multiplication by a function of absolute value 1. To show U_h is multiplicative is a straightforward calculation. The hard part of the theorem is showing the uniqueness which is beyond the scope of the article. However, below we sketch a proof of the corresponding Stone–von Neumann theorem for certain finite Heisenberg groups.

In particular, irreducible representations π, π' of the Heisenberg group H_n which are non-trivial on the center of H_n are unitarily equivalent if and only if $\pi(z) = \pi'(z)$ for any z in the center of H_n .

One representation of the Heisenberg group that is important in number theory and the theory of modular forms is the theta representation, so named because the Jacobi theta function is invariant under the action of the discrete subgroup of the Heisenberg group.

Relation to the Fourier transform

For any non-zero h , the mapping

$$\alpha_h : M(a, b, c) \rightarrow M(-h^{-1}b, ha, c - ab)$$

is an automorphism of H_n which is the identity on the center of H_n . In particular, the representations U_h and $U_h \alpha$ are unitarily equivalent. This means that there is a unitary operator W on $L^2(\mathbf{R}^n)$ such that for any g in H_n ,

$$WU_h(g)W^* = U_h\alpha(g).$$

Moreover, by irreducibility of the representations U_h , it follows that up to a scalar, such an operator W is unique (cf. Schur's lemma).

Theorem. The operator W is, up to a scalar multiple, the Fourier transform on $L^2(\mathbf{R}^n)$.

This means that (ignoring the factor of $(2\pi)^{n/2}$ in the definition of the Fourier transform)

$$\int_{\mathbf{R}^n} e^{-ix \cdot p} e^{i(b \cdot x + hc)} \psi(x + ha) dx = e^{i(ha \cdot p + h(c - b \cdot a))} \int_{\mathbf{R}^n} e^{-iy \cdot (p - b)} \psi(y) dy.$$

The previous theorem can actually be used to prove the unitary nature of the Fourier transform, also known as the Plancherel theorem. Moreover, note that

$$(\alpha_h)^2 M(a, b, c) = M(-a, -b, c).$$

Theorem. The operator W_1 such that

$$W_1 U_h W_1^* = U_h \alpha^2(g)$$

is the reflection operator

$$[W_1 \psi](x) = \psi(-x).$$

From this fact the Fourier inversion formula easily follows.

Representations of finite Heisenberg groups

The Heisenberg group $H_n(\mathbf{K})$ is defined for any commutative ring $\mathbf{K}$. In this section let us specialize to the field $\mathbf{K} = \mathbf{Z}/p\mathbf{Z}$ for p a prime. This field has the property that there is an imbedding ω of $\mathbf{K}$ as an additive group into the circle group $\mathbf{T}$. Note that $H_n(\mathbf{K})$ is finite with cardinality $|\mathbf{K}|^{2n+1}$. For finite Heisenberg group $H_n(\mathbf{K})$ one can give a simple proof of the Stone–von Neumann theorem using simple properties of character functions of representations. These properties follow from the orthogonality relations for characters of representations of finite groups.

For any non-zero h in $\mathbf{K}$ define the representation U_h on the finite-dimensional inner product space $l^2(\mathbf{K}^n)$ by

$$[U_h M(a, b, c) \psi](x) = \omega(b \cdot x + hc) \psi(x + ha).$$

Theorem. For a fixed non-zero h , the character function χ of U_h is given by:

$$\chi(M(a, b, c)) = \begin{cases} |\mathbf{K}|^n \omega(hc) & \text{if } a = b = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\frac{1}{|H_n(\mathbf{K})|} \sum_{g \in H_n(\mathbf{K})} |\chi(g)|^2 = \frac{1}{|\mathbf{K}|^{2n+1}} |\mathbf{K}|^{2n} |\mathbf{K}| = 1.$$

By the orthogonality relations for characters of representations of finite groups this fact implies the corresponding Stone–von Neumann theorem for Heisenberg groups $H_n(\mathbf{Z}/p\mathbf{Z})$, particularly:

- Irreducibility of U_h
- Pairwise inequivalence of all the representations U_h .

Generalizations

The Stone–von Neumann theorem admits numerous generalizations. Much of the early work of George Mackey was directed at obtaining a formulation of the theory of induced representations developed originally by Frobenius for finite groups to the context of unitary representations of locally compact topological groups.

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Peter-Weyl theorem

In mathematics, the **Peter–Weyl theorem** is a basic result in the theory of harmonic analysis, applying to topological groups that are compact, but are not necessarily abelian. It was initially proved by Hermann Weyl, with his student Fritz Peter, in the setting of a compact topological group G (Peter & Weyl 1927). The theorem is a collection of results generalizing the significant facts about the decomposition of the regular representation of any finite group, as discovered by F. G. Frobenius and Issai Schur.

The theorem has three parts. The first part states that the matrix coefficients of G are dense in the space $C(G)$ of continuous complex-valued functions on G , and thus also in the space $L^2(G)$ of square-integrable functions. The second part asserts the complete reducibility of unitary representations of G . The third part then asserts that the regular representation of G on $L^2(G)$ decomposes as the direct sum of all irreducible unitary representations. Moreover, the matrix coefficients of the irreducible unitary representations form an orthonormal basis of $L^2(G)$.

Matrix coefficients

A **matrix coefficient** of the group G is a complex-valued function ϕ on G given as the composition

$$\varphi = L \circ \pi$$

where $\pi : G \rightarrow \text{GL}(V)$ is a finite-dimensional (continuous) group representation of G , and L is a linear functional on the vector space of endomorphisms of V (e.g. trace), which contains $\text{GL}(V)$ as an open subset. Matrix coefficients are continuous, since representations are by definition continuous, and linear functionals on finite-dimensional spaces are also continuous.

The first part of the Peter–Weyl theorem asserts (Bump 2004, §4.1; Knapp 1986, Theorem 1.12):

- The set of matrix coefficients of G is dense in the space of continuous complex functions $C(G)$ on G , equipped with the uniform norm.

This first result resembles the Stone-Weierstrass theorem in that it indicates the density of a set of functions in the space of all continuous functions, subject only to an *algebraic* characterization. In fact, if G is a matrix group, then the result follows easily from the Stone-Weierstrass theorem (Knapp 1986, p. 17). Conversely, it is a consequence of the subsequent conclusions of the theorem that any compact Lie group is isomorphic to a matrix group (Knapp 1986, Theorem 1.15).

A corollary of this result is that the matrix coefficients of G are dense in $L^2(G)$.

Decomposition of a unitary representation

The second part of the theorem gives the existence of a decomposition of a unitary representation of G into finite-dimensional representations. Now, intuitively groups were conceived as rotations on geometric objects, so it is only natural to study representations which essentially arise from continuous **actions** on Hilbert spaces. (For those who were first introduced to dual groups consisting of characters which are the continuous homomorphisms into the circle group $\{z \in \mathbb{C} : |z|=1\}$, this approach is similar except that the circle group is (ultimately) generalised to the group of unitary operators on a given Hilbert space.)

Let G be a topological group and H a complex Hilbert space.

Given a continuous action $*: G \times H \rightarrow H$, it gives rise to a map $\rho_*: G \rightarrow {}^H H$ defined in the obvious way: $\rho_*(v) = gv$. This map is clearly an homomorphism from G into $GL(H)$, the homeomorphic automorphisms of H . And given such a map, we can uniquely recover the action in the obvious way.

Thus we define the **representations of G on an Hilbert space H** to be those group homomorphisms, ρ , which arise from continuous actions of G on H . We say that a representation ρ is **unitary** if $\rho(g)$ is a unitary operator for all $g \in G$; i.e., $\langle gv, gw \rangle = \langle v, w \rangle$ for all $v, w \in H$. (I.e. it is unitary if $\rho: G \rightarrow U(H)$). Notice how this generalises the special

case of the one-dimensional Hilbert space, where $U(\mathbb{C}^1)$ is just the circle group.)

Given these definitions, we can state the Peter–Weyl theorem. The second part of this theorem asserts (Knapp 1986, Theorem 1.14):

- Let ρ be a unitary representation of a compact group G on a complex Hilbert space H . Then H splits into an orthogonal direct sum of irreducible finite-dimensional unitary representations of G .

Decomposition of square-integrable functions

To state the third and final part of the theorem, there is a natural Hilbert space over G consisting of square-integrable functions, $L^2(G)$; this makes sense because Haar measure exists on G . Calling this Hilbert space H , the group G has a unitary representation ρ on H by acting on the left, via

$$\rho(h)f(g) = f(h^{-1}g).$$

The final statement of the Peter–Weyl theorem (Knapp 1986, Theorem 1.14) gives an explicit orthonormal basis of $L^2(G)$. Roughly it asserts that the matrix coefficients for G , suitably renormalized, are an orthonormal basis of $L^2(G)$. In particular, $L^2(G)$ decomposes into an orthogonal direct sum of all the irreducible unitary representations, in which the multiplicity of each irreducible representation is equal to its degree (that is, the dimension of the underlying space of the representation). Thus,

$$L^2(G) = \widehat{\bigoplus_{\pi \in \Sigma} E_{\pi}}$$

where Σ denotes the set of (isomorphism classes of) irreducible unitary representations of G , and the summation denotes the closure of the direct sum of the total spaces E_{π} of the representations π .

More precisely, suppose that a representative π is chosen for each isomorphism class of irreducible unitary representation, and denote the collection of all such π by Σ . Let $u_{ij}^{(\pi)}$ be the matrix coefficients of π in an orthonormal basis, in other words

$$u_{ij}^{(\pi)}(g) = \langle \pi(g)e_i, e_j \rangle.$$

for each $g \in G$. Finally, let $d^{(\pi)}$ be the degree of the representation π . The theorem now asserts that the set of functions

$$\left\{ \sqrt{d^{(\pi)}} u_{ij}^{(\pi)} \mid \pi \in \Sigma, 1 \leq i, j \leq d^{(\pi)} \right\}$$

is an orthonormal basis of $L^2(G)$.

Consequences

Structure of compact topological groups

From the theorem, one can deduce a significant general structure theorem. Let G be a compact topological group, which we assume Hausdorff. For any finite-dimensional G -invariant subspace V in $L^2(G)$, where G acts on the left, we consider the image of G in $\text{GL}(V)$. It is closed, since G is compact, and a subgroup of the Lie group $\text{GL}(V)$. It follows by a basic theorem (of Élie Cartan) that the image of G is a Lie group also.

If we now take the limit (in the sense of category theory) over all such spaces V , we get a result about G - because G acts faithfully on $L^2(G)$. We can say that G is an *inverse limit of Lie groups*. It may of course not itself be a Lie group: it may for example be a profinite group.

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Supersymmetry

In particle physics, **supersymmetry** (often abbreviated **SUSY**) is a symmetry that relates elementary particles of one spin to other particles that differ by half a unit of spin and are known as superpartners. In a theory with unbroken supersymmetry, for every type of boson there exists a corresponding type of fermion with the same mass and internal quantum numbers, and vice-versa.

So far, there is only indirect evidence for the existence of supersymmetry.^[1] Since the superpartners of the Standard Model particles have not been observed, supersymmetry, if it exists, must be a broken symmetry, allowing the superparticles to be heavier than the corresponding Standard Model particles.

If supersymmetry exists close to the TeV energy scale, it allows for a solution of the hierarchy problem of the Standard Model, i.e., the fact that the Higgs boson mass is subject to quantum corrections which — barring extremely fine-tuned cancellations among independent contributions — would make it so large as to undermine the internal consistency of the theory. In supersymmetric theories, on the other hand, the contributions to the quantum corrections coming from Standard Model particles are naturally canceled by the contributions of the corresponding superpartners. Other attractive features of TeV-scale supersymmetry are the fact that it allows for the high-energy unification of the weak interactions, the strong interactions and electromagnetism, and the fact that it provides a candidate for Dark Matter and a natural mechanism for electroweak symmetry breaking.

Another advantage of supersymmetry is that supersymmetric quantum field theory can sometimes be solved. Supersymmetry is also a feature of most versions of string theory, though it can exist in nature even if string theory is incorrect.

The Minimal Supersymmetric Standard Model is one of the best studied candidates for physics beyond the Standard Model. Theories of gravity that are also invariant under supersymmetry are known as supergravity theories.

History

A supersymmetry relating mesons and baryons was first proposed, in the context of hadronic physics, by Hironari Miyazawa in 1966, but his work was ignored at the time.^{[2] [3] [4] [5]} In the early 1970s, J. L. Gervais and B. Sakita (in 1971), Yu. A. Golfand and E.P. Likhtman (also in 1971), D.V. Volkov and V.P. Akulov (in 1972) and J. Wess and B. Zumino (in 1974) independently rediscovered supersymmetry, a radically new type of symmetry of spacetime and fundamental fields, which establishes a relationship between elementary particles of different quantum nature,

bosons and fermions, and unifies spacetime and internal symmetries of the microscopic world. Supersymmetry first arose in the context of an early version of string theory by Pierre Ramond, John H. Schwarz and Andre Neveu, but the mathematical structure of supersymmetry has subsequently been applied successfully to other areas of physics; firstly by Wess, Zumino, and Abdus Salam and their fellow researchers to particle physics, and later to a variety of fields, ranging from quantum mechanics to statistical physics. It remains a vital part of many proposed theories of physics.

The first realistic supersymmetric version of the Standard Model was proposed in 1981 by Howard Georgi and Savas Dimopoulos and is called the Minimal Supersymmetric Standard Model or MSSM for short. It was proposed to solve the hierarchy problem and predicts superpartners with masses between 100 GeV and 1 TeV. As of 2009 there is no irrefutable experimental evidence that supersymmetry is a symmetry of nature. In 2010 the Large Hadron Collider at CERN is scheduled to produce the world's highest energy collisions and offers the best chance at discovering superparticles for the foreseeable future. Recently prediction markets like intrade offered scientific contracts that give estimates for that probability.

Applications

Extension of possible symmetry groups

One reason that physicists explored supersymmetry is because it offers an extension to the more familiar symmetries of quantum field theory. These symmetries are grouped into the Poincaré group and internal symmetries and the Coleman–Mandula theorem showed that under certain assumptions, the symmetries of the S-matrix must be a direct product of the Poincaré group with a compact internal symmetry group or if there is no mass gap, the conformal group with a compact internal symmetry group. In 1971 Golfand and Likhtman were the first to show that the Poincaré algebra can be extended through introduction of four anticommuting spinor generators (in four dimensions), which later became known as supercharges. In 1975 the Haag-Lopuszanski-Sohnius theorem analyzed all possible superalgebras in the general form, including those with an extended number of the supergenerators and central charges. This extended super-Poincaré algebra paved the way for obtaining a very large and important class of supersymmetric field theories.

The supersymmetry algebra

Traditional symmetries in physics are generated by objects that transform under the tensor representations of the Poincaré group and internal symmetries. Supersymmetries, on the other hand, are generated by objects that transform under the spinor representations. According to the spin-statistics theorem, bosonic fields commute while fermionic fields anticommute. Combining the two kinds of fields into a single algebra requires the introduction of a $\mathbb{Z}_2$ -grading under which the bosons are the even elements and the fermions are the odd elements. Such an algebra is called a Lie superalgebra.

The simplest supersymmetric extension of the Poincaré algebra is the Super-Poincaré algebra. Expressed in terms of two Weyl spinors, has the following anti-commutation relation:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu$$

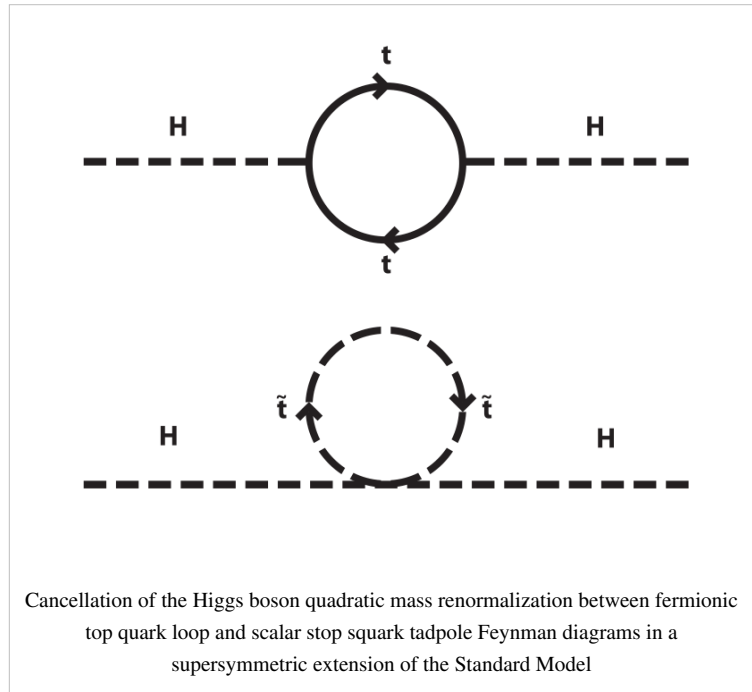
and all other anti-commutation relations between the Q s and commutation relations between the Q s and P s vanish. In the above expression $P_\mu = -i\partial_\mu$ are the generators of translation and σ^μ are the Pauli matrices.

There are representations of a Lie superalgebra that are analogous to representations of a Lie algebra. Each Lie algebra has an associated Lie group and a Lie superalgebra can sometimes be extended into representations of a Lie supergroup.

The Supersymmetric Standard Model

Incorporating supersymmetry into the Standard Model requires doubling the number of particles since there is no way that any of the particles in the Standard Model can be superpartners of each other. With the addition of new particles, there are many possible new interactions. The simplest possible supersymmetric model consistent with the Standard Model is the Minimal Supersymmetric Standard Model (MSSM) which can include the necessary additional new particles that are able to be superpartners of those in the Standard Model.

One of the main motivations for SUSY comes from the quadratically divergent contributions to the Higgs mass squared. The quantum mechanical interactions of the Higgs boson causes a large renormalization of the Higgs mass and unless there is an accidental cancellation, the natural size of the Higgs mass is the highest scale possible. This problem is known as the hierarchy problem. Supersymmetry reduces the size of the quantum corrections by having automatic cancellations between fermionic and bosonic Higgs interactions. If supersymmetry is restored at the weak scale, then the Higgs mass is related to supersymmetry breaking which can be induced from small non-perturbative effects explaining the vastly different scales in the weak interactions and gravitational interactions.



In many supersymmetric Standard Models there is a heavy stable particle (such as neutralino) which could serve as a Weakly interacting massive particle (WIMP) dark matter candidate. The existence of a supersymmetric dark matter candidate is closely tied to R-parity.

The standard paradigm for incorporating supersymmetry into a realistic theory is to have the underlying dynamics of the theory be supersymmetric, but the ground state of the theory does not respect the symmetry and supersymmetry is broken spontaneously. The supersymmetry break can not be done permanently by the particles of the MSSM as they currently appear. This means that there is a new sector of the theory that is responsible for the breaking. The only constraint on this new sector is that it must break supersymmetry permanently and must give superparticles TeV scale masses. There are many models that can do this and most of their details do not currently matter. In order to parameterize the relevant features of supersymmetry breaking, arbitrary soft SUSY breaking terms are added to the theory which temporarily break SUSY explicitly but could never arise from a complete theory of supersymmetry breaking.

Gauge Coupling Unification

One piece of evidence for supersymmetry existing is gauge coupling unification. The renormalization group evolution of the three gauge coupling constants of the Standard Model is somewhat sensitive to the present particle content of the theory. These coupling constants do not quite meet together at a common energy scale if we run the renormalization group using the Standard Model.^[1] With the addition of minimal SUSY joint convergence of the coupling constants is projected at approximately 10^{16} GeV.^[1]

Supersymmetric quantum mechanics

Supersymmetric quantum mechanics adds the SUSY superalgebra to quantum mechanics as opposed to quantum field theory. Supersymmetric quantum mechanics often comes up when studying the dynamics of supersymmetric solitons and due to the simplified nature of having fields only functions of time (rather than space-time), a great deal of progress has been made in this subject and is now studied in its own right.

SUSY quantum mechanics involves pairs of Hamiltonians which share a particular mathematical relationship, which are called *partner Hamiltonians*. (The potential energy terms which occur in the Hamiltonians are then called *partner potentials*.) An introductory theorem shows that for every eigenstate of one Hamiltonian, its partner Hamiltonian has a corresponding eigenstate with the same energy. This fact can be exploited to deduce many properties of the eigenstate spectrum. It is analogous to the original description of SUSY, which referred to bosons and fermions. We can imagine a "bosonic Hamiltonian", whose eigenstates are the various bosons of our theory. The SUSY partner of this Hamiltonian would be "fermionic", and its eigenstates would be the theory's fermions. Each boson would have a fermionic partner of equal energy.

SUSY concepts have provided useful extensions to the WKB approximation. In addition, SUSY has been applied to non-quantum statistical mechanics through the Fokker-Planck equation.

Mathematics

SUSY is also sometimes studied mathematically for its intrinsic properties. This is because it describes complex fields satisfying a property known as holomorphy, which allows holomorphic quantities to be exactly computed. This makes supersymmetric models useful toy models of more realistic theories. A prime example of this has been the demonstration of S-duality in four-dimensional gauge theories that interchanges particles and monopoles.

General supersymmetry

Supersymmetry appears in many different contexts in theoretical physics that are closely related. It is possible to have multiple supersymmetries and also have supersymmetric extra dimensions.

Extended supersymmetry

It is possible to have more than one kind of supersymmetry transformation. Theories with more than one supersymmetry transformation are known as extended supersymmetric theories. The more supersymmetry a theory has, the more constrained the field content and interactions are. Typically the number of copies of a supersymmetry is a power of 2, i.e. 1, 2, 4, 8. In four dimensions, a spinor has four degrees of freedom and thus the minimal number of supersymmetry generators is four in four dimensions and having eight copies of supersymmetry means that there are 32 supersymmetry generators.

The maximal number of supersymmetry generators possible is 32. Theories with more than 32 supersymmetry generators automatically have massless fields with spin greater than 2. It is not known how to make massless fields with spin greater than two interact, so the maximal number of supersymmetry generators considered is 32. This corresponds to an $N = 8$ supersymmetry theory. Theories with 32 supersymmetries automatically have a graviton.

In four dimensions there are the following theories, with the corresponding multiplets ^[6] (CPT adds a copy, whenever they are not invariant under such symmetry)

- $N = 1$

Chiral multiplet: $(0, \frac{1}{2})$ Vector multiplet: $(\frac{1}{2}, 1)$ Gravitino multiplet: $(1, \frac{3}{2})$ Graviton multiplet: $(\frac{3}{2}, 2)$

- $N = 2$

hypermultiplet: $(-\frac{1}{2}, 0^2, \frac{1}{2})$ vector multiplet: $(0, \frac{1}{2}^2, 1)$ supergravity multiplet: $(1, \frac{3}{2}^2, 2)$

- $N = 4$

Vector multiplet: $(-1, -\frac{1}{2}, 0, \frac{6}{2}, \frac{1}{2}, 1)$ Supergravity multiplet: $(0, \frac{1}{2}, 1, \frac{6}{2}, \frac{3}{2}, 2)$

- $N = 8$

Supergravity multiplet: $(-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{56}{2}, \frac{1}{2}, \frac{56}{2}, 1, \frac{28}{2}, \frac{3}{2}, 2)$

Supersymmetry in alternate numbers of dimensions

It is possible to have supersymmetry in dimensions other than four. Because the properties of spinors change drastically between different dimensions, each dimension has its characteristic. In d dimensions, the size of spinors is roughly $2^{d/2}$ or $2^{(d-1)/2}$. Since the maximum number of supersymmetries is 32, the greatest number of dimensions in which a supersymmetric theory can exist is eleven.

Supersymmetry as a quantum group

Supersymmetry can be reinterpreted in the language of noncommutative geometry and quantum groups. In particular, it involves a mild form of noncommutativity, namely supercommutativity. See the main article for more details.

Supersymmetry in quantum gravity

Supersymmetry is part of a larger enterprise of theoretical physics to unify everything we know about the physical world into a single fundamental framework of physical laws, known as the quest for a Theory of Everything (TOE). A significant part of this larger enterprise is the quest for a theory of quantum gravity, which would unify the classical theory of general relativity and the Standard Model, which explains the other three basic forces in physics (electromagnetism, the strong interaction, and the weak interaction), and provides a palette of fundamental particles upon which all four forces act. Two of the most active approaches to forming a theory of quantum gravity are string theory and loop quantum gravity (LQG), although in theory, supersymmetry could be a component of other theoretical approaches as well.

For string theory to be consistent, supersymmetry appears to be required at some level (although it may be a strongly broken symmetry). In particle theory, supersymmetry is recognized as a way to stabilize the hierarchy between the unification scale and the electroweak scale (or the Higgs boson mass), and can also provide a natural dark matter candidate. String theory also requires extra spatial dimensions which have to be compactified as in Kaluza-Klein theory.

Loop quantum gravity (LQG), in its current formulation, predicts no additional spatial dimensions, nor anything else about particle physics. These theories can be formulated in three spatial dimensions and one dimension of time, although in some LQG theories dimensionality is an emergent property of the theory, rather than a fundamental assumption of the theory. Also, LQG is a theory of quantum gravity which does not require supersymmetry. Lee Smolin, one of the originators of LQG, has proposed that a loop quantum gravity theory incorporating either supersymmetry or extra dimensions, or both, be called "loop quantum gravity II".

If experimental evidence confirms supersymmetry in the form of supersymmetric particles such as the neutralino that is often believed to be the lightest superpartner, some people believe this would be a major boost to string theory. Since supersymmetry is a required component of string theory, any discovered supersymmetry would be consistent with string theory. If the Large Hadron Collider and other major particle physics experiments fail to detect supersymmetric partners or evidence of extra dimensions, many versions of string theory which had predicted certain low mass superpartners to existing particles may need to be significantly revised. The failure of experiments to discover either supersymmetric partners or extra spatial dimensions, as of 2009, has encouraged loop quantum gravity researchers.

Current Limits

The tightest limits will of course come from direct production at colliders. Both the Large Electron–Positron Collider and Tevatron have set limits for specific models which have not yet been exceeded by the Large Hadron Collider. Searches are only applicable for a finite set of tested points because simulation using the Monte Carlo method must be made so that limits for that particular model can be calculated. This complicates matters because different experiments have looked at different sets of points. Some extrapolation between points can be made within particular models but it is difficult to set general limits even for the Minimal Supersymmetric Standard Model.

The first mass limits for squarks and gluinos were made at CERN by the UA1 experiment and the UA2 experiment at the Super Proton Synchrotron. LEP later set very strong limits which are still relevant today ^[7]. Most recently these limits were extended by the D0 experiment ^{[8] [9]}

See also

- *Concise Encyclopedia of Supersymmetry* (book)
- Minimal Supersymmetric Standard Model
- Quantum group
- Supercharge
- Supergeometry
- Supergravity
- Supergroup
- Superspace

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Further reading

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External links

- "Particle wobble shakes up supersymmetry" (<http://www.cosmosmagazine.com/node/714>), *Cosmos* magazine, September 2006

Invariance mechanics

In physics, **invariance mechanics**, in its simplest form, is the rewriting of the laws of quantum field theory in terms of invariant quantities only. For example, the positions of a set of particles in a particular coordinate system is not invariant under translations of the system. However, the (4-dimensional) distances between the particles is invariant under translations, rotations and Lorentz transformations of the system.

The invariant quantities made from the input and output states of a system are the only quantities needed to give a probability amplitude to a given system. This is what is meant by the system obeying a symmetry. Since all the quantities involved are relative quantities, invariance mechanics, can be thought of as taking relativity theory to its natural limit.

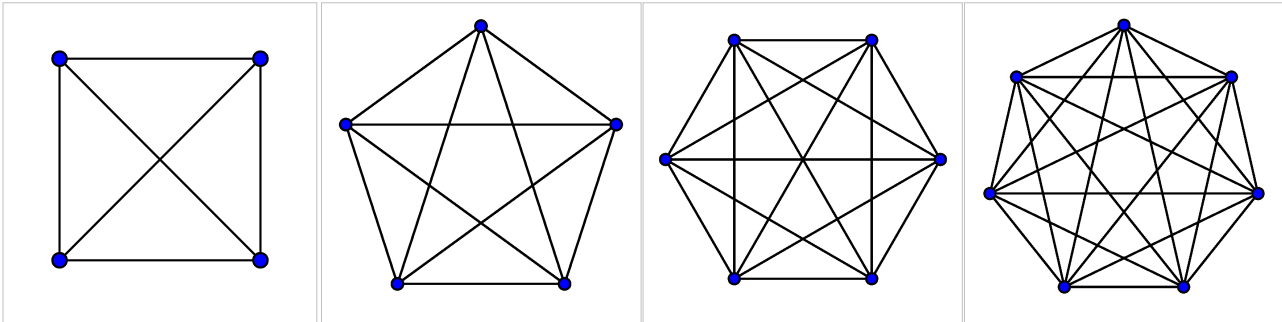
Invariance mechanics has strong links with loop quantum gravity in which the invariant quantities are based on angular momentum. In invariance mechanics, space and time come secondary to the invariants and are seen as useful concepts that emerge only in the large scale limit.

Feynman rules

The Feynman rules of a quantum system can be rewritten in terms of invariant quantities (plus constants such as mass, charge, etc.) The invariant quantities depend on the type of particle, scalar, vector or spinor. The rules often involve geometric quantities such as the volumes of simplices formed from vertices of the Feynman graphs.

Scalar particles

In a system of scalar particles, the only invariant quantities are the 4-dimensional distances (intervals) between the starting points (x) and ending points (x') of the particle paths. These points form a complete graph:



The invariants are the numbers

$$R = |x - x'|.$$

Vector particles

In a system of vector particles such as photons, the invariants are the 4-dimensional distances between the starting points and ending points of the particle paths, and the angles between the starting and ending polarisation vectors of the photons (ρ_μ)

There are four invariants on each line:

$$R = |x - x'|$$

$$S = \rho_\mu \rho'_\mu$$

$$T = (x - x')_\mu \rho'_\mu$$

$$U = \rho_\mu (x - x')_\mu$$

Yang–Mills vector particles

Yang–Mills vector fields of a given gauge group also involve the angle representing a rotation of the gauge group (ρ_μ^ν).

There are three invariants on each line:

$$\begin{aligned} R &= |x - x'| \\ S &= \rho_\mu^\nu \rho_\mu^\nu \\ T &= \rho_\mu^\nu (x - x')_\mu (x - x')_\tau \rho_\tau^\nu. \end{aligned}$$

Spinor fields

These involve the angles between the spinor vectors. The invariants are:

$$\begin{aligned} R &= |x - x'|, \\ S &= u^\alpha \gamma_5^{\alpha\beta} \bar{u}^\beta, \\ T &= u^\alpha \gamma_\mu^{\alpha\beta} (x - x')_\mu \bar{u}^\beta. \end{aligned}$$

So for example, the fermion propagator is defined in relation to the massless scalar propagator as

$$G^\psi(R, S, T, m) = \left(\frac{\partial}{\partial T} + \frac{T}{2R} \frac{\partial}{\partial R} - mS \right) G^F(R, m).$$

Mixed systems

Systems usually consist of a mixture of scalar, spinor and vector fields and the invariants can depend on angles between spinors and vectors. To simplify this process ideas from Twistor theory are often used which enables one to decompose a null-vector into a pair of spinors. Alternatively 3-point invariants can be introduced such as the *spinor-spinor-vector* triangle invariant:

$$V = u^\alpha \gamma_\mu^{\alpha\beta} \rho_\mu \bar{u}^\beta.$$

It is important to note that some types of invariants are combinations of other types invariants, for example the angles in a complete-graph are invariants but they can be found as combinations of distance invariants.

In chromodynamics, for example, there are 4-point invariants also. So for a completely specified system you would have several numbers assigned to each line, triangle and tetrahedron in a complete graph representing the system.

One outstanding problem is that of enumerating all the possible invariants which can be made from the various spin and polarisation vectors.

Constraints

A system represented by a complete graph contains many invariant quantities. For large graphs, however, not all these quantities are independent and we must specify dimensional and gauge constraints. Why the particular number of dimensions or particular gauge group is chosen is still not known. The constraints and whether they are satisfied exactly or approximately is the key to invariance mechanics and the difference between it and conventional field theory. Work is being done to see whether the breaking of these constraints is a consequence of the gravitational field. If the constraints are satisfied only approximately, i.e. if there is a quantum uncertainty in the constraints then they are best thought of as local maxima of the amplitudes of a system which occur due to the specific Feynman rules used.

Dimensions

Since invariance mechanics does not explicitly use coordinate systems, the definition of dimension is slightly different. The equivalent way of expressing the number of dimensions is given, as in distance geometry, as specifying that the volume of any $(D + 2)$ -simplex made from the points in the system is zero. The volume of a simplex is given by a formula involving the invariant distances (the R 's) between the points which is given by the Cayley–Menger determinants. If this determinant is exactly 0 for all simplices then the geometry is Euclidean. If the determinant is only approximately 0 then at small distances space-time is non-Euclidean. This has deep connections with quantum foam and loop quantum gravity.

For Minkowski space, or for any space with signature $(+ + + \dots + -)$ this makes no difference to the formulae for invariance mechanics.

Gravity

By allowing quantum uncertainty in the dimensional constraints (which involves replacing delta functions with reciprocal functions in the equations), the geometry is no longer confined to flat space-time, this break from flat space-time can be seen as a curvature and, just as in General Relativity can be seen as the cause of gravity. This is called 'off-dimension' physics in analogy to off-shell physics.

Gauge group

In a similar way to expressing the number of dimensions, the dimension and type of the gauge group is given by an identity involving the polarisation (or spin) invariants (the S , T and U 's). In the simple cases such as for the photon, these are simply spherical versions of the Cayley–Menger determinants. The gauge group is an internal symmetry because the gauge identity involves far more quantities than the dimensional identity. A simple gauge group such as $SU(5)$ or E_6 involves less invariants than a non-simple gauge group such as $U(1) \times SU(2) \times SU(3)$ (see: Standard Model). There has been recent work on combining the dimensional and gauge constraints into a single equation to produce a unified theory. It is thought that this will be achieved by combining of the invariants on each line into a single complex number (or hypercomplex number).

Supersymmetry

In the supersymmetric model, some of the spinor invariants and vector invariants are combined together into a single invariant. Having less invariants means that there is more symmetry and more transformations are possible such as transformations between fermions and bosons. It is believed, although currently unproved, that the minimum number of invariants on each line of a complete graph representing a system is two – those being the 4-dimensional distances (the R 's) and an angle representing the rotation from one particle 'flavour' to another particle 'flavour' (the T 's). Some have suggested that even *these* invariants can be combined into one by saying that the 4 dimensions of space and time are just 4 more flavours that a particle can have, albeit ones which can change very little (compared with the size of the Universe as a whole). Models of this type imply that the universe has an overall spherical geometry. The mixing of space-time and flavour symmetries adds an additional degree of freedom to a particle's light-cone which appears as a unique mass for each particle depending on the flavour.

Having a small number of invariants doesn't necessarily make a simpler model since all the complexity of the model is bound up in the constraints which can be polynomials of hundreds of variables. One of the primary aims of invariance mechanics is to find these polynomial(s) and to find which symmetry group they correspond to. Many believe that the permutation of the variables of these polynomial(s) will correspond to one of the special sporadic groups. (Interestingly, only the largest sporadic group, the monster group is big enough to incorporate the constraints for the Standard Model). The other main aim is to find appropriate Feynman rules on the invariants which both accurately describe nature and don't lead to infinities.

M-theory

Although invariance mechanics was born out of trying to understand point particle theory, possible connections with superstring and M-theory have emerged. The argument is that the smallest simplex which needs a constraint to be 4-dimensional is the 6-simplex. This can be viewed as the endpoints of a 3-simplex (a triangular membrane) moving through time. The propagator function of this would be $1/V_6$ which is the inverse of the volume of a 6-simplex. In other words the greatest probability would be when the volume of this 6-simplex is 0 and hence it is embedded in 4 dimensions. Hence the propagator for a particle would be the same as the dimensional constraint. So if the Universe is built out of 6 simplices then the dimensional constraint can be applied to all simplices. Other fields of work are investigating whether the distance invariants may take only discrete values and whether areas or volumes should be taken as the fundamental invariants. (The dual of loop quantum gravity involves quantized areas).

Others take the view that in invariance mechanics it should be irrelevant whether you view the fundamental constituents as particles or strings or membranes and a more formal approach is called for.

History

The history of invariance mechanics is difficult to pinpoint since many people have been working on it without realising that they were working on invariance mechanics. Notable milestones include the 4-dimensional invariant found by Einstein in special relativity (1905), Yang–Mills gauge invariants theory. Roger Penrose and his spin-networks (1960's) influenced the subject. Cayley–Menger and their invariant based metric theory was an important milestone. Recently Baratin–Freidel (2006) have demonstrated the connection between invariance mechanics and loop quantum gravity.

External links

- Introduction to Invariance Mechanics ^[1]
- Renormalization of Crumpled Manifolds ^[2]
- Hidden Quantum Gravity in 4d Feynman Diagrams ^[3]

References

- [1] <http://xxx.lanl.gov/abs/0705.2558>
- [2] <http://arxiv.org/pdf/hep-th/9212102>
- [3] <http://arxiv.org/abs/hep-th/0611042>

Wigner's theorem

Wigner's theorem is a cornerstone of the mathematical formulation of quantum mechanics. It was proved by Eugene Wigner in 1931^[1]. The theorem specifies how physical symmetries such as rotations, translations, CPT, etc. act on the Hilbert space of states. According to the theorem any symmetry acts as a unitary or anti-unitary transformation in the Hilbert space.

More precisely, it states that a surjective map $T : H \rightarrow H$ on a complex Hilbert space H , which satisfies

$$|\langle Tx, Ty \rangle| = |\langle x, y \rangle|$$

for all $x, y \in H$, has the form $Tx = \varphi(x)Ux$ for all $x \in H$, where $\varphi : H \rightarrow \mathbb{C}$ has modulus one and $U : H \rightarrow H$ is either unitary or antiunitary.

References

[1] E. P. Wigner, Gruppentheorie (Friedrich Vieweg und Sohn, Braunschweig, Germany, 1931), pp. 251-254; Group Theory (Academic Press Inc., New York, 1959), pp. 233-236

- Bargmann, V. "Note on Wigner's Theorem on Symmetry Operations". Journal of Mathematical Physics Vol 5, no. 7, Jul 1964.
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Wigner's classification

In mathematics and theoretical physics, **Wigner's classification** is a classification of the nonnegative energy irreducible unitary representations of the Poincaré group, which have sharp mass eigenvalues. It was proposed by Eugene Wigner, for reasons coming from physics—see the article particle physics and representation theory.

The mass $m \equiv \sqrt{P^2}$ is a Casimir invariant of the Poincaré group. So, we can classify the representations according to whether $m > 0$, $m = 0$ but $P_0 > 0$ and $m = 0$ and $\mathbf{P} = 0$.

For the first case, we note that the eigenspace (see generalized eigenspaces of unbounded operators) associated with $P_0 = m$ and $P_i = 0$ is a representation of $SO(3)$. In the ray interpretation, we can go over to $Spin(3)$ instead. So, massive states are classified by an irreducible $Spin(3)$ unitary and a positive mass, m .

For the second case, we look at the stabilizer of $P_0 = k$, $P_3 = -k$, $P_i = 0$, $i = 1, 2$. This is the double cover of $SE(2)$ (see unit ray representation). We have two cases, one where irreps are described by an integral multiple of $1/2$, called the helicity and the other called the "continuous spin" representation.

The last case describes the vacuum. The only finite dimensional unitary solution is the trivial representation called the vacuum.

The double cover of the Poincaré group admits no central extensions.

Note: This classification leaves out tachyonic solutions, solutions with no fixed mass, infraparticles with no fixed mass, etc.

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- Wigner, E. P. (1939), "On unitary representations of the inhomogeneous Lorentz group" ^[1], *Annals of Mathematics* (Annals of Mathematics) **40** (1): 149–204, doi:10.2307/1968551, MR1503456.

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Quantum hydrodynamics

Quantum hydrodynamics (QHD) is most generally the study of hydrodynamic systems which demonstrate behavior implicit in quantum subsystems (usually quantum tunneling). They arise in semiclassical mechanics in the study of semiconductor devices, in which case being derived from the Wigner-Boltzmann equation. In quantum chemistry they arise as solutions to chemical kinetic systems, in which case they are derived from the Schrödinger equation by way of Madelung equations.

An important system of study in quantum hydrodynamics is that of superfluidity. Some other topics of interest in quantum hydrodynamics are quantum turbulence, quantized vortices, first, second and third sound, and quantum solvents. The quantum hydrodynamic equation is an equation in Bohmian mechanics, which, it turns out, has a mathematical relationship to classical fluid dynamics (see Madelung equations). This is a rich theoretical field.

Some common experimental applications of these studies are in liquid helium (He-3 and He-4), and of the interior of neutron stars and the quark-gluon plasma. Many famous scientists have worked in quantum hydrodynamics, including Richard Feynman, Lev Landau, and Pyotr L. Kapitsa.

References

- Wyatt R.E. Quantum Dynamics with Trajectories: Introduction to Quantum Hydrodynamics (Springer, 2005)
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Quantum magnetodynamics

Quantum electrodynamics (QED) is the relativistic quantum field theory of electrodynamics. In essence, it describes how light and matter interact and is the first theory where full agreement between quantum mechanics and special relativity is achieved. QED mathematically describes all phenomena involving electrically charged particles interacting by means of exchange of photons and represents the quantum counterpart of classical electrodynamics giving a complete account of matter and light interaction. One of the founding fathers of QED, Richard Feynman, has called it "the jewel of physics" for its extremely accurate predictions of quantities like the anomalous magnetic moment of the electron, and the Lamb shift of the energy levels of hydrogen.^[1]

In technical terms, QED can be described as a perturbation theory of the electromagnetic quantum vacuum.

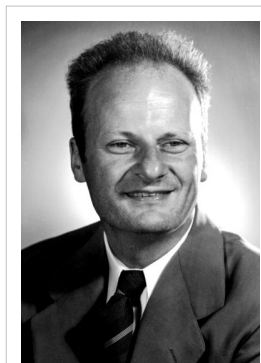
History

The first formulation of a quantum theory describing radiation and matter interaction is due to Paul Adrien Maurice Dirac, who, during 1920, was first able to compute the coefficient of spontaneous emission of an atom.^[2]

Dirac described the quantization of the electromagnetic field as an ensemble of harmonic oscillators with the introduction of the concept of creation and annihilation operators of particles. In the following years, with contributions from Wolfgang Pauli, Eugene Wigner, Pascual Jordan, Werner Heisenberg and an elegant formulation of quantum electrodynamics due to Enrico Fermi,^[3] physicists came to believe that, in principle, it would be possible to perform any computation for any physical process involving photons and charged particles. However, further studies by Felix Bloch with Arnold Nordsieck,^[4] and Victor Weisskopf,^[5] in 1937 and 1939, revealed that such computations were reliable only at a first order of perturbation theory, a problem already pointed out by Robert Oppenheimer.^[6] At higher orders in the series infinities emerged, making such computations meaningless and casting serious doubts on the internal consistency of the theory itself. With no solution for this problem known at the time, it appeared that a fundamental incompatibility existed between special relativity and quantum mechanics.



Paul Dirac



Hans Bethe

Difficulties with the theory increased through the end of 1940. Improvements in microwave technology made it possible to take more precise measurements of the shift of the levels of a hydrogen atom,^[7] now known as the Lamb shift and magnetic moment of the electron.^[8] These experiments unequivocally exposed discrepancies which the theory was unable to explain.

A first indication of a possible way out was given by Hans Bethe. In 1947, while he was traveling by train to reach Schenectady from New York,^[9] after giving a talk at the conference at Shelter Island on the subject, Bethe completed the first non-relativistic computation of the shift of the lines of the hydrogen atom as measured by Lamb and Retherford.^[10] Despite the limitations of the computation, agreement was excellent. The idea was simply to attach infinities to corrections at mass and charge that were actually fixed to a finite value by experiments. In this way, the infinities get absorbed in those constants and yield a finite result in good agreement with experiments. This procedure was named renormalization.

Based on Bethe's intuition and fundamental papers on the subject by Sin-Itiro Tomonaga,^[11] Julian Schwinger,^{[12] [13]} Richard Feynman^{[14] [15] [16]} and Freeman Dyson^{[17] [18]}, it was finally possible to get fully covariant formulations that were finite at any order in a perturbation series of quantum electrodynamics. Sin-Itiro Tomonaga, Julian Schwinger and Richard Feynman were jointly awarded with a Nobel prize in physics in 1965 for their work in this area.^[19] Their contributions, and those of Freeman Dyson, were about covariant and gauge invariant formulations of quantum electrodynamics that allow computations of observables at any order of perturbation theory. Feynman's mathematical technique, based on his diagrams, initially seemed very different from the field-theoretic, operator-based approach of Schwinger and Tomonaga, but Freeman Dyson later showed that the two approaches were equivalent.^[17] Renormalization, the need to attach a physical meaning at certain divergences appearing in the theory through integrals, has subsequently become one of the fundamental aspects of quantum field theory and has come to be seen as a criterion for a theory's general acceptability. Even though renormalization works very well in practice, Feynman was never entirely comfortable with its mathematical validity, even referring to renormalization as a "shell game" and "hocus pocus".^[20]

QED has served as the model and template for all subsequent quantum field theories. One such subsequent theory is quantum chromodynamics, which began in the early 1960s and attained its present form in the 1975 work by H. David Politzer, Sidney Coleman, David Gross and Frank Wilczek. Building on the pioneering work of Schwinger, Gerald Guralnik, Dick Hagen, and Tom Kibble,^{[21] [22]} Peter Higgs, Jeffrey Goldstone, and others, Sheldon Glashow, Steven Weinberg and Abdus Salam independently showed how the weak nuclear force and quantum electrodynamics could be merged into a single electroweak force.



Shelter Island Conference group photo (Courtesy of Archives, National Academy of Sciences).



Feynman (center) and Oppenheimer (left) at Los Alamos.

Feynman's view of quantum electrodynamics

Introduction

Near the end of his life, Richard P. Feynman gave a series of lectures on QED intended for the lay public. These lectures were transcribed and published as Feynman (1985), *QED: The strange theory of light and matter*,^{[1] [20]} a classic non-mathematical exposition of QED from the point of view articulated below.

The key components of Feynman's presentation of QED are three basic actions.

- A photon goes from one place and time to another place and time.
- An electron goes from one place and time to another place and time.
- An electron emits or absorbs a photon at a certain place and time.

These actions are represented in a form of visual shorthand by the three basic elements of Feynman diagrams: a wavy line for the photon, a straight line for the electron and a junction of two straight lines and a wavy one for a vertex representing emission or absorption of a photon by an electron. These may all be seen in the adjacent diagram.

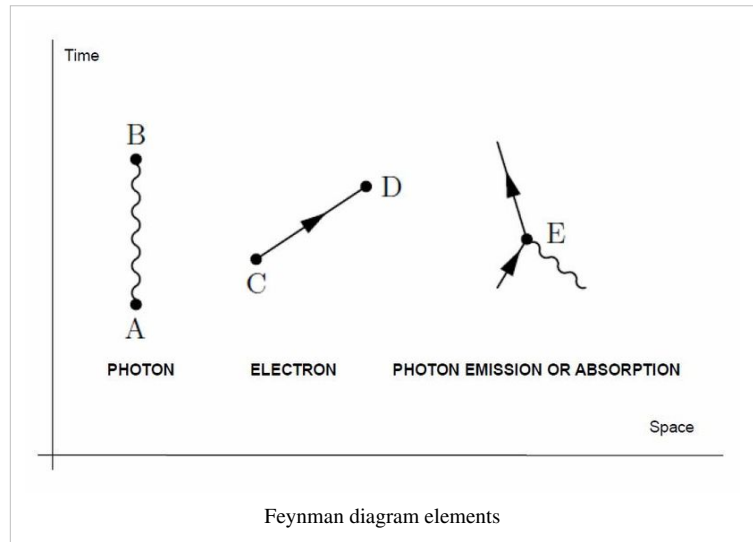
It is important not to over-interpret these diagrams. Nothing is implied about *how* a particle gets from one point to another. The diagrams do *not* imply that the particles are moving in straight or curved lines. They do *not* imply that the particles are moving with

fixed speeds. The fact that the photon is often represented, by convention, by a wavy line and not a straight one does *not* imply that it is thought that it is more wavelike than is an electron. The images are just symbols to represent the actions above: photons and electrons do, somehow, move from point to point and electrons, somehow, emit and absorb photons. We do not know how these things happen, but the theory tells us about the probabilities of these things happening. Trajectory is a meaningless concept in quantum mechanics.

As well as the visual shorthand for the actions Feynman introduces another kind of shorthand for the numerical quantities which tell us about the probabilities. If a photon moves from one place and time – in shorthand, A – to another place and time – shorthand, B – the associated quantity is written in Feynman's shorthand as $P(A \text{ to } B)$. The similar quantity for an electron moving from C to D is written $E(C \text{ to } D)$. The quantity which tells us about the probability for the emission or absorption of a photon he calls 'j'. This is related to, but not the same as, the measured electron charge 'e'.

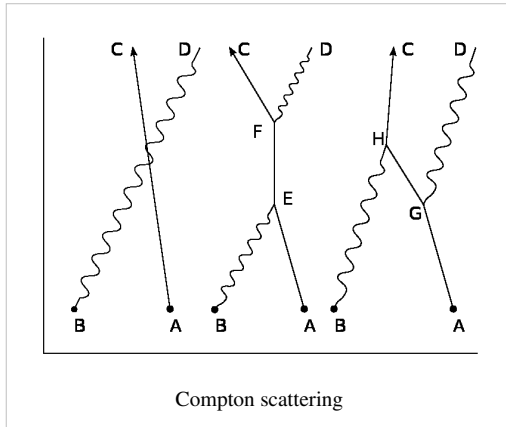
QED is based on the assumption that complex interactions of many electrons and photons can be represented by fitting together a suitable collection of the above three building blocks, and then using the probability-quantities to calculate the probability of any such complex interaction. It turns out that the basic idea of QED can be communicated while making the assumption that the quantities mentioned above are just our everyday probabilities. (A simplification of Feynman's book.) Later on this will be corrected to include specifically quantum mathematics, following Feynman.

The basic rules of probabilities that will be used are that a) if an event can happen in a variety of different ways then its probability is the **sum** of the probabilities of the possible ways and b) if a process involves a number of independent subprocesses then its probability is the **product** of the component probabilities.



Basic constructions

Suppose we start with one electron at a certain place and time (this place and time being given the arbitrary label A) and a photon at another place and time (given the label B). A typical question from a physical standpoint is: 'What is the probability of finding an electron at C (another place and a later time) and a photon at D (yet another place and time)?'. The simplest process to achieve this end is for the electron to move from A to C (an elementary action) and that the photon moves from B to D (another elementary action). From a knowledge of the probabilities of each of these subprocesses – $E(A \text{ to } C)$ and $P(B \text{ to } D)$ – then we would expect to calculate the probability of both happening by multiplying them, using rule b) above. This gives a simple estimated answer to our question.



But there are other ways in which the end result could come about. The electron might move to a place and time E where it absorbs the photon; then move on before emitting another photon at F; then move on to C where it is detected, while the new photon moves on to D. The probability of this complex process can again be calculated by knowing the probabilities of each of the individual actions: three electron actions, two photon actions and two vertexes – one emission and one absorption. We would expect to find the total probability by multiplying the probabilities of each of the actions, for any chosen positions of E and F. We then, using rule a) above, have to add up all these probabilities for all the alternatives for E and F. (This is not elementary in practice, and

involves integration.) But there is another possibility: that is that the electron first moves to G where it emits a photon which goes on to D, while the electron moves on to H, where it absorbs the first photon, before moving on to C. Again we can calculate the probability of these possibilities (for all points G and H). We then have a better estimation for the total probability by adding the probabilities of these two possibilities to our original simple estimate. Incidentally the name given to this process of a photon interacting with an electron in this way is Compton Scattering.

There are an *infinite number* of other intermediate processes in which more and more photons are absorbed and/or emitted. For each of these possibilities there is a Feynman diagram describing it. This implies a complex computation for the resulting probabilities, but provided it is the case that the more complicated the diagram the less it contributes to the result, it is only a matter of time and effort to find as accurate an answer as one wants to the original question. This is the basic approach of QED. To calculate the probability of **any** interactive process between electrons and photons it is a matter of first noting, with Feynman diagrams, all the possible ways in which the process can be constructed from the three basic elements. Each diagram involves some calculation involving definite rules to find the associated probability.

That basic scaffolding remains when one moves to a quantum description but some conceptual changes are requested. One is that whereas we might expect in our everyday life that there would be some constraints on the points to which a particle can move, that is **not** true in full quantum electrodynamics. There is a certain possibility of an electron or photon at A moving as a basic action to *any other place and time in the universe*. That includes places that could only be reached at speeds greater than that of light and also *earlier times*. (An electron moving backwards in time can be viewed as a positron moving forward in time.)

Probability amplitudes

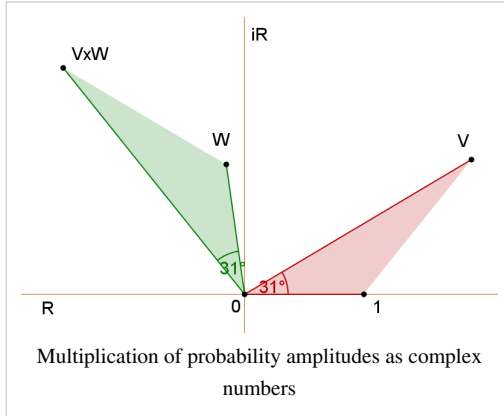
Quantum mechanics introduces an important change on the way probabilities are computed. It has been found that the quantities which we have to use to represent the probabilities are not the usual real numbers we use for probabilities in our everyday world, but complex numbers which are called probability amplitudes. Feynman avoids exposing the reader to the mathematics of complex numbers by using a simple but accurate representation of them as arrows on a piece of paper or screen. (These must not be confused with the arrows of Feynman diagrams which are actually simplified representations in two dimensions of a relationship between points in three dimensions of space and one of time.) The amplitude-arrows are fundamental to the description of the world given by quantum theory. No satisfactory reason has been given for *why* they are needed. But pragmatically we have to accept that they are an essential part of our description of all quantum phenomena. They are related to our everyday ideas of probability by the simple rule that the probability of an event is the **square** of the length of the corresponding amplitude-arrow. So, for a given process, if two probability amplitudes, $\mathbf{v}$ and $\mathbf{w}$, are involved, the probability of the process will be given either by

$$P = |\mathbf{v} + \mathbf{w}|^2$$

or

$$P = |\mathbf{v} \times \mathbf{w}|^2.$$

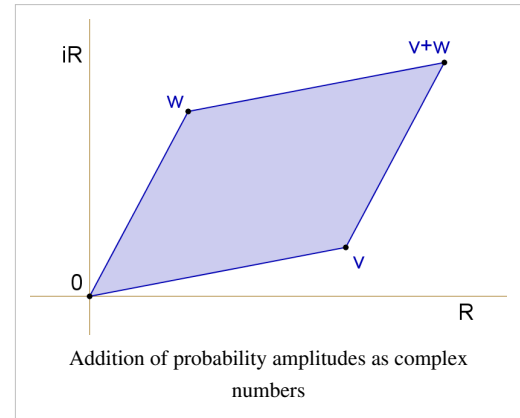
The rules as regards adding or multiplying, however, are the same as above. But where you would expect to add or multiply probabilities, instead you add or multiply probability amplitudes that now are complex numbers.



Addition and multiplication are familiar operations in the theory of complex numbers and are given in the figures. The sum is found as follows. Let the start of the second arrow be at the end of the first. The sum is then a third arrow that goes directly from the start of the first to the end of the second. The product of two arrows is an arrow whose length is the product of the two lengths. The direction of the product is found by adding the angles that each of the two have been turned through relative to a reference direction: that gives the angle that the product is turned relative to the reference direction.

That change, from probabilities to probability amplitudes, complicates the mathematics without changing the basic approach. But that change is still not quite enough because it fails to take into account the fact that both photons and electrons can be polarized, which is to say that their orientation in space and time have to be taken into account. Therefore $P(A \text{ to } B)$ actually consists of 16 complex numbers, or probability amplitude arrows. There are also some minor changes to do with the quantity " j ", which may have to be rotated by a multiple of 90° for some polarizations, which is only of interest for the detailed bookkeeping.

Associated with the fact that the electron can be polarized is another small necessary detail which is connected with the fact that an electron is a Fermion and obeys Fermi-Dirac statistics. The basic rule is that if we have the probability amplitude for a given complex process involving more than one electron, then when we include (as we always must) the complementary Feynman diagram in which we just exchange two electron events, the resulting amplitude is the reverse – the negative – of the first. The simplest case would be two electrons starting at A and B ending at C and D. The amplitude would be calculated as the "difference", $E(A \text{ to } B) \times E(C \text{ to } D) - E(A \text{ to } C) \times E(B \text{ to } D)$, where we would expect, from our everyday idea of probabilities, that it would be a sum.



Propagators

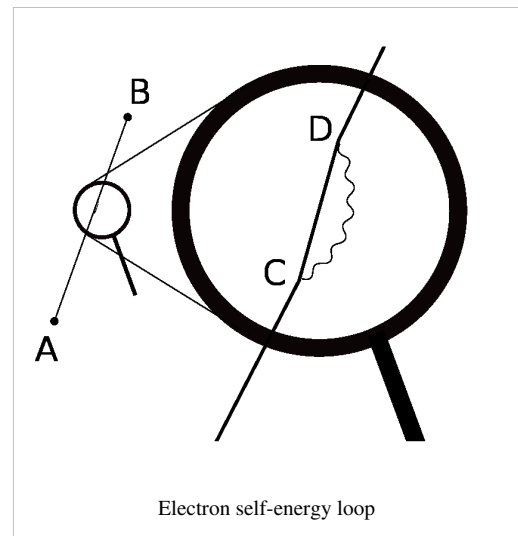
Finally, one has to compute $P(A \text{ to } B)$ and $E(C \text{ to } D)$ corresponding to the probability amplitudes for the photon and the electron respectively. These are essentially the solutions of the Dirac Equation which describes the behavior of the electron's probability amplitude and the Klein-Gordon equation which describes the behavior of the photon's probability amplitude. These are called Feynman propagators. The translation to a notation commonly used in the standard literature is as follows:

$$P(A \text{ to } B) \rightarrow D_F(x_B - x_A), \quad E(C \text{ to } D) \rightarrow S_F(x_D - x_C)$$

where a shorthand symbol such as x_A stands for the four real numbers which give the time and position in three dimensions of the point labeled A.

Mass renormalization

A problem arose historically which held up progress for twenty years: although we start with the assumption of three basic "simple" actions, the rules of the game say that if we want to calculate the probability amplitude for an electron to get from A to B we must take into account **all** the possible ways: all possible Feynman diagrams with those end points. Thus there will be a way in which the electron travels to C, emits a photon there and then absorbs it again at D before moving on to B. Or it could do this kind of thing twice, or more. In short we have a fractal-like situation in which if we look closely at a line it breaks up into a collection of "simple" lines, each of which, if looked at closely, are in turn composed of "simple" lines, and so on *ad infinitum*. This is a very difficult situation to handle. If adding that detail only altered things slightly then it would not have been too bad, but disaster struck when it was found that the simple correction mentioned above led to *infinite* probability amplitudes. In time this problem was "fixed" by the technique of renormalization (see below and the article on mass renormalization). However, Feynman himself remained unhappy about it, calling it a "dippy process".^[20]



Conclusions

Within the above framework physicists were then able to calculate to a high degree of accuracy some of the properties of electrons, such as the anomalous magnetic dipole moment. However, as Feynman points out, it fails totally to explain why particles such as the electron have the masses they do. "There is no theory that adequately explains these numbers. We use the numbers in all our theories, but we don't understand them – what they are, or where they come from. I believe that from a fundamental point of view, this is a very interesting and serious problem."^[23]

Mathematics

Mathematically, QED is an abelian gauge theory with the symmetry group $U(1)$. The gauge field, which mediates the interaction between the charged spin-1/2 fields, is the electromagnetic field. The QED Lagrangian for a spin-1/2 field interacting with the electromagnetic field is given by the real part of

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

where

γ^μ are Dirac matrices;

ψ a bispinor field of spin-1/2 particles (e.g. electron-positron field);

$\bar{\psi} \equiv \psi^\dagger \gamma_0$, called "psi-bar", is sometimes referred to as Dirac adjoint;

$D_\mu = \partial_\mu + ieA_\mu + ieB_\mu$ is the gauge covariant derivative;

e is the coupling constant, equal to the electric charge of the bispinor field;

A_μ is the covariant four-potential of the electromagnetic field generated by the electron itself;

B_μ is the external field imposed by external source;

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor.

Equations of motion

To begin, substituting the definition of D into the Lagrangian gives us:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - e\bar{\psi}\gamma_\mu(A^\mu + B^\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Next, we can substitute this Lagrangian into the Euler-Lagrange equation of motion for a field:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad (2)$$

to find the field equations for QED.

The two terms from this Lagrangian are then:

$$\begin{aligned} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) &= \partial_\mu (i\bar{\psi}\gamma^\mu) \\ \frac{\partial \mathcal{L}}{\partial \psi} &= -e\bar{\psi}\gamma_\mu(A^\mu + B^\mu) - m\bar{\psi}. \end{aligned}$$

Substituting these two back into the Euler-Lagrange equation (2) results in:

$$i\partial_\mu \bar{\psi}\gamma^\mu + e\bar{\psi}\gamma_\mu(A^\mu + B^\mu) + m\bar{\psi} = 0$$

with complex conjugate:

$$i\gamma^\mu \partial_\mu \psi - e\gamma_\mu(A^\mu + B^\mu)\psi - m\psi = 0.$$

Bringing the middle term to the right-hand side transforms this second equation into:

$$\boxed{i\gamma^\mu \partial_\mu \psi - m\psi = e\gamma_\mu(A^\mu + B^\mu)\psi.}$$

The left-hand side is like the original Dirac equation and the right-hand side is the interaction with the electromagnetic field.

One further important equation can be found by substituting the Lagrangian into another Euler-Lagrange equation, this time for the field, A^μ :

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0. \quad (3)$$

The two terms this time are:

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} \right) = \partial_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -e \bar{\psi} \gamma^\mu \psi$$

and these two terms, when substituted back into (3) give us:

$$\partial_\nu F^{\nu\mu} = e \bar{\psi} \gamma^\mu \psi.$$

Interaction picture

This theory can be straightforwardly quantized treating bosonic and fermionic sectors as free. This permits to build a set of asymptotic states to start a computation of the probability amplitudes for different processes. In order to be able to do so, we have to compute an evolution operator that, for a given initial state, will give a final state in such a way to have

$$M_{fi} = \langle f | U | i \rangle.$$

This technique is also known as the S-Matrix. Evolution operator is obtained in the interaction picture where time evolution is given by the interaction Hamiltonian. So, from equations above is

$$V = e \int d^3x \bar{\psi} \gamma^\mu \psi A_\mu$$

and so, one has

$$U = T \exp \left[-\frac{i}{\hbar} \int_{t_0}^t dt' V(t') \right]$$

being T the time ordering operator. This evolution operator has only a meaning as a series and what we get here is a perturbation series with a development parameter being fine structure constant. This series is named Dyson series.

Feynman diagrams

Despite the conceptual clarity of this Feynman approach to QED, almost no textbooks follow him in their presentation. When performing calculations it is much easier to work with the Fourier transforms of the propagators. Quantum physics considers particle's momenta rather than their positions, and it is convenient to think of particles as being created or annihilated when they interact. Feynman diagrams then *look* the same, but the lines have different interpretations. The electron line represents an electron with a given energy and momentum, with a similar interpretation of the photon line. A vertex diagram represents the annihilation of one electron and the creation of another together with the absorption or creation of a photon, each having specified energies and momenta.

Using Wick theorem on the terms of the Dyson series, all the terms of the S-matrix for quantum electrodynamics can be computed through the technique of Feynman diagrams. In this case rules for drawing are the following

$$\alpha \longrightarrow \beta \quad \rightarrow \quad \left(\frac{i}{\not{p} - m + i\varepsilon} \right)_{\beta\alpha}$$

$$\mu \text{ (wavy line)} \nu \quad \rightarrow \quad \frac{-i\eta_{\mu\nu}}{p^2 + i\varepsilon}$$

$$\begin{array}{c} \beta \\ \nearrow \\ \alpha \end{array} \text{ (fermion lines)} \text{ (wavy line)} \mu \quad \rightarrow \quad -ie\gamma_{\beta\alpha}^{\mu} (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3).$$

$$\text{Incoming fermion: } \alpha \longrightarrow \bullet \quad \rightarrow \quad u_{\alpha}(\vec{p}, s)$$

$$\text{Incoming antifermion: } \alpha \longleftarrow \bullet \quad \rightarrow \quad \bar{v}_{\alpha}(\vec{p}, s)$$

$$\text{Outgoing fermion: } \bullet \longrightarrow \alpha \quad \rightarrow \quad \bar{u}_{\alpha}(\vec{p}, s)$$

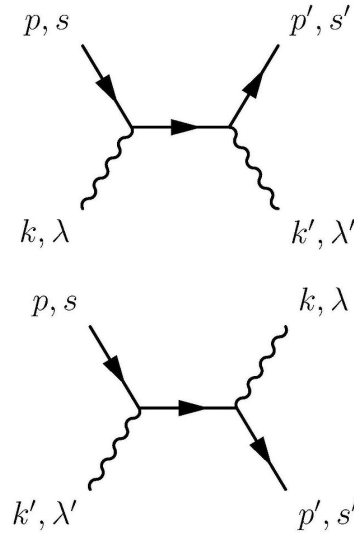
$$\text{Outgoing antifermion: } \bullet \longleftarrow \alpha \quad \rightarrow \quad v_{\alpha}(p, s)$$

$$\text{Incoming photon: } \mu \text{ (wavy line)} \bullet \quad \rightarrow \quad \epsilon_{\mu}(\vec{k}, \lambda)$$

$$\text{Outgoing photon: } \bullet \text{ (wavy line)} \mu \quad \rightarrow \quad \epsilon_{\mu}(\vec{k}, \lambda)^*$$

To these rules we must add a further one for closed loops that implies an integration on momenta $\int d^4p/(2\pi)^4$.

From them, computations of probability amplitudes are straightforwardly given. An example is Compton scattering, with an electron and a photon undergoing elastic scattering. Feynman diagrams are in this case



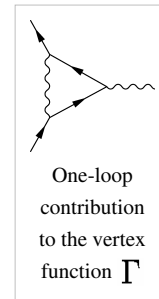
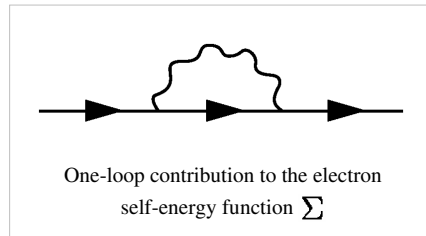
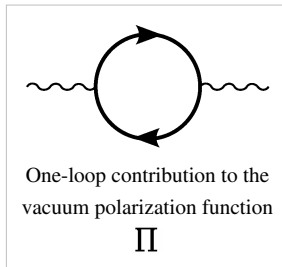
and so we are able to get the corresponding amplitude at the first order of a perturbation series for S-matrix:

$$M_{fi} = (ie)^2 \bar{u}(\vec{p}', s') \not{\epsilon}'(\vec{k}', \lambda')^* \frac{\not{p} + \not{k} + m_e}{(p+k)^2 - m_e^2} \not{\epsilon}(\vec{k}, \lambda) u(\vec{p}, s) + (ie)^2 \bar{u}(\vec{p}', s') \not{\epsilon}(\vec{k}, \lambda) \frac{\not{p} - \not{k}' + m_e}{(p-k')^2 - m_e^2} \not{\epsilon}'(\vec{k}', \lambda')$$

from which we are able to compute the cross section for this scattering.

Renormalizability

Higher order terms can be straightforwardly computed for the evolution operator but these terms display diagrams containing the following simpler ones



that, being closed loops, imply the presence of diverging integrals having no mathematical meaning. To overcome this difficulty, a technique like renormalization has been devised, producing finite results in very close agreement with experiments. It is important to note that a criterion for theory being meaningful after renormalization is that the number of diverging diagrams is finite. In this case the theory is said **renormalizable**. The reason for this is that to get observables renormalized one needs a finite number of constants to maintain the predictive value of the theory untouched. This is exactly the case of quantum electrodynamics displaying just three diverging diagrams. This procedure gives observables in very close agreement with experiment as seen e.g. for electron gyromagnetic ratio.

Renormalizability has become an essential criterion for a quantum field theory to be considered as a viable one. All the theories describing fundamental interactions, except gravitation whose quantum counterpart is presently under very active research, are renormalizable theories.

Nonconvergence of series

An argument by Freeman Dyson shows that the radius of convergence of the perturbation series in QED is zero.^[24] The basic argument goes as follows: if the coupling constant were negative, this would be equivalent to the Coulomb force constant being negative. This would "reverse" the electromagnetic interaction so that *like* charges would *attract* and *unlike* charges would *repel*. This would render the vacuum unstable against decay into a cluster of electrons on one side of the universe and a cluster of positrons on the other side of the universe. Because the theory is sick for any negative value of the coupling constant, the series do not converge, but are an asymptotic series. This can be taken as a need for a new theory, a problem with perturbation theory, or ignored by taking a "shut-up-and-calculate" approach.

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External links

- Feynman's Nobel Prize lecture describing the evolution of QED and his role in it (<http://nobelprize.org/physics/laureates/1965/feynman-lecture.html>)
 - Feynman's New Zealand lectures on QED for non-physicists (<http://www.vega.org.uk/video/subseries/8>)
-

Nonabelian group

In mathematics, a **non-abelian group**, also sometimes called a **non-commutative group**, is a group $(G, *)$ in which there are at least two elements a and b of G such that $a * b \neq b * a$.^[1] ^[2] The term *non-abelian* is used to distinguish from the idea of an abelian group, where all of the elements of the group commute.

Non-abelian groups are pervasive in mathematics and physics. One of the simplest examples of a non-abelian group is the dihedral group of order 6. A common example from physics is the rotation group in three dimensions.

Both discrete groups and continuous groups may be non-abelian; most of the interesting Lie groups are non-abelian. The term *non-abelian* is used primarily by physicists, rather than mathematicians, and is frequently taken to be a synonym for the collection of Lie groups. This usage is particularly common in gauge theory.

Concepts in group theory
category of groups
subgroups, normal subgroups
group homomorphisms, kernel, image, quotient
direct product, direct sum
semidirect product, wreath product
Types of groups
simple, finite, infinite
discrete, continuous
multiplicative, additive
cyclic, abelian, dihedral
nilpotent, solvable
list of group theory topics
glossary of group theory

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Higher Dimensional Algebra

Algebraic Topology

Algebraic topology is a branch of mathematics which uses tools from abstract algebra to study topological spaces. The basic goal is to find algebraic invariants that classify topological spaces up to homeomorphism, though usually most classify up to homotopy equivalence.

Although algebraic topology primarily uses algebra to study topological problems, using topology to solve algebraic problems is sometimes also possible. Algebraic topology, for example, allows for a convenient proof that any subgroup of a free group is again a free group.

The method of algebraic invariants

An older name for the subject was combinatorial topology, implying an emphasis on how a space X was constructed from simpler ones (the modern standard tool for such construction is the CW-complex). The basic method now applied in algebraic topology is to investigate spaces via algebraic invariants by mapping them, for example, to groups which have a great deal of manageable structure in a way that respects the relation of homeomorphism (or more general homotopy) of spaces. This allows one to recast statements about topological spaces into statements about groups, which are often easier to prove.

Two major ways in which this can be done are through fundamental groups, or more generally homotopy theory, and through homology and cohomology groups. The fundamental groups give us basic information about the structure of a topological space, but they are often nonabelian and can be difficult to work with. The fundamental group of a (finite) simplicial complex does have a finite presentation.

Homology and cohomology groups, on the other hand, are abelian and in many important cases finitely generated. Finitely generated abelian groups are completely classified and are particularly easy to work with.

Setting in category theory

In general, all constructions of algebraic topology are functorial; the notions of category, functor and natural transformation originated here. Fundamental groups and homology and cohomology groups are not only *invariants* of the underlying topological space, in the sense that two topological spaces which are homeomorphic have the same associated groups, but their associated morphisms also correspond — a continuous mapping of spaces induces a group homomorphism on the associated groups, and these homomorphisms can be used to show non-existence (or, much more deeply, existence) of mappings.

Results on homology

Several useful results follow immediately from working with finitely generated abelian groups. The free rank of the n -th homology group of a simplicial complex is equal to the n -th Betti number, so one can use the homology groups of a simplicial complex to calculate its Euler-Poincaré characteristic. As another example, the top-dimensional integral homology group of a closed manifold detects orientability: this group is isomorphic to either the integers or 0, according as the manifold is orientable or not. Thus, a great deal of topological information is encoded in the homology of a given topological space.

Beyond simplicial homology, which is defined only for simplicial complexes, one can use the differential structure of smooth manifolds via de Rham cohomology, or Čech or sheaf cohomology to investigate the solvability of

differential equations defined on the manifold in question. De Rham showed that all of these approaches were interrelated and that, for a closed, oriented manifold, the Betti numbers derived through simplicial homology were the same Betti numbers as those derived through de Rham cohomology. This was extended in the 1950s, when Eilenberg and Steenrod generalized this approach. They defined homology and cohomology as functors equipped with natural transformations subject to certain axioms (e.g., a weak equivalence of spaces passes to an isomorphism of homology groups), verified that all existing (co)homology theories satisfied these axioms, and then proved that such an axiomatization uniquely characterized the theory.

A new approach uses a functor from filtered spaces to crossed complexes defined directly and homotopically using relative homotopy groups; a higher homotopy van Kampen theorem proved for this functor enables basic results in algebraic topology, especially on the border between homology and homotopy, to be obtained without using singular homology or simplicial approximation. This approach is also called nonabelian algebraic topology, and generalises to higher dimensions ideas coming from the fundamental group.

Applications of algebraic topology

Classic applications of algebraic topology include:

- The Brouwer fixed point theorem: every continuous map from the unit n -disk to itself has a fixed point.
- The n -sphere admits a nowhere-vanishing continuous unit vector field if and only if n is odd. (For $n = 2$, this is sometimes called the "hairy ball theorem".)
- The Borsuk–Ulam theorem: any continuous map from the n -sphere to Euclidean n -space identifies at least one pair of antipodal points.
- Any subgroup of a free group is free. This result is quite interesting, because the statement is purely algebraic yet the simplest proof is topological. Namely, any free group G may be realized as the fundamental group of a graph X . The main theorem on covering spaces tells us that every subgroup H of G is the fundamental group of some covering space Y of X ; but every such Y is again a graph. Therefore its fundamental group H is free.

On the other hand this type of application is also handled by the use of covering morphisms of groupoids, and that technique has yielded subgroup theorems not yet proved by methods of algebraic topology (see the book by Higgins listed under groupoids).

- Topological combinatorics

Notable algebraic topologists

- Frank Adams
- Karol Borsuk
- Luitzen Egbertus Jan Brouwer
- William Browder
- Ronald Brown (mathematician)
- Nicolas Bourbaki
- Henri Cartan
- Hermann K nneth
- Samuel Eilenberg
- Hans Freudenthal
- Peter Freyd
- Alexander Grothendieck
- Friedrich Hirzebruch
- Heinz Hopf
- Michael J. Hopkins

- Witold Hurewicz
- Egbert van Kampen
- Saunders Mac Lane
- Jean Leray
- Mark Mahowald
- J. Peter May
- John Coleman Moore
- Sergei Novikov
- Lev Pontryagin
- Mikhail Postnikov
- Daniel Quillen
- Jean-Pierre Serre
- Stephen Smale
- Norman Steenrod
- Dennis Sullivan
- René Thom
- Leopold Vietoris
- Hassler Whitney
- J. H. C. Whitehead
- Brandon Blaha

Important theorems in algebraic topology

- Borsuk-Ulam theorem
- Brouwer fixed point theorem
- Cellular approximation theorem
- Eilenberg–Zilber theorem
- Freudenthal suspension theorem
- Hurewicz theorem
- Kunnet theorem
- Poincaré duality theorem
- Universal coefficient theorem
- Van Kampen's theorem
- Generalized van Kampen's theorems^[1]
- Higher homotopy, generalized van Kampen's theorem^[2]
- Whitehead's theorem

Notes

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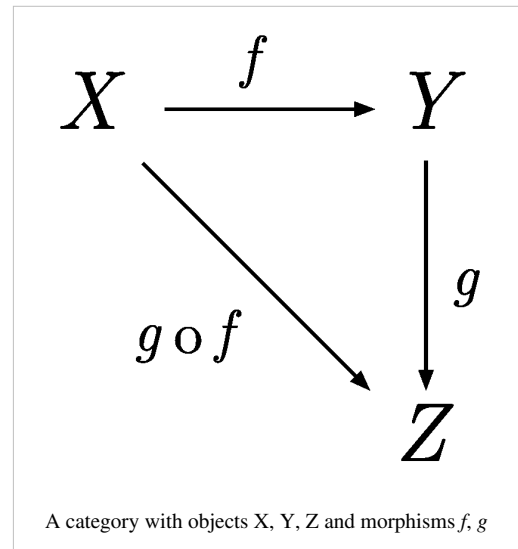
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Category Theory

Category theory is an area of study in mathematics that examines in an abstract way the properties of particular mathematical concepts, by formalising them as collections of *objects* and *arrows* (also called morphisms, although this term also has a specific, non category-theoretical sense), where these collections satisfy certain basic conditions. Many significant areas of mathematics can be formalised as categories, and the use of category theory allows many intricate and subtle mathematical results in these fields to be stated, and proved, in a much simpler way than without the use of categories.

The most accessible example of a category is the Category of sets, where the objects are sets and the arrows are functions from one set to another. However it is important to note that the objects of a category need not be sets nor the arrows functions; any way of formalising a mathematical concept such that it meets the basic conditions on the behaviour of objects and arrows is a valid category, and all the results of category theory will apply to it.



One of the simplest examples of a category (which is a very important concept in topology) is that of groupoid, defined as a category whose arrows or morphisms are all invertible. Categories now appear in most branches of mathematics, some areas of theoretical computer science where they correspond to types, and mathematical physics where they can be used to describe vector spaces. Categories were first introduced by Samuel Eilenberg and Saunders Mac Lane in 1942–45, in connection with algebraic topology.

Category theory has several faces known not just to specialists, but to other mathematicians. A term dating from the 1940s, "general abstract nonsense", refers to its high level of abstraction, compared to more classical branches of mathematics. Homological algebra is category theory in its aspect of organising and suggesting manipulations in abstract algebra. Diagram chasing is a visual method of arguing with abstract "arrows" joined in diagrams. Note that arrows between categories are called functors, subject to specific defining commutativity conditions; moreover, categorical diagrams and sequences can be defined as functors (viz. Mitchell, 1965). An arrow between two functors is a natural transformation when it is subject to certain naturality or commutativity conditions. Functors and natural transformations ('naturality') are the key concepts in category theory^[1]. Topos theory is a form of abstract sheaf theory, with geometric origins, and leads to ideas such as pointless topology. A topos can also be considered as a specific type of category with two additional topos axioms.

Background

The study of categories is an attempt to *axiomatically* capture what is commonly found in various classes of related *mathematical structures* by relating them to the *structure-preserving functions* between them. A systematic study of category theory then allows us to prove general results about any of these types of mathematical structures from the axioms of a category.

Consider the following example. The class **Grp** of groups consists of all objects having a "group structure". One can proceed to prove theorems about groups by making logical deductions from the set of axioms. For example, it is immediately proved from the axioms that the identity element of a group is unique.

Instead of focusing merely on the individual objects (e.g., groups) possessing a given structure, category theory emphasizes the morphisms – the structure-preserving mappings – *between* these objects; by studying these morphisms, we are able to learn more about the structure of the objects. In the case of groups, the morphisms are the

group homomorphisms. A group homomorphism between two groups "preserves the group structure" in a precise sense – it is a "process" taking one group to another, in a way that carries along information about the structure of the first group into the second group. The study of group homomorphisms then provides a tool for studying general properties of groups and consequences of the group axioms.

A similar type of investigation occurs in many mathematical theories, such as the study of continuous maps (morphisms) between topological spaces in topology (the associated category is called **Top**), and the study of smooth functions (morphisms) in manifold theory.

If one axiomatizes relations instead of functions, one obtains the theory of allegories.

Functors

Abstracting again, a category is *itself* a type of mathematical structure, so we can look for "processes" which preserve this structure in some sense; such a process is called a functor. A functor associates to every object of one category an object of another category, and to every morphism in the first category a morphism in the second.

In fact, what we have done is define a category *of categories and functors* – the objects are categories, and the morphisms (between categories) are functors.

By studying categories and functors, we are not just studying a class of mathematical structures and the morphisms between them; we are studying the *relationships between various classes of mathematical structures*. This is a fundamental idea, which first surfaced in algebraic topology. Difficult *topological* questions can be translated into *algebraic* questions which are often easier to solve. Basic constructions, such as the fundamental group or fundamental groupoid ^[2] of a topological space, can be expressed as fundamental functors ^[2] to the category of groupoids in this way, and the concept is pervasive in algebra and its applications.

Natural transformation

Abstracting yet again, constructions are often "naturally related" – a vague notion, at first sight. This leads to the clarifying concept of natural transformation, a way to "map" one functor to another. Many important constructions in mathematics can be studied in this context. "Naturality" is a principle, like general covariance in physics, that cuts deeper than is initially apparent.

Historical notes

In 1942–45, Samuel Eilenberg and Saunders Mac Lane introduced categories, functors, and natural transformations as part of their work in topology, especially algebraic topology. Their work was an important part of the transition from intuitive and geometric homology to axiomatic homology theory. Eilenberg and Mac Lane later wrote that their goal was to understand natural transformations; in order to do that, functors had to be defined, which required categories.

Stanisław Ulam, and some writing on his behalf, have claimed that related ideas were current in the late 1930s in Poland. Eilenberg was Polish, and studied mathematics in Poland in the 1930s. Category theory is also, in some sense, a continuation of the work of Emmy Noether (one of Mac Lane's teachers) in formalizing abstract processes; Noether realized that in order to understand a type of mathematical structure, one needs to understand the processes preserving that structure. In order to achieve this understanding, Eilenberg and Mac Lane proposed an axiomatic formalization of the relation between structures and the processes preserving them.

The subsequent development of category theory was powered first by the computational needs of homological algebra, and later by the axiomatic needs of algebraic geometry, the field most resistant to being grounded in either axiomatic set theory or the Russell-Whitehead view of united foundations. General category theory, an extension of universal algebra having many new features allowing for semantic flexibility and higher-order logic, came later; it is now applied throughout mathematics.

Certain categories called topoi (singular *topos*) can even serve as an alternative to axiomatic set theory as a foundation of mathematics. These foundational applications of category theory have been worked out in fair detail as a basis for, and justification of, constructive mathematics. More recent efforts to introduce undergraduates to categories as a foundation for mathematics include Lawvere and Rosebrugh (2003) and Lawvere and Schanuel (1997).

Categorical logic is now a well-defined field based on type theory for intuitionistic logics, with applications in functional programming and domain theory, where a cartesian closed category is taken as a non-syntactic description of a lambda calculus. At the very least, category theoretic language clarifies what exactly these related areas have in common (in some abstract sense).

Categories, objects, and morphisms

A category C consists of the following three mathematical entities:

- A class $\text{ob}(C)$, whose elements are called *objects*;
- A class $\text{hom}(C)$, whose elements are called morphisms or maps or *arrows*. Each morphism f has a unique *source object* a and *target object* b . We write $f: a \rightarrow b$, and we say " f is a morphism from a to b ". We write $\text{hom}(a, b)$ (or $\text{Hom}(a, b)$, or $\text{hom}_C(a, b)$, or $\text{Mor}(a, b)$, or $C(a, b)$) to denote the *hom-class* of all morphisms from a to b .
- A binary operation $\circ$, called *composition of morphisms*, such that for any three objects a, b , and c , we have $\text{hom}(a, b) \times \text{hom}(b, c) \rightarrow \text{hom}(a, c)$. The composition of $f: a \rightarrow b$ and $g: b \rightarrow c$ is written as $g \circ f$ or gf ^[3], governed by two axioms:
 - Associativity: If $f: a \rightarrow b$, $g: b \rightarrow c$ and $h: c \rightarrow d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
 - Identity: For every object x , there exists a morphism $1_x: x \rightarrow x$ called the *identity morphism* for x , such that for every morphism $f: a \rightarrow b$, we have $1_b \circ f = f = f \circ 1_a$.

From these axioms, it can be proved that there is exactly one identity morphism for every object. Some authors deviate from the definition just given by identifying each object with its identity morphism.

Relations among morphisms (such as $fg = h$) are often depicted using commutative diagrams, with "points" (corners) representing objects and "arrows" representing morphisms.

Properties of morphisms

Some morphisms have important properties. A morphism $f: a \rightarrow b$ is:

- a monomorphism (or *monic*) if $fg_1 = fg_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2: x \rightarrow a$.
- an epimorphism (or *epic*) if $g_1f = g_2f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2: b \rightarrow x$.
- an isomorphism if there exists a morphism $g: b \rightarrow a$ with $fg = 1_b$ and $gf = 1_a$.^[4]
- an endomorphism if $a = b$. $\text{end}(a)$ denotes the class of endomorphisms of a .
- an automorphism if f is both an endomorphism and an isomorphism. $\text{aut}(a)$ denotes the class of automorphisms of a .

Functors

Functors are structure-preserving maps between categories. They can be thought of as morphisms in the category of all (small) categories.

A (**covariant**) functor F from a category C to a category D , written $F:C \rightarrow D$, consists of:

- for each object x in C , an object $F(x)$ in D ; and
- for each morphism $f: x \rightarrow y$ in C , a morphism $F(f): F(x) \rightarrow F(y)$,

such that the following two properties hold:

- For every object x in C , $F(1_x) = 1_{F(x)}$;
- For all morphisms $f: x \rightarrow y$ and $g: y \rightarrow z$, $F(g \circ f) = F(g) \circ F(f)$.

A **contravariant** functor $F: C \rightarrow D$, is like a covariant functor, except that it "turns morphisms around" ("reverses all the arrows"). More specifically, every morphism $f: x \rightarrow y$ in C must be assigned to a morphism $F(f): F(y) \rightarrow F(x)$ in D . In other words, a contravariant functor is a covariant functor from the opposite category C^{op} to D .

Natural transformations and isomorphisms

A *natural transformation* is a relation between two functors. Functors often describe "natural constructions" and natural transformations then describe "natural homomorphisms" between two such constructions. Sometimes two quite different constructions yield "the same" result; this is expressed by a natural isomorphism between the two functors.

If F and G are (covariant) functors between the categories C and D , then a natural transformation η from F to G associates to every object x in C a morphism $\eta_x: F(x) \rightarrow G(x)$ in D such that for every morphism $f: x \rightarrow y$ in C , we have $\eta_y \circ F(f) = G(f) \circ \eta_x$; this means that the following diagram is commutative:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \eta_X \downarrow & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

The two functors F and G are called *naturally isomorphic* if there exists a natural transformation from F to G such that η_x is an isomorphism for every object x in C .

Universal constructions, limits, and colimits

Using the language of category theory, many areas of mathematical study can be cast into appropriate categories, such as the categories of all sets, groups, topologies, and so on. These categories surely have some objects that are "special" in a certain way, such as the empty set or the product of two topologies, yet in the definition of a category, objects are considered to be atomic, i.e., we *do not know* whether an object A is a set, a topology, or any other abstract concept – hence, the challenge is to define special objects without referring to the internal structure of those objects. But how can we define the empty set without referring to elements, or the product topology without referring to open sets?

The solution is to characterize these objects in terms of their relations to other objects, as given by the morphisms of the respective categories. Thus, the task is to find *universal properties* that uniquely determine the objects of interest.

Indeed, it turns out that numerous important constructions can be described in a purely categorical way. The central concept which is needed for this purpose is called categorical *limit*, and can be dualized to yield the notion of a *colimit*.

Equivalent categories

It is a natural question to ask: under which conditions can two categories be considered to be "essentially the same", in the sense that theorems about one category can readily be transformed into theorems about the other category? The major tool one employs to describe such a situation is called *equivalence of categories*, which is given by appropriate functors between two categories. Categorical equivalence has found numerous applications in mathematics.

Further concepts and results

The definitions of categories and functors provide only the very basics of categorical algebra; additional important topics are listed below. Although there are strong interrelations between all of these topics, the given order can be considered as a guideline for further reading.

- The functor category D^C has as objects the functors from C to D and as morphisms the natural transformations of such functors. The Yoneda lemma is one of the most famous basic results of category theory; it describes representable functors in functor categories.
- Duality: Every statement, theorem, or definition in category theory has a *dual* which is essentially obtained by "reversing all the arrows". If one statement is true in a category C then its dual will be true in the dual category C^{op} . This duality, which is transparent at the level of category theory, is often obscured in applications and can lead to surprising relationships.
- Adjoint functors: A functor can be left (or right) adjoint to another functor that maps in the opposite direction. Such a pair of adjoint functors typically arises from a construction defined by a universal property; this can be seen as a more abstract and powerful view on universal properties.

Higher-dimensional categories

Many of the above concepts, especially equivalence of categories, adjoint functor pairs, and functor categories, can be situated into the context of *higher-dimensional categories*. Briefly, if we consider a morphism between two objects as a "process taking us from one object to another", then higher-dimensional categories allow us to profitably generalize this by considering "higher-dimensional processes".

For example, a (strict) 2-category is a category together with "morphisms between morphisms", i.e., processes which allow us to transform one morphism into another. We can then "compose" these "bimorphisms" both horizontally and vertically, and we require a 2-dimensional "exchange law" to hold, relating the two composition laws. In this context, the standard example is **Cat**, the 2-category of all (small) categories, and in this example, bimorphisms of morphisms are simply natural transformations of morphisms in the usual sense. Another basic example is to consider a 2-category with a single object; these are essentially monoidal categories. Bicategories are a weaker notion of 2-dimensional categories in which the composition of morphisms is not strictly associative, but only associative "up to" an isomorphism.

This process can be extended for all natural numbers n , and these are called n -categories. There is even a notion of ω -category corresponding to the ordinal number ω .

Higher-dimensional categories are part of the broader mathematical field of higher-dimensional algebra, a concept introduced by Ronald Brown. For a conversational introduction to these ideas, see John Baez, 'A Tale of n -categories' (1996).^[5]

See also

- Domain theory
- Enriched category theory
- Glossary of category theory
- Higher category theory
- Higher-dimensional algebra
- Important publications in category theory
- Timeline of category theory and related mathematics

Notes

- [1] *Categories for the Working Mathematician*, 2nd Edition, p 18: "As Eilenberg-Mac Lane first observed, 'category' has been defined in order to be able to define 'functor' and 'functor' has been defined in order to be able to define 'natural transformation'".
- [2] <http://planetphysics.org/encyclopedia/FundamentalGroupoidFunctor.html>
- [3] Some authors compose in the opposite order, writing fg or $f \circ g$ for $g \circ f$. Computer scientists using category theory very commonly write $f;g$ for $g \circ f$.
- [4] Note that a morphism that is both epic and monic is not necessarily an isomorphism! For example, in the category consisting of two objects A and B , the identity morphisms, and a single morphism f from A to B , f is both epic and monic but is not an isomorphism.
- [5] <http://math.ucr.edu/home/baez/week73.html>

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External links

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- Stanford Encyclopedia of Philosophy: "Category Theory" (<http://plato.stanford.edu/entries/category-theory/>) -- by Jean-Pierre Marquis. Extensive bibliography.
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- Video archive (<http://categorieslogicphysics.wikidot.com/events>) of recorded talks relevant to categories, logic and the foundations of physics.
- Interactive Web page (<http://www.j-paine.org/cgi-bin/webcats/webcats.php>) which generates examples of categorical constructions in the category of finite sets.

Double Groupoid

In mathematics, especially in higher-dimensional algebra and homotopy theory, a **double groupoid** generalises the notion of groupoid and of category to a higher dimension.

Definition

A **double groupoid** $\mathbf{D}$ is a higher-dimensional groupoid involving a relationship for both 'horizontal' and 'vertical' groupoid structures^[1]. (A double groupoid can also be considered as a generalization of certain higher-dimensional groups^[2].) The geometry of squares and their compositions leads to a common representation of a *double groupoid* in the following diagram:

$$\mathbf{D} = \begin{array}{ccc} & \xrightarrow{s^1} & \\ S & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} & H \\ & \xleftarrow{t^1} & \\ \downarrow s_2 & & \downarrow s \\ \uparrow t_2 & & \uparrow t \\ V & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} & M \\ & \xleftarrow{t} & \end{array}$$

where $\mathbf{M}$ is a set of 'points', $\mathbf{H}$ and $\mathbf{V}$ are, respectively, 'horizontal' and 'vertical' groupoids, and $\mathbf{S}$ is a set of 'squares' with two compositions. The composition laws for a double groupoid $\mathbf{D}$ make it also describable as a groupoid internal to the category of groupoids.

Given two groupoids $\mathbf{H}, \mathbf{V}$ over a set $\mathbf{M}$, there is a double groupoid $\square(\mathbf{H}, \mathbf{V})$ with $\mathbf{H}, \mathbf{V}$ as horizontal and vertical edge groupoids, and squares given by quadruples

$$\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix}$$

for which we assume always that h, h' are in $\mathbf{H}$, v, v' are in $\mathbf{V}$, and that the initial and final points of these edges match in $\mathbf{M}$ as suggested by the notation, that is for example $sh = sv, th = sv', \dots$, etc. The compositions are to be inherited from those of $\mathbf{H}, \mathbf{V}$, that is:

$$\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \circ_1 \begin{pmatrix} & h' & \\ w & & w' \\ & h'' & \end{pmatrix} = \begin{pmatrix} & h & \\ vw & & v'w' \\ & h'' & \end{pmatrix}, \quad \begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \circ_2 \begin{pmatrix} & k & \\ v' & & v'' \\ & k' & \end{pmatrix} = \begin{pmatrix} & hk & \\ v & & v'' \\ & h'k' & \end{pmatrix}$$

This construction is the right adjoint to the forgetful functor which takes the double groupoid as above, to the pair of groupoids $\mathbf{H}, \mathbf{V}$ over $\mathbf{M}$.

Other related constructions are that of a double groupoid with connection^[3] and homotopy double groupoids^[4].

Convolution algebra

A *convolution C^* -algebra* of a double groupoid can also be constructed by employing the square diagram $\mathbf{D}$ of a double groupoid^[5].

Double Groupoid Category

The category whose objects are double groupoids and whose morphisms are double groupoid homomorphisms that are double groupoid diagram ($\mathbf{D}$) functors is called the **double groupoid category**, or the **category of double groupoids**.

Notes

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This article incorporates material from higher dimensional algebra (<http://planetmath.org/encyclopedia/ExtensionOfAlgebraicTopology.html>), which is licensed under the Creative Commons Attribution/Share-Alike License.

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Higher dimensional algebra

*This article is about **higher-dimensional algebra and supercategories** in generalized category theory, super-category theory, and also its extensions in nonabelian algebraic topology and metamathematics^[1].*

Supercategories were first introduced in 1970,^[2] and were subsequently developed for applications in theoretical physics (especially quantum field theory and topological quantum field theory) and mathematical biology or mathematical biophysics.^[3]

Double groupoids, fundamental groupoids, 2-categories, categorical QFTs and TQFTs

In **higher-dimensional algebra (HDA)**, a double groupoid is a generalisation of a one-dimensional groupoid to two dimensions^[4], and the latter groupoid can be considered as a special case of a category with all invertible arrows, or morphisms.

Double groupoids are often used to capture information about geometrical objects such as higher-dimensional manifolds (or n -dimensional manifolds)^[5]. In general, an n -dimensional manifold is a space that locally looks like an n -dimensional Euclidean space, but whose global structure may be non-Euclidean. A first step towards defining higher dimensional algebras is the concept of 2-category of higher category theory, followed by the more 'geometric' concept of double category^{[6][7][8]}. Other pathways in HDA involve: bicategories, homomorphisms of bicategories, variable categories (*aka*, indexed, or parametrized categories), topoi, effective descent, enriched and internal categories, as well as quantum categories^{[9][10][11]} and quantum double groupoids^[12]. In the latter case, by considering fundamental groupoids defined via a 2-functor allows one to think about the physically interesting case of quantum fundamental groupoids (QFGs) in terms of the bicategory **Span(Groupoids)**, and then constructing 2-Hilbert spaces and 2-linear maps for manifolds and cobordisms. At the next step, one obtains cobordisms with corners via natural transformations of such 2-functors. A claim was then made that, with the gauge group $SU(2)$, "the extended TQFT, or ETQFT, gives a theory equivalent to the Ponzano-Regge model of quantum gravity"^[13]; similarly, the Turaev-Viro model would be then obtained with representations of $SU_q(2)$. Therefore, according to the construction proposed by Jeffrey Morton, one can describe the state space of a gauge theory – or many kinds of quantum field theories (QFTs) and local quantum physics, in terms of the transformation groupoids given by symmetries, as for example in the case of a gauge theory, by the gauge transformations acting on states that are, in this case, connections. In the case of symmetries related to quantum groups, one would obtain structures that are

representation categories of quantum groupoids^[14], instead of the 2-vector spaces that are representation categories of groupoids.

Double categories, Category of categories and Supercategories

A higher level concept is thus defined as a category of categories, or **super-category**, which generalises to higher dimensions the notion of category – regarded as any structure which is an interpretation of Lawvere's axioms of the *elementary theory of abstract categories* (ETAC)^{[15] [16] [17] [18]}. Thus, a supercategory and also a super-category, can be regarded as natural extensions of the concepts of meta-category,^[19] multicategory, and multi-graph, k-partite graph, or colored graph (see a color figure, and also its definition in graph theory).

Double groupoids were first introduced by Ronald Brown in 1976, in ref.^[20] and were further developed towards applications in nonabelian algebraic topology^{[21] [22] [23] [24]}. A related, 'dual' concept is that of a double algebroid, and the more general concept of R-algebroid.

Nonabelian algebraic topology

Many of the higher dimensional algebraic structures are noncommutative and, therefore, their study is a very significant part of nonabelian category theory, and also of Nonabelian Algebraic Topology (NAAT)^{[25] [26]} which generalises to higher dimensions ideas coming from the fundamental group^[27]. Such algebraic structures in dimensions greater than 1 develop the nonabelian character of the fundamental group, and they are in a precise sense '*more nonabelian than the groups*'^{[28] [29]}. These noncommutative, or more specifically, nonabelian structures reflect more accurately the geometrical complications of higher dimensions than the known homology and homotopy groups commonly encountered in classical algebraic topology. An important part of nonabelian algebraic topology is concerned with the properties and applications of homotopy groupoids and filtered spaces. Noncommutative double groupoids and double algebroids are only the first examples of such higher dimensional structures that are nonabelian. The new methods of Nonabelian Algebraic Topology (NAAT) '*can be applied to determine homotopy invariants of spaces, and homotopy classification of maps, in cases which include some classical results, and allow results not available by classical methods*'^[30]. Cubical omega-groupoids, higher homotopy groupoids, crossed modules, crossed complexes and Galois groupoids are key concepts in developing applications related to homotopy of filtered spaces, higher dimensional space structures, the construction of the fundamental groupoid of a topos $\mathcal{E}$ in the general theory of topoi, and also in their physical applications in nonabelian quantum theories, and recent developments in quantum gravity, as well as categorical and topological dynamics^[31]. Further examples of such applications include the generalisations of noncommutative geometry formalizations of the noncommutative standard models *via* fundamental double groupoids and spacetime structures even more general than topoi or the lower-dimensional noncommutative spacetimes encountered in several topological quantum field theories and noncommutative geometry theories of quantum gravity.

A fundamental result in NAAT is the generalised, higher homotopy van Kampen theorem proven by R. Brown which states that '*the homotopy type of a topological space can be computed by a suitable colimit or homotopy colimit over homotopy types of its pieces*'. A related example is that of van Kampen theorems for categories of covering morphisms in left extensive categories^[32]. Other reports of generalisations of the van Kampen theorem include statements for 2-categories^[33] and a topos of topoi [34]. Important results in HDA are also the extensions of the Galois theory in categories and variable categories, or indexed/'parametrized' categories^{[35] [36]}. The Joyal-Tierney representation theorem for topoi is also a generalisation of the Galois theory^[37]. Thus, indexing by bicategories in the sense of Benabou one also includes here the Joyal-Tierney theory^[38].

Notes

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- [12] [http://theoreticalatlas.wordpress.com/2009/03/18/a-note-on-quantum-groupoids/March 18, 2009. A Note on Quantum Groupoids,](http://theoreticalatlas.wordpress.com/2009/03/18/a-note-on-quantum-groupoids/March%2018%2C%202009.%20A%20Note%20on%20Quantum%20Groupoids,%20posted%20by%20Jeffrey%20Morton%20under%20C*-algebras,%20deformation%20theory,%20groupoids,%20noncommutative%20geometry,%20quantization) posted by Jeffrey Morton under C*-algebras, deformation theory, groupoids, noncommutative geometry, quantization
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Higher category theory

Higher category theory is the part of category theory at a *higher order*, which means that some equalities are replaced by explicit arrows in order to be able to explicitly study the structure behind those equalities.

Strict higher categories

N-categories are defined inductively using the enriched category theory: 0-categories are sets, and (n+1)-categories are categories enriched over the monoidal category of n-categories (with the monoidal structure given by finite products).^[1] This construction is well defined, as shown in the article on n-categories. This concept introduces higher arrows, higher compositions and higher identities, which must well behave together. For example, the category of small categories is in fact a 2-category, with natural transformations as second degree arrows. However this concept is too strict for some purposes (for example, homotopy theory), where "weak" structures arise in the form of higher categories.^[2]

Weak higher categories

In weak n-categories, the associativity and identity conditions are no longer strict (that is, they are not given by equalities), but rather are satisfied up to an isomorphism of the next level. An example in topology is the composition of paths, which is associative only up to homotopy. These isomorphisms must well behave between hom-sets and expressing this is the difficulty in the definition of weak n-categories. Weak 2-categories, also called bicategories, were the first to be defined explicitly. A particularity of these is that a bicategory with one object is exactly a monoidal category, so that bicategories can be said to be "monoidal categories with many objects." Weak 3-categories, also called tricategories, and higher-level generalizations are increasingly harder to define explicitly. Several definitions have been given, and telling when they are equivalent, and in what sense, has become a new object of study in category theory.

Quasicategories

Weak Kan complexes, or quasi-categories, are semisimplicial complexes satisfying a weak version of the Kan condition. Joyal showed that they are a good foundation for higher category theory. Recently the theory has been sistematized further by Jacob Lurie who simply call them infinity categories, though the latter term is also a generic term for all models of (infinity,k) categories for any k.

Simplicially enriched category

Simplicially enriched categories, or simplicial categories, are categories enriched over simplicial sets. However, when we look at them as a model for (infinity,1)-categories, then many categorical notions, say limits do not agree with the corresponding notions in the sense of enriched categories. The same for other enriched models like topologically enriched categories.

Topologically enriched categories

Topologically enriched categories (sometimes simply topological categories) are categories enriched over some convenient category of topological spaces, e.g. the category of compactly generated Hausdorff topological spaces.

Segal categories

These are models of higher categories introduced by Hirschowitz and Simpson in 1988^[3], partly inspired by results of Graeme Segal in 1974.

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- *nlab* (<http://ncatlab.org/nlab/show/HomePage>), the collective and open wiki notebook project on higher category theory and applications in physics, mathematics and philosophy
- Joyal's Catlab (<http://ncatlab.org/joyalscatlab/show/HomePage>), a wiki dedicated to polished expositions of categorical and higher categorical mathematics with proofs

External links

- John Baez Tale of n -Categories (<http://math.ucr.edu/home/baez/week73.html>)
- The n -Category Cafe (<http://golem.ph.utexas.edu/category/>) - a group blog devoted to higher category theory.

Duality (mathematics)

In mathematics, *duality* has numerous meanings, and although it is “a very pervasive and important concept in (modern) mathematics”^[1] and “an important general theme that has manifestations in almost every area of mathematics”,^[2] there is no single universally agreed definition that unifies all concepts of duality.^[2]

Generally speaking, a duality translates concepts, theorems or mathematical structures into other concepts, theorems or structures, in a one-to-one fashion, often (but not always) by means of an involution operation: if the dual of A is B , then the dual of B is A . As involutions sometimes have fixed points, the dual of A is sometimes A itself. For example, Desargues' theorem in projective geometry is self-dual in this sense.

Many mathematical dualities between objects of two types correspond to pairings, bilinear functions from an object of one type and another object of the second type to some family of scalars. For instance, linear algebra duality corresponds in this way to bilinear maps from pairs of vector spaces to scalars, the duality between distributions and the associated test functions corresponds to the pairing in which one integrates a distribution against a test function, and Poincaré duality corresponds similarly to intersection number, viewed as a pairing between submanifolds of a given manifold.^[3]

Order-reversing dualities

A particularly simple form of duality comes from order theory. The dual of a poset $P = (X, \leq)$ is the poset $P^d = (X, \geq)$ comprising the same ground set but the converse relation. Familiar examples of dual partial orders include

- the subset and superset relations $\subset$ and $\supset$ on any collection of sets,
- the *divides* and *multiple-of* relations on the integers, and
- the *descendant-of* and *ancestor-of* relations on the set of humans.

A concept defined for a partial order P will correspond to a *dual concept* on the dual poset P^d . For instance, a minimal element of P will be a maximal element of P^d : minimality and maximality are dual concepts in order theory. Other pairs of dual concepts are upper and lower bounds, lower sets and upper sets, and ideals and filters.

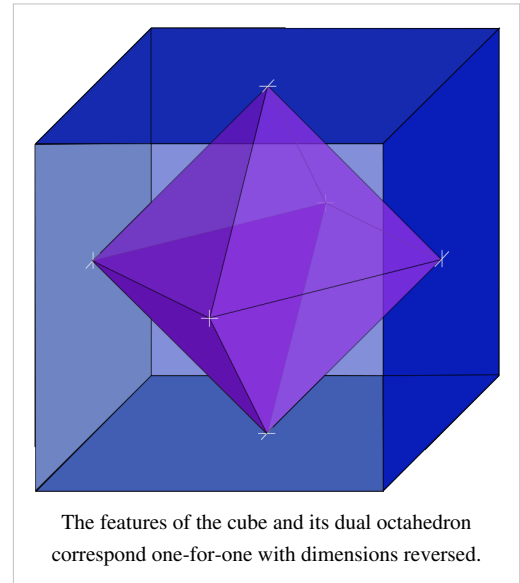
A particular order reversal of this type occurs in the family of all subsets of some set S : if $\bar{A} = S \setminus A$ denotes the complement set, then $A \subset B$ if and only if $\bar{B} \subset \bar{A}$. In topology, open sets and closed sets are dual concepts: the complement of an open set is closed, and vice versa. In matroid theory, the family of sets complementary to the independent sets of a given matroid themselves form another matroid, called the dual matroid. In logic, one may represent a truth assignment to the variables of an unquantified formula as a set, the variables that are true for the assignment. A truth assignment satisfies the formula if and only if the complementary truth assignment satisfies the De Morgan dual of its formula. The existential and universal quantifiers in logic are similarly dual.

A partial order may be interpreted as a category in which there is an arrow from x to y in the category if and only if $x \leq y$ in the partial order. The order-reversing duality of partial orders can be extended to the concept of a dual category, the category formed by reversing all the arrows in a given category. Many of the specific dualities described later are dualities of categories in this sense.

According to Artstein-Avidan and Milman,^{[4] [5]} a *duality transform* is just an involutive antiautomorphism $\mathcal{T}$ of a partially ordered set S , that is, an order reversing involution $\mathcal{T} : S \rightarrow S$. Surprisingly, in several important cases these simple properties determine the transform uniquely up to some simple symmetries. If $\mathcal{T}_1, \mathcal{T}_2$ are two duality transforms then their composition is an order automorphism of S ; thus, any two duality transforms differ only by an order automorphism. For example, all order automorphisms of a power set $S=2^R$ are induced by permutations of R . The papers cited above treat only sets S of functions on R^n satisfying some condition of convexity and prove that all order automorphisms are induced by linear or affine transformations of R^n .

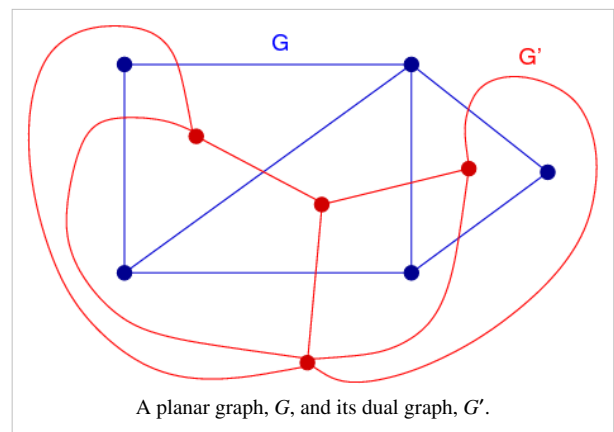
Dimension-reversing dualities

There are many distinct but interrelated dualities in which geometric or topological objects correspond to other objects of the same type, but with a reversal of the dimensions of the features of the objects. A classical example of this is the duality of the platonic solids, in which the cube and the octahedron form a dual pair, the dodecahedron and the icosahedron form a dual pair, and the tetrahedron is **self-dual**. The dual polyhedron of any of these polyhedra may be formed as the convex hull of the center points of each face of the primal polyhedron, so the vertices of the dual correspond one-for-one with the faces of the primal. Similarly, each edge of the dual corresponds to an edge of the primal, and each face of the dual corresponds to a vertex of the primal. These correspondences are incidence-preserving: if two parts of the primal polyhedron touch each other, so do the corresponding two parts of the dual polyhedron. More generally, using the concept of polar reciprocation, any convex polyhedron, or more generally any



convex polytope, corresponds to a dual polyhedron or dual polytope, with an i -dimensional feature of an n -dimensional polytope corresponding to an $(n - i - 1)$ -dimensional feature of the dual polytope. The incidence-preserving nature of the duality is reflected in the fact that the face lattices of the primal and dual polyhedra or polytopes are themselves order-theoretic duals. Duality of polytopes and order-theoretic duality are both involutions: the dual polytope of the dual polytope of any polytope is the original polytope, and reversing all order-relations twice returns to the original order. Choosing a different center of polarity leads to geometrically different dual polytopes, but all have the same combinatorial structure.

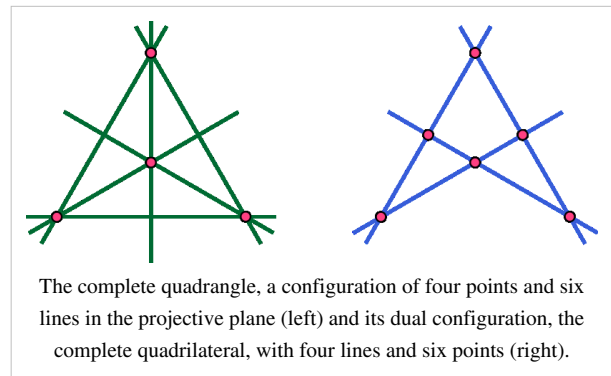
From any three-dimensional polyhedron, one can form a planar graph, the graph of its vertices and edges. The dual polyhedron has a dual graph, a graph with one vertex for each face of the polyhedron and with one edge for every two adjacent faces. The same concept of planar graph duality may be generalized to graphs that are drawn in the plane but that do not come from a three-dimensional polyhedron, or more generally to graph embeddings on surfaces of higher genus: one may draw a dual graph by placing one vertex within each region bounded by a cycle of edges in the embedding, and drawing an edge connecting any two regions that share a boundary edge.



An important example of this type comes from computational geometry: the duality for any finite set S of points in the plane between the Delaunay triangulation of S and the Voronoi diagram of S . As with dual polyhedra and dual polytopes, the duality of graphs on surfaces is a dimension-reversing involution: each vertex in the primal embedded graph corresponds to a region of the dual embedding, each edge in the primal is crossed by an edge in the dual, and each region of the primal corresponds to a vertex of the dual. The dual graph depends on how the primal graph is embedded: different planar embeddings of a single graph may lead to different dual graphs. Matroid duality is an algebraic extension of planar graph duality, in the sense that the dual matroid of the graphic matroid of a planar graph is isomorphic to the graphic matroid of the dual graph.

In topology, Poincaré duality also reverses dimensions; it corresponds to the fact that, if a topological manifold is represented as a cell complex, then the dual of the complex (a higher dimensional generalization of the planar graph dual) represents the same manifold. In Poincaré duality, this homeomorphism is reflected in an isomorphism of the k th homology group and the $(n - k)$ th cohomology group.

Another example of a dimension-reversing duality arises in projective geometry.^[6] In the projective plane, it is possible to find geometric transformations that map each point of the projective plane to a line, and each line of the projective plane to a point, in an incidence-preserving way: in terms of the incidence matrix of the points and lines in the plane, this operation is just that of forming the transpose. Transformations of this type exist also in any higher dimension; one way to construct them is to use the same polar transformations that generate polyhedron and polytope duality. Due to this ability to replace any configuration of points and lines with a corresponding configuration of lines and points, there arises a general principle of duality in projective geometry: given any theorem in plane projective geometry, exchanging the terms "point" and "line" everywhere results in a new, equally valid theorem.^[7]



The points, lines, and higher dimensional subspaces n -dimensional projective space may be interpreted as describing the linear subspaces of an $(n + 1)$ -dimensional vector space; if this vector space is supplied with an inner product the transformation from any linear subspace to its perpendicular subspace is an example of a projective duality. The Hodge dual extends this duality within an inner product space by providing a canonical correspondence between the elements of the exterior algebra.

A kind of geometric duality also occurs in optimization theory, but not one that reverses dimensions. A linear program may be specified by a system of real variables (the coordinates for a point in Euclidean space $\mathbf{R}^n$), a system of linear constraints (specifying that the point lie in a halfspace; the intersection of these halfspaces is a convex polytope, the feasible region of the program), and a linear function (what to optimize). Every linear program has a dual problem with the same optimal solution, but the variables in the dual problem correspond to constraints in the primal problem and vice versa.

Duality in logic and set theory

In logic, functions or relations A and B are considered dual if $A(\neg x) = \neg B(x)$, where $\neg$ is logical negation. The basic duality of this type is the duality of the $\exists$ and $\forall$ quantifiers. These are dual because $\exists x. \neg P(x)$ and $\neg \forall x. P(x)$ are equivalent for all predicates P : if there exists an x for which P fails to hold, then it is false that P holds for all x . From this fundamental logical duality follow several others:

- A formula is said to be *satisfiable* in a certain model if there are assignments to its free variables that render it true; it is *valid* if *every* assignment to its free variables makes it true. Satisfiability and validity are dual because the invalid formulas are precisely those whose negations are satisfiable, and the unsatisfiable formulas are those whose negations are valid. This can be viewed as a special case of the previous item, with the quantifiers ranging over interpretations.
- In classical logic, the $\wedge$ and $\vee$ operators are dual in this sense, because $(\neg x \wedge \neg y)$ and $\neg(x \vee y)$ are equivalent. This means that for every theorem of classical logic there is an equivalent dual theorem. De Morgan's laws are examples. More generally, $\bigwedge (\neg x_i) = \neg \bigvee x_i$. The left side is true if and only if $\forall i. \neg x_i$, and the right side if and only if $\neg \exists i. x_i$.

- In modal logic, $\Box p$ means that the proposition p is "necessarily" true, and $\Diamond p$ that p is "possibly" true. Most interpretations of modal logic assign dual meanings to these two operators. For example in Kripke semantics, " p is possibly true" means "there exists some world W in which p is true", while " p is necessarily true" means "for all worlds W , p is true". The duality of $\Box$ and $\Diamond$ then follows from the analogous duality of $\forall$ and $\exists$. Other dual modal operators behave similarly. For example, temporal logic has operators denoting "will be true at some time in the future" and "will be true at all times in the future" which are similarly dual.

Other analogous dualities follow from these:

- Set-theoretic union and intersection are dual under the set complement operator C . That is, $(A^C \cap B^C) = (A \cup B)^C$, and more generally, $\bigcap A_\alpha^C = \left(\bigcup A_\alpha\right)^C$. This follows from the duality of $\forall$ and $\exists$: an element x is a member of $\bigcap A_\alpha^C$ if and only if $\forall \alpha. \neg x \in A_\alpha$, and is a member of $\left(\bigcup A_\alpha\right)^C$ if and only if $\neg \exists \alpha. x \in A_\alpha$.

Topology inherits a duality between open and closed subsets of some fixed topological space X : a subset U of X is closed if and only if its complement in X is open. Because of this, many theorems about closed sets are dual to theorems about open sets. For example, any union of open sets is open, so dually, any intersection of closed sets is closed. The interior of a set is the largest open set contained in it, and the closure of the set is the smallest closed set that contains it. Because of the duality, the complement of the interior of any set U is equal to the closure of the complement of U .

The collection of all open subsets of a topological space X forms a complete Heyting algebra. There is a duality, known as Stone duality, connecting sober spaces and spatial locales.

- Birkhoff's representation theorem relating distributive lattices and partial orders

Dual objects

A group of dualities can be described by endowing, for any mathematical object X , the set of morphisms $\text{Hom}(X, D)$ into some fixed object D , with a structure similar to the one of X . This is sometimes called internal Hom. In general, this yields a true duality only for specific choices of D , in which case $X^* = \text{Hom}(X, D)$ is referred to as the *dual* of X . It may or may not be true that the *bidual*, that is to say, the dual of the dual, $X^{**} = (X^*)^*$ is isomorphic to X , as the following example, which is underlying many other dualities, shows: the dual vector space V^* of a K -vector space V is defined as

$$V^* = \text{Hom}(V, K).$$

The set of morphisms, i.e., linear maps, is a vector space in its own right. There is always a natural, injective map $V \rightarrow V^{**}$ given by $v \mapsto (f \mapsto f(v))$, where f is an element of the dual space. That map is an isomorphism if and only if the dimension of V is finite.

In the realm of topological vector spaces, a similar construction exists, replacing the dual by the topological dual vector space. A topological vector space that is canonically isomorphic to its bidual is called reflexive space.

The dual lattice of a lattice L is given by

$$\text{Hom}(L, \mathbf{Z}),$$

which is used in the construction of toric varieties.^[8] The Pontryagin dual of locally compact topological groups G is given by

$$\text{Hom}(G, S^1),$$

continuous group homomorphisms with values in the circle (with multiplication of complex numbers as group operation).

Dual categories

Opposite category and adjoint functors

In another group of dualities, the objects of one theory are translated into objects of another theory and the maps between objects in the first theory are translated into morphisms in the second theory, but with direction reversed. Using the parlance of category theory, this amounts to a contravariant functor between two categories C and D :

$$F: C \rightarrow D$$

which for any two objects X and Y of C gives a map

$$\mathrm{Hom}_C(X, Y) \rightarrow \mathrm{Hom}_D(F(Y), F(X))$$

That functor may or may not be an equivalence of categories. There are various situations, where such a functor is an equivalence between the opposite category C^{op} of C , and D . Using a duality of this type, every statement in the first theory can be translated into a "dual" statement in the second theory, where the direction of all arrows has to be reversed.^[9] Therefore, any duality between categories C and D is formally the same as an equivalence between C and D^{op} (C^{op} and D). However, in many circumstances the opposite categories have no inherent meaning, which makes duality an additional, separate concept.^[10]

Many category-theoretic notions come in pairs in the sense that they correspond to each other while considering the opposite category. For example, Cartesian products $Y_1 \times Y_2$ and disjoint unions $Y_1 \sqcup Y_2$ of sets are dual to each other in the sense that

$$\mathrm{Hom}(X, Y_1 \times Y_2) = \mathrm{Hom}(X, Y_1) \times \mathrm{Hom}(X, Y_2)$$

and

$$\mathrm{Hom}(Y_1 \sqcup Y_2, X) = \mathrm{Hom}(Y_1, X) \times \mathrm{Hom}(Y_2, X)$$

for any set X . This is a particular case of a more general duality phenomenon, under which limits in a category C correspond to colimits in the opposite category C^{op} ; further concrete examples of this are epimorphisms vs. monomorphism, in particular factor modules (or groups etc.) vs. submodules, direct products vs. direct sums (also called coproducts to emphasize the duality aspect). Therefore, in some cases, proofs of certain statements can be halved, using such a duality phenomenon. Further notions displaying related by such a categorical duality are projective and injective modules in homological algebra,^[11] fibrations and cofibrations in topology and more generally model categories.^[12]

Two functors $F: C \rightarrow D$ and $G: D \rightarrow C$ are adjoint if for all objects c in C and d in D

$$\mathrm{Hom}_D(F(c), d) \cong \mathrm{Hom}_C(c, G(d)),$$

in a natural way. Actually, the correspondence of limits and colimits is an example of adjoints, since there is an adjunction

$$\mathrm{colim} : C^I \leftrightarrow C : \Delta$$

between the colimit functor that assigns to any diagram in C indexed by some category I its colimit and the diagonal functor that maps any object c of C to the constant diagram which has c at all places. Dually,

$$\Delta : C^I \leftrightarrow C : \lim.$$

Examples

For example, there is a duality between commutative rings and affine schemes: to every commutative ring A there is an affine spectrum, $\text{Spec } A$, conversely, given an affine scheme S , one gets back a ring by taking global sections of the structure sheaf $\mathcal{O}_S$. In addition, ring homomorphisms are in one-to-one correspondence with morphisms of affine schemes, thereby there is an equivalence

$$(\text{Commutative rings})^{\text{op}} \cong (\text{affine schemes})^{[13]}$$

Compare with noncommutative geometry and Gelfand duality.

In a number of situations, the objects of two categories linked by a duality are partially ordered, i.e., there is some notion of an object "being smaller" than another one. In such a situation, a duality that respects the orderings in question is known as a Galois connection. An example is the standard duality in Galois theory (fundamental theorem of Galois theory) between field extensions and subgroups of the Galois group: a bigger field extension corresponds—under the mapping that assigns to any extension $L \supset K$ (inside some fixed bigger field Ω) the Galois group $\text{Gal}(\Omega / L)$ —to a smaller group.^[14]

Pontryagin duality gives a duality on the category of locally compact abelian groups: given any such group G , the character group

$$\chi(G) = \text{Hom}(G, S^1)$$

given by continuous group homomorphisms from G to the circle group S^1 can be endowed with the compact-open topology. Pontryagin duality states that the character group is again locally compact abelian and that

$$G \cong \chi(\chi(G)).^{[15]}$$

Moreover, discrete groups correspond to compact abelian groups; finite groups correspond to finite groups. Pontryagin is the background to Fourier analysis, see below.

- Tannaka-Krein duality, a non-commutative analogue of Pontryagin duality^[16]
- Gelfand duality relating commutative C^* -algebras and compact Hausdorff spaces

Both Gelfand and Pontryagin duality can be deduced in a largely formal, category-theoretic way.^[17]

Analytic dualities

In analysis, problems are frequently solved by passing to the dual description of functions and operators.

Fourier transform switches between functions on a vector space and its dual:

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx,$$

and conversely

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi.$$

If f is an L^2 -function on $\mathbf{R}$ or $\mathbf{R}^N$, say, then so is $\hat{f}$ and $f(-x) = \hat{\hat{f}}(x)$. Moreover, the transform interchanges operations of multiplication and convolution on the corresponding function spaces. A conceptual explanation of the Fourier transform is obtained by the aforementioned Pontryagin duality, applied to the locally compact groups $\mathbf{R}$ (or $\mathbf{R}^N$ etc.): any character of $\mathbf{R}$ is given by $\xi \mapsto e^{-2\pi i x \xi}$. The dualizing character of Fourier transform has many other manifestations, for example, in alternative descriptions of quantum mechanical systems in terms of coordinate and momentum representations.

- Laplace transform is similar to Fourier transform and interchanges operators of multiplication by polynomials with constant coefficient linear differential operators.
- Legendre transformation is an important analytic duality which switches between velocities in Lagrangian mechanics and momenta in Hamiltonian mechanics.

Poincaré-style dualities

Theorems showing that certain objects of interest are the dual spaces (in the sense of linear algebra) of other objects of interest are often called *dualities*. Many of these dualities are given by a bilinear pairing of two K -vector spaces

$$A \otimes B \rightarrow K.$$

For perfect pairings, there is, therefore, an isomorphism of A to the dual of B .

For example, Poincaré duality of a smooth compact complex manifold X is given by a pairing of singular cohomology with $\mathbf{C}$ -coefficients (equivalently, sheaf cohomology of the constant sheaf $\mathbf{C}$)

$$H^i(X) \otimes H^{2n-i}(X) \rightarrow \mathbf{C},$$

where n is the (complex) dimension of X .^[18] Poincaré duality can also be expressed as a relation of singular homology and de Rham cohomology, by asserting that the map

$$(\gamma, \omega) \mapsto \int_{\gamma} \omega$$

(integrating a differential k -form over an $2n-k$ -(real)-dimensional cycle) is a perfect pairing.

The same duality pattern holds for a smooth projective variety over a separably closed field, using ℓ -adic cohomology with $\mathbf{Q}_{\ell}$ -coefficients instead.^[19] This is further generalized to possibly singular varieties, using intersection cohomology instead, a duality called Verdier duality.^[20] With increasing level of generality, it turns out, an increasing amount of technical background is helpful or necessary to understand these theorems: the modern formulation of both these dualities can be done using derived categories and certain direct and inverse image functors of sheaves, applied to locally constant sheaves (with respect to the classical analytical topology in the first case, and with respect to the étale topology in the second case).

Yet another group of similar duality statements is encountered in arithmetics: étale cohomology of finite, local and global fields (also known as Galois cohomology, since étale cohomology over a field is equivalent to group cohomology of the (absolute) Galois group of the field) admit similar pairings. The absolute Galois group $G(\mathbf{F}_q)$ of a finite field, for example, is isomorphic to $\hat{\mathbf{Z}}$, the profinite completion of $\mathbf{Z}$, the integers. Therefore, the perfect pairing (for any G -module M)

$$H^n(G, M) \times H^{1-n}(G, \operatorname{Hom}(M, \mathbf{Q}/\mathbf{Z})) \rightarrow \mathbf{Q}/\mathbf{Z}^{[21]}$$

is a direct consequence of Pontryagin duality of finite groups. For local and global fields, similar statements exist (local duality and global or Poitou–Tate duality).^[22]

Serre duality or coherent duality are similar to the statements above, but applies to cohomology of coherent sheaves instead.^[23]

- Alexander duality

Notes

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- [2] Gowers 2008, p. 187, col. 1
- [3] Gowers 2008, p. 189, col. 2
- [4] Artstein-Avidan & Milman 2007
- [5] Artstein-Avidan & Milman 2008
- [6] Veblen & Young 1965.
- [7] (Veblen & Young 1965, Ch. I, Theorem 11)
- [8] Fulton 1993
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- [10] (Lam 1999, §19C)
- [11] Weibel (1994)
- [12] Dwyer and Spaliński (1995)
- [13] Hartshorne 1966, Ch. II.2, esp. Prop. II.2.3
- [14] See (Lang 2002, Theorem VI.1.1) for finite Galois extensions.

- [15] (Loomis 1953, p. 151, section 37D)
- [16] Joyal and Street (1991)
- [17] Negreponis 1971.
- [18] Griffiths & Harris 1994, p. 56
- [19] Milne 1980, Ch. VI.11
- [20] Iversen 1986, Ch. VII.3, VII.5
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Anabelian geometry

Anabelian geometry is a proposed theory in mathematics, describing the way the algebraic fundamental group G of an algebraic variety V , or some related geometric object, determines how V can be mapped into another geometric object W , under the assumption that G is very far from being an abelian group, in a sense to be made more precise. The word *anabelian* (an alpha privative *an-* before *abelian*) was introduced in *Esquisse d'un Programme*, an influential manuscript of Alexander Grothendieck, circulated in the 1980s.^[1]

While the work of Grothendieck was for many years unpublished, and unavailable through the traditional formal scholarly channels, the formulation and predictions of the proposed theory received much attention, and some alterations, at the hands of a number of mathematicians. Those who have researched in this area have obtained some expected and related results, and in the 21st century the beginnings of such a theory started to be available.

Formulation of a conjecture of Grothendieck on curves

The "anabelian question" has been formulated as

“how much information about the isomorphism class of the variety X is contained in the knowledge of the étale fundamental group?”^[2]

A concrete example is the case of curves, which may be affine as well as projective. Suppose given a hyperbolic curve C , i.e. the complement of n points in a projective algebraic curve of genus g , taken to be smooth and irreducible, defined over a field K that is finitely generated (over its prime field), such that

$$2 - 2g - n < 0.$$

Grothendieck conjectured that the algebraic fundamental group G of C , a profinite group, determines C itself (i.e. the isomorphism class of G determines that of C). This was proved by Shinichi Mochizuki.^[3] An example is for the case of $g = 0$ (the projective line) and $n = 4$, when the isomorphism class of C is determined by the cross-ratio in K of the four points removed (almost, there being an order to the four points in a cross-ratio, but not in the points removed).^[4] There are also results for the case of K a local field.^[5]

Notes

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External links

- *Heidelberg Lectures on Fundamental Groups* (<http://www.renyi.hu/~szamuely/heid.pdf>), section 5.

Noncommutative geometry

Noncommutative geometry (NCG), is a branch of mathematics concerned with geometric approach to noncommutative algebras, and with construction of *spaces* which are locally presented by noncommutative algebras of functions (possibly in some generalized sense). A noncommutative algebra is here an associative algebra in which the multiplication is not commutative, that is, for which xy does not always equal yx ; or more generally an algebraic structure in which one of the principal binary operations is not commutative; one also allows additional structures, e.g. topology or norm to be possibly carried by the noncommutative algebra of functions. The leading direction in noncommutative geometry has been laid by French mathematician Alain Connes since his involvement from about 1979.

Motivation

Main motivation is to extend the commutative duality between spaces and functions to the noncommutative setting. In mathematics, there is a close relationship between *spaces*, which are geometric in nature, and the numerical functions on them. In general, such functions will form a commutative ring. For instance, one may take the ring $C(X)$ of continuous complex-valued functions on a topological space X . In many important cases (e.g., if X is a compact Hausdorff space), we can recover X from $C(X)$, and therefore it makes some sense to say that X has *commutative geometry*.

More specifically, in topology, compact Hausdorff topological spaces can be reconstructed from the Banach algebra of functions on the space (Gel'fand-Neimark). In commutative algebraic geometry, algebraic schemes are locally prime spectra of commutative unital rings (A. Grothendieck), and schemes can be reconstructed from the categories of quasicoherent sheaves of modules on them (P. Gabriel-A. Rosenberg). For Grothendieck topologies, the cohomological properties of a site are invariant of the corresponding category of sheaves of sets viewed abstractly as a topos (A. Grothendieck). In all these cases, a space is reconstructed from the algebra of functions or its categorified version—some category of sheaves on that space.

Functions on a topological space can be multiplied and added pointwise hence they form a commutative algebra; in fact these operations are local in the topology of the base space, hence the functions form a sheaf of commutative rings over the base space.

The dream of noncommutative geometry is to generalize this duality to the duality between

- noncommutative algebras, or sheaves of noncommutative algebras, or sheaf-like noncommutative algebraic or operator-algebraic structures
- and geometric entities of certain kind,

and interact between the algebraic and geometric description of those via this duality.

Regarding that the commutative rings correspond to usual affine schemes, and commutative C^* -algebras to usual topological spaces, the extension to noncommutative rings and algebras requires non-trivial generalization of topological spaces, as "non-commutative spaces". For this reason, some talk about non-commutative topology, though the term has also other meanings.

Applications in mathematical physics

Some applications in particle physics are described on the entries Noncommutative standard model and Noncommutative quantum field theory. Sudden rise in interest in noncommutative geometry in physics, follows after the speculations of its role in M-theory made in 1997^[1].

Motivation from ergodic theory

Some of the theory developed by Alain Connes to handle noncommutative geometry at a technical level has roots in older attempts, in particular in ergodic theory. The proposal of George Mackey to create a *virtual subgroup* theory, with respect to which ergodic group actions would become homogeneous spaces of an extended kind, has by now been subsumed.

Non-commutative C^* -algebras, von Neumann algebras

(The formal duals of) non-commutative C^* -algebras are often now called non-commutative spaces. This is by analogy with the Gelfand representation, which shows that commutative C^* -algebras are dual to locally compact Hausdorff spaces. In general, one can associate to any C^* -algebra S a topological space $\hat{S}$; see spectrum of a C^* -algebra.

For the duality between σ -finite measure spaces and commutative von Neumann algebras, noncommutative von Neumann algebras are called *non-commutative measure spaces*.

Non-commutative differentiable manifolds

A smooth Riemannian manifold M is a topological space with a lot of extra structure. From its algebra of continuous functions $C(M)$ we only recover M topologically. The algebraic invariant that recovers the Riemannian structure is a spectral triple. It is constructed from a smooth vector bundle E over M , e.g. the exterior algebra bundle. The Hilbert space $L^2(M, E)$ of square integrable sections of E carries a representation of $C(M)$ by multiplication operators, and we consider an unbounded operator D in $L^2(M, E)$ with compact resolvent (e.g. the signature operator), such that the commutators $[D, f]$ are bounded whenever f is smooth. A recent deep theorem states that M as a Riemannian manifold can be recovered from this data.

This suggests that one might define a noncommutative Riemannian manifold as a spectral triple (A, H, D) , consisting of a representation of a C^* -algebra A on a Hilbert space H , together with an unbounded operator D on H , with compact resolvent, such that $[D, a]$ is bounded for all a in some dense subalgebra of A . Research in spectral triples is very active, and many examples of noncommutative manifolds have been constructed.

Non-commutative affine and projective schemes

In analogy to the duality between affine schemes and commutative rings, we define a category of **noncommutative affine schemes** as the dual of the category of associative unital rings. There are certain analogues of Zariski topology in that context so that one can glue such affine schemes to more general objects.

There are also generalizations of the Cone and of the Proj of a commutative graded ring, mimicking a Serre's theorem on Proj. Namely the category of quasicoherent sheaves of $\mathcal{O}$ -modules on a Proj of a commutative graded algebra is equivalent to the category of graded modules over the ring localized on Serre's subcategory of graded modules of finite length; there is also analogous theorem for coherent sheaves when the algebra is Noetherian. This theorem is extended as a definition of **noncommutative projective geometry** by Michael Artin and J. J. Zhang^[2], who add also some general ring-theoretic conditions (e.g. Artin-Schelter regularity).

Many properties of projective schemes extend to this context. For example, there exist an analog of the celebrated Serre duality for noncommutative projective schemes of Artin and Zhang^[3].

A. L. Rosenberg has created a rather general relative concept of **noncommutative quasicompact scheme** (over a base category), abstracting the Grothendieck's study of morphisms of schemes and covers in terms of categories of quasicoherent sheaves and flat localization functors^[4]. There is also another interesting approach via localization theory, due to Fred Van Oystaeyen, Luc Willaert and Alain Verschooren, where the main concept is that of a **schematic algebra**^[5].

Invariants for noncommutative spaces

Some of the motivating questions of the theory are concerned with extending known topological invariants to formal duals of noncommutative (operator) algebras and other replacements and candidates for noncommutative spaces. One of the main starting points of the Alain Connes' direction in noncommutative geometry is his spectacular discovery (and independently by Boris Tsygan) of a very important new homology theory associated to noncommutative associative algebras and noncommutative operator algebras, namely the cyclic homology and its relations to the algebraic K-theory (primarily via Connes-Chern character map).

The theory of characteristic classes of smooth manifolds has been extended to spectral triples, employing the tools of operator K-theory and cyclic cohomology. Several generalizations of now classical index theorems allow for effective extraction of numerical invariants from spectral triples. The fundamental characteristic class in cyclic cohomology, the JLO cocycle, generalizes the classical Chern character.

Examples of non-commutative spaces

- In Weyl quantization, the symplectic phase space of classical mechanics is deformed into a non-commutative phase space generated by the position and momentum operators.
- The standard model of particle physics is another example of a noncommutative geometry, cf noncommutative standard model.
- The noncommutative torus, deformation of the function algebra of the ordinary torus, can be given the structure of a spectral triple. This class of examples has been studied intensively and still functions as a test case for more complicated situations.
- Snyder space^[6]
- Noncommutative algebras arising from foliations.
- Examples related to dynamical systems arising from number theory, such as the Gauss shift on continued fractions, give rise to noncommutative algebras that appear to have interesting noncommutative geometries.

Notes

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External links

- Introduction to Quantum Geometry (<http://www.matem.unam.mx/~micho/papers/qgeom.pdf>) by Micho Dürdevich
- Lectures on Noncommutative Geometry (<http://arxiv.org/abs/math/0506603>) by Victor Ginzburg
- Very Basic Noncommutative Geometry (<http://arxiv.org/abs/math/0408416>) by Masoud Khalkhali
- Lectures on Arithmetic Noncommutative Geometry (<http://arxiv.org/abs/math.qa/0409520>) by Matilde Marcolli
- Noncommutative Geometry for Pedestrians (<http://arxiv.org/abs/gr-qc/9906059>) by J. Madore
- An informal introduction to the ideas and concepts of noncommutative geometry (<http://arxiv.org/abs/math-ph/0612012>) by Thierry Masson (an easier introduction that is still rather technical)
- Noncommutative geometry on arxiv.org (<http://xstructure.inr.ac.ru/x-bin/subthemes3.py?level=2&index1=173391&skip=0>)
- MathOverflow, Theories of Noncommutative Geometry (<http://mathoverflow.net/questions/10512/theories-of-noncommutative-geometry>)
- S. Mahanta, On some approaches towards non-commutative algebraic geometry, math.QA/0501166 (<http://arxiv.org/abs/math/0501166>)

Quantum Logics and Quantum Computers

Multi-valued logic

Multi-valued logics are 'logical calculi' in which there are more than two truth values. Traditionally, in Aristotle's logical calculus, there were only two possible values (i.e., "true" and "false") for any proposition. An obvious extension to classical two-valued logic is an n -valued logic for $n > 2$. Those most popular in the literature are three-valued (e.g., Łukasiewicz's and Kleene's),—which accept the values "true", "false", and "unknown",—the finite-valued with more than 3 values, and the infinite-valued (e.g. fuzzy logic) logics.

Relation to classical logic

Logics are usually systems intended to codify rules for preserving some semantic property of propositions across transformations. In classical logic, this property is "truth." In a valid argument, the truth of the derived proposition is guaranteed if the premises are jointly true, because the application of valid steps preserves the property. However, that property doesn't have to be that of "truth"; instead, it can be some other concept.

Multi-valued logics are intended to preserve the property of designationhood (or being designated). Since there are more than two truth values, rules of inference may be intended to preserve more than just whichever corresponds (in the relevant sense) to truth. For example, in a three-valued logic, sometimes the two greatest truth-values (when they are represented as e.g. positive integers) are designated and the rules of inference preserve these values. Precisely, a valid argument will be such that the value of the premises taken jointly will always be less than or equal to the conclusion.

For example, the preserved property could be *justification*, the foundational concept of intuitionistic logic. Thus, a proposition is not true or false; instead, it is justified or flawed. A key difference between justification and truth, in this case, is that the law of excluded middle doesn't hold: a proposition that is not flawed is not necessarily justified; instead, it's only not proven that it's flawed. The key difference is the determinacy of the preserved property: One may prove that P is justified, that P is flawed, or be unable to prove either. A valid argument preserves justification across transformations, so a proposition derived from justified propositions is still justified. However, there are proofs in classical logic that depend upon the law of excluded middle; since that law is not usable under this scheme, there are propositions that cannot be proven that way.

Relation to fuzzy logic

Multi-valued logic is strictly related with fuzzy set theory and fuzzy logic. The notion of fuzzy subset was introduced by Lotfi Zadeh as a formalization of vagueness; i.e., the phenomenon that a predicate may apply to an object not absolutely, but to a certain degree, and that there may be borderline cases. Indeed, as in multi-valued logic, fuzzy logic admits truth values different from "true" and "false". As an example, usually the set of possible truth values is the whole interval $[0,1]$. Nevertheless, the main difference between fuzzy logic and multi-valued logic is in the aims. In fact, in spite of its philosophical interest (it can be used to deal with the Sorites paradox), fuzzy logic is devoted mainly to the applications. More precisely, there are two approaches to fuzzy logic. The first one is very closely linked with multi-valued logic tradition (Hajek school). So a set of designed values is fixed and this enables us to define an entailment relation. The deduction apparatus is defined by a suitable set of logical axioms and suitable inference rules. Another approach (Goguen, Pavelka and others) is devoted to defining a deduction apparatus in which *approximate reasonings* are admitted. Such an apparatus is defined by a suitable fuzzy subset of logical axioms and by a suitable set of fuzzy inference rules. In the first case the logical consequence operator gives the set

of logical consequence of a given set of axioms. In the latter the logical consequence operator gives the fuzzy subset of logical consequence of a given fuzzy subset of hypotheses.

Another example of an infinitely-valued logic is probability logic.

History

The first known classical logician who didn't fully accept the law of excluded middle was Aristotle (who, ironically, is also generally considered to be the first classical logician and the "father of logic"^[1]), who admitted that his laws did not all apply to future events (*De Interpretatione*, ch. IX). But he didn't create a system of multi-valued logic to explain this isolated remark. The later logicians until the coming of the 20th century followed Aristotelian logic, which includes or assumes the law of the excluded middle.

The 20th century brought the idea of multi-valued logic back. The Polish logician and philosopher Jan Łukasiewicz began to create systems of many-valued logic in 1920, using a third value "possible" to deal with Aristotle's paradox of the sea battle. Meanwhile, the American mathematician Emil L. Post (1921) also introduced the formulation of additional truth degrees with $n \geq 2$, where n are the truth values. Later Jan Łukasiewicz and Alfred Tarski together formulated a logic on n truth values where $n \geq 2$ and in 1932 Hans Reichenbach formulated a logic of many truth values where $n \rightarrow \text{infinity}$. Kurt Gödel in 1932 showed that intuitionistic logic is not a finitely-many valued logic, and defined a system of Gödel logics intermediate between classical and intuitionistic logic; such logics are known as intermediate logics.

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External links

- Stanford Encyclopedia of Philosophy: "Many-Valued Logic ^[2]" -- by Siegfried Gottwald.

Notes

[1] Hurley, Patrick. *A Concise Introduction to Logic*, 9th edition. (2006).

[2] <http://plato.stanford.edu/entries/logic-manyvalued/>

Quantum information

In quantum mechanics, **quantum information** is physical information that is held in the "state" of a quantum system. The most popular unit of quantum information is the qubit, a two-level quantum system. However, unlike classical digital states (which are discrete), a two-state quantum system can actually be in a superposition of the two states at any given time.

Quantum information differs from classical information in several respects, among which we note the following:

- It cannot be read without the state becoming the measured value,
- An arbitrary state cannot be cloned,
- The state may be in a superposition of basis values.

However, despite this, the amount of information that can be retrieved in a single qubit is equal to one bit. It is in the *processing* of information (quantum computation) that a difference occurs.

The ability to manipulate quantum information enables us to perform tasks that would be unachievable in a classical context, such as unconditionally secure transmission of information. Quantum information processing is the most general field that is concerned with quantum information. There are certain tasks which classical computers cannot perform "efficiently" (that is, in polynomial time) according to any known algorithm. However, a quantum computer can compute the answer to some of these problems in polynomial time; one well-known example of this is Shor's factoring algorithm. Other algorithms can speed up a task less dramatically - for example, Grover's search algorithm which gives a quadratic speed-up over the best possible classical algorithm.

Quantum information, and changes in quantum information, can be quantitatively measured by using an analogue of Shannon entropy, called the von Neumann Entropy. Given a statistical ensemble of quantum mechanical systems with the density matrix ρ , it is given by

$$S(\rho) = -\text{Tr}(\rho \ln \rho).$$

Many of the same entropy measures in classical information theory can also be generalized to the quantum case, such as Holevo entropy ^[1] and the conditional quantum entropy.

Quantum information theory

The theory of quantum information is a result of the effort to generalise classical information theory to the quantum world. Quantum information theory aims to answer the following question:

What happens if information is stored in a state of a quantum system?

One of the strengths of classical information theory is that physical representation of information can be disregarded: There is no need for an 'ink-on-paper' information theory or a 'DVD information' theory. This is because it is always possible to efficiently transform information from one representation to another. However, this is not the case for quantum information: it is not possible, for example, to write down on paper the previously unknown information contained in the polarisation of a photon.

In general, quantum mechanics does not allow us to read out the state of a quantum system with arbitrary precision. The existence of Bell correlations between quantum systems cannot be converted into classical information. It is only possible to transform quantum information between quantum systems of sufficient information capacity. The information content of a message $\mathcal{M}$ can, for this reason, be measured in terms of the minimum number n of two-level systems which are needed to store the message: $\mathcal{M}$ consists of n qubits. In its original theoretical sense, the term qubit is thus a measure for the amount of information. A two-level quantum system can carry at most one qubit, in the same sense a classical binary digit can carry at most one classical bit.

As a consequence of the noisy-channel coding theorem, noise limits the information content of an analog information carrier to be finite. It is very difficult to protect the remaining finite information content of analog information carriers against noise. The example of classical analog information shows that quantum information processing schemes must necessarily be tolerant against noise, otherwise there would not be a chance for them to be useful. It was a big breakthrough for the theory of quantum information, when quantum error correction codes and fault-tolerant quantum computation schemes were discovered.

External links and references

- Lectures at the Institut Henri Poincaré (slides and videos) ^[2]
- Quantum Information Theory at ETH Zurich ^[3]
- Quantum Information ^[4] Perimeter Institute for Theoretical Physics
- Center for Quantum Computation ^[5] - The CQC, part of Cambridge University, is a group of researchers studying quantum information, and is a useful portal for those interested in this field.
- Quantum Information Group ^[6] The quantum information research group at the University of Nottingham.
- Qwiki ^[7] - A quantum physics wiki devoted to providing technical resources for practicing quantum information scientists.
- Quantiki ^[8] - A wiki portal for quantum information with introductory tutorials.
- Charles H. Bennett and Peter W. Shor, "Quantum Information Theory," IEEE Transactions on Information Theory, Vol 44, pp 2724-2742, Oct 1998
- Institute for Quantum Computing ^[9] - The Institute for Quantum Computing, based in Waterloo, ON Canada, is a research institute working in conjunction with the University of Waterloo ^[10] and Perimeter Institute ^[11] on the subject of Quantum Information.
- Quantum information can be negative ^[12]
- Gregg Jaeger's book on Quantum Information ^[13] (published by Springer, New York, 2007, ISBN 0-387-35725-4)
- The International Conference on Quantum Information (ICQI) ^[14]
- Institute of Quantum Information ^[15] Caltech
- Quantum Information Theory ^[16] Imperial College
- Quantum Information Technology ^[17] Toshiba Research
- International Journal of Quantum Information ^[18] World Scientific
- USC Center for Quantum Information Science & Technology ^[19]

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- [10] <http://www.uwaterloo.ca>
- [11] <http://www.perimeterinstitute.ca/>
- [12] <http://www.damtp.cam.ac.uk/user/jono/negative-information.html>
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Quantum logic

In quantum mechanics, **quantum logic** is a set of rules for reasoning about propositions which takes the principles of quantum theory into account. This research area and its name originated in the 1936 paper by Garrett Birkhoff and John von Neumann, who were attempting to reconcile the apparent inconsistency of classical boolean logic with the facts concerning the measurement of complementary variables in quantum mechanics, such as position and momentum.

Quantum logic can be formulated either as a modified version of propositional logic or as a noncommutative and non-associative many-valued (MV) logic^{[1] [2] [3] [4] [5]}.

Quantum logic has some properties which clearly distinguish it from classical logic, most notably, the failure of the distributive law of propositional logic:

$$p \text{ and } (q \text{ or } r) = (p \text{ and } q) \text{ or } (p \text{ and } r),$$

where the symbols p , q and r are propositional variables. To illustrate why the distributive law fails, consider a particle moving on a line and let

$$p = \text{"the particle is moving to the right"}$$

$$q = \text{"the particle is in the interval } [-1, 1] \text{"}$$

$$r = \text{"the particle is not in the interval } [-1, 1] \text{"}$$

then the proposition " q or r " is true, so

$$p \text{ and } (q \text{ or } r) = p$$

On the other hand, the propositions " p and q " and " p and r " are both false, since they assert tighter restrictions on simultaneous values of position and momentum than is allowed by the uncertainty principle. So,

$$(p \text{ and } q) \text{ or } (p \text{ and } r) = \text{false}$$

Thus the distributive law fails.

Quantum logic has been proposed as the correct logic for propositional inference generally, most notably by the philosopher Hilary Putnam, at least at one point in his career. This thesis was an important ingredient in Putnam's paper *Is Logic Empirical?* in which he analysed the epistemological status of the rules of propositional logic. Putnam

attributes the idea that anomalies associated to quantum measurements originate with anomalies in the logic of physics itself to the physicist David Finkelstein. However, this idea had been around for some time and had been revived several years earlier by George Mackey's work on group representations and symmetry.

The more common view regarding quantum logic, however, is that it provides a formalism for relating observables, system preparation filters and states. In this view, the quantum logic approach resembles more closely the C^* -algebraic approach to quantum mechanics; in fact with some minor technical assumptions it can be subsumed by it. The similarities of the quantum logic formalism to a system of deductive logic may then be regarded more as a curiosity than as a fact of fundamental philosophical importance. A more modern approach to the structure of quantum logic is to assume that it is a diagram – in the sense of category theory – of classical logics (see David Edwards).

Introduction

In his classic treatise *Mathematical Foundations of Quantum Mechanics*, John von Neumann noted that projections on a Hilbert space can be viewed as propositions about physical observables. The set of principles for manipulating these quantum propositions was called *quantum logic* by von Neumann and Birkhoff. In his book (also called *Mathematical Foundations of Quantum Mechanics*) G. Mackey attempted to provide a set of axioms for this propositional system as an orthocomplemented lattice. Mackey viewed elements of this set as potential *yes or no questions* an observer might ask about the state of a physical system, questions that would be settled by some measurement. Moreover Mackey defined a physical observable in terms of these basic questions. Mackey's axiom system is somewhat unsatisfactory though, since it assumes that the partially ordered set is actually given as the orthocomplemented closed subspace lattice of a separable Hilbert space. Piron, Ludwig and others have attempted to give axiomatizations which do not require such explicit relations to the lattice of subspaces.

The remainder of this article assumes the reader is familiar with the spectral theory of self-adjoint operators on a Hilbert space. However, the main ideas can be understood using the finite-dimensional spectral theorem.

Projections as propositions

The so-called *Hamiltonian* formulations of classical mechanics have three ingredients: *states*, *observables* and *dynamics*. In the simplest case of a single particle moving in $\mathbf{R}^3$, the state space is the position-momentum space $\mathbf{R}^6$. We will merely note here that an observable is some real-valued function f on the state space. Examples of observables are position, momentum or energy of a particle. For classical systems, the value $f(x)$, that is the value of f for some particular system state x , is obtained by a process of measurement of f . The propositions concerning a classical system are generated from basic statements of the form

- Measurement of f yields a value in the interval $[a, b]$ for some real numbers a, b .

It follows easily from this characterization of propositions in classical systems that the corresponding logic is identical to that of some Boolean algebra of subsets of the state space. By logic in this context we mean the rules that relate set operations and ordering relations, such as de Morgan's laws. These are analogous to the rules relating boolean conjunctives and material implication in classical propositional logic. For technical reasons, we will also assume that the algebra of subsets of the state space is that of all Borel sets. The set of propositions is ordered by the natural ordering of sets and has a complementation operation. In terms of observables, the complement of the proposition $\{f \geq a\}$ is $\{f < a\}$.

We summarize these remarks as follows:

- The proposition system of a classical system is a lattice with a distinguished *orthocomplementation* operation: The lattice operations of *meet* and *join* are respectively set intersection and set union. The orthocomplementation operation is set complement. Moreover this lattice is *sequentially complete*, in the sense that any sequence $\{E_i\}_i$ of elements of the lattice has a least upper bound, specifically the set-theoretic union:

$$\text{LUB}(\{E_i\}) = \bigcup_{i=1}^{\infty} E_i.$$

In the Hilbert space formulation of quantum mechanics as presented by von Neumann, a physical observable is represented by some (possibly unbounded) densely-defined self-adjoint operator A on a Hilbert space H . A has a spectral decomposition, which is a projection-valued measure E defined on the Borel subsets of $\mathbf{R}$. In particular, for any bounded Borel function f , the following equation holds:

$$f(A) = \int_{\mathbf{R}} f(\lambda) dE(\lambda).$$

In case f is the indicator function of an interval $[a, b]$, the operator $f(A)$ is a self-adjoint projection, and can be interpreted as the quantum analogue of the classical proposition

- Measurement of A yields a value in the interval $[a, b]$.

The propositional lattice of a quantum mechanical system

This suggests the following quantum mechanical replacement for the orthocomplemented lattice of propositions in classical mechanics. This is essentially Mackey's *Axiom VII*:

- The orthocomplemented lattice Q of propositions of a quantum mechanical system is the lattice of closed subspaces of a complex Hilbert space H where orthocomplementation of V is the orthogonal complement $V^\perp$.

Q is also sequentially complete: any pairwise disjoint sequence $\{V_i\}_i$ of elements of Q has a least upper bound. Here disjointness of W_1 and W_2 means W_2 is a subspace of $W_1^\perp$. The least upper bound of $\{V_i\}_i$ is the closed internal direct sum.

Henceforth we identify elements of Q with self-adjoint projections on the Hilbert space H .

The structure of Q immediately points to a difference with the partial order structure of a classical proposition system. In the classical case, given a proposition p , the equations

$$I = p \vee q$$

$$0 = p \wedge q$$

have exactly one solution, namely the set-theoretic complement of p . In these equations I refers to the atomic proposition which is identically true and 0 the atomic proposition which is identically false. In the case of the lattice of projections there are infinitely many solutions to the above equations.

Having made these preliminary remarks, we turn everything around and attempt to define observables within the projection lattice framework and using this definition establish the correspondence between self-adjoint operators and observables: A *Mackey observable* is a countably additive homomorphism from the orthocomplemented lattice of the Borel subsets of $\mathbf{R}$ to Q . To say the mapping φ is a countably additive homomorphism means that for any sequence $\{S_i\}_i$ of pairwise disjoint Borel subsets of $\mathbf{R}$, $\{\varphi(S_i)\}_i$ are pairwise orthogonal projections and

$$\varphi\left(\bigcup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} \varphi(S_i).$$

Theorem. There is a bijective correspondence between Mackey observables and densely-defined self-adjoint operators on H .

This is the content of the spectral theorem as stated in terms of spectral measures.

Statistical structure

Imagine a forensics lab which has some apparatus to measure the speed of a bullet fired from a gun. Under carefully controlled conditions of temperature, humidity, pressure and so on the same gun is fired repeatedly and speed measurements taken. This produces some distribution of speeds. Though we will not get exactly the same value for each individual measurement, for each cluster of measurements, we would expect the experiment to lead to the same distribution of speeds. In particular, we can expect to assign probability distributions to propositions such as $\{a \leq \text{speed} \leq b\}$. This leads naturally to propose that under controlled conditions of preparation, the measurement of a classical system can be described by a probability measure on the state space. This same statistical structure is also present in quantum mechanics.

A *quantum probability measure* is a function P defined on Q with values in $[0,1]$ such that $P(0)=0$, $P(I)=1$ and if $\{E_i\}_i$ is a sequence of pairwise orthogonal elements of Q then

$$P\left(\sum_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

The following highly non-trivial theorem is due to Andrew Gleason:

Theorem. Suppose H is a separable Hilbert space of complex dimension at least 3. Then for any quantum probability measure on Q there exists a unique trace class operator S such that

$$P(E) = \text{Tr}(SE)$$

for any self-adjoint projection E .

The operator S is necessarily non-negative (that is all eigenvalues are non-negative) and of trace 1. Such an operator is often called a *density operator*.

Physicists commonly regard a density operator as being represented by a (possibly infinite) density matrix relative to some orthonormal basis.

For more information on statistics of quantum systems, see quantum statistical mechanics.

Automorphisms

An *automorphism* of Q is a bijective mapping $\alpha:Q \rightarrow Q$ which preserves the orthocomplemented structure of Q , that is

$$\alpha\left(\sum_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \alpha(E_i)$$

for any sequence $\{E_i\}_i$ of pairwise orthogonal self-adjoint projections. Note that this property implies monotonicity of α . If P is a quantum probability measure on Q , then $E \rightarrow \alpha(E)$ is also a quantum probability measure on Q . By the Gleason theorem characterizing quantum probability measures quoted above, any automorphism α induces a mapping α^* on the density operators by the following formula:

$$\text{Tr}(\alpha^*(S)E) = \text{Tr}(S\alpha(E)).$$

The mapping α^* is bijective and preserves convex combinations of density operators. This means

$$\alpha^*(r_1 S_1 + r_2 S_2) = r_1 \alpha^*(S_1) + r_2 \alpha^*(S_2)$$

whenever $1 = r_1 + r_2$ and r_1, r_2 are non-negative real numbers. Now we use a theorem of Richard V. Kadison:

Theorem. Suppose β is a bijective map from density operators to density operators which is convexity preserving. Then there is an operator U on the Hilbert space which is either linear or conjugate-linear, preserves the inner product and is such that

$$\beta(S) = USU^*$$

for every density operator S . In the first case we say U is unitary, in the second case U is anti-unitary.

Remark. This note is included for technical accuracy only, and should not concern most readers. The result quoted above is not directly stated in Kadison's paper, but can be reduced to it by noting first that β extends to a positive trace preserving map on the trace class operators, then applying duality and finally applying a result of Kadison's paper.

The operator U is not quite unique; if r is a complex scalar of modulus 1, then rU will be unitary or anti-unitary if U is and will implement the same automorphism. In fact, this is the only ambiguity possible.

It follows that automorphisms of $\mathcal{Q}$ are in bijective correspondence to unitary or anti-unitary operators modulo multiplication by scalars of modulus 1. Moreover, we can regard automorphisms in two equivalent ways: as operating on states (represented as density operators) or as operating on $\mathcal{Q}$.

Non-relativistic dynamics

In non-relativistic physical systems, there is no ambiguity in referring to time evolution since there is a global time parameter. Moreover an isolated quantum system evolves in a deterministic way: if the system is in a state S at time t then at time $s > t$, the system is in a state $F_{s,t}(S)$. Moreover, we assume

- The dependence is reversible: The operators $F_{s,t}$ are bijective.
- The dependence is homogeneous: $F_{s,t} = F_{s-t,0}$.
- The dependence is convexity preserving: That is, each $F_{s,t}(S)$ is convexity preserving.
- The dependence is weakly continuous: The mapping $\mathbf{R} \rightarrow \mathbf{R}$ given by $t \rightarrow \text{Tr}(F_{s,t}(S)E)$ is continuous for every E in $\mathcal{Q}$.

By Kadison's theorem, there is a 1-parameter family of unitary or anti-unitary operators $\{U_t\}_t$ such that

$$F_{s,t}(S) = U_{s-t} S U_{s-t}^*$$

In fact,

Theorem. Under the above assumptions, there is a strongly continuous 1-parameter group of unitary operators $\{U_t\}_t$ such that the above equation holds.

Note that it easily from uniqueness from Kadison's theorem that

$$U_{t+s} = \sigma(t,s) U_t U_s$$

where $\sigma(t,s)$ has modulus 1. Now the square of an anti-unitary is a unitary, so that all the U_t are unitary. The remainder of the argument shows that $\sigma(t,s)$ can be chosen to be 1 (by modifying each U_t by a scalar of modulus 1.)

Pure states

A convex combination of statistical states S_1 and S_2 is a state of the form $S = p_1 S_1 + p_2 S_2$ where p_1, p_2 are non-negative and $p_1 + p_2 = 1$. Considering the statistical state of system as specified by lab conditions used for its preparation, the convex combination S can be regarded as the state formed in the following way: toss a biased coin with outcome probabilities p_1, p_2 and depending on outcome choose system prepared to S_1 or S_2 .

Density operators form a convex set. The convex set of density operators has extreme points; these are the density operators given by a projection onto a one-dimensional space. To see that any extreme point is such a projection, note that by the spectral theorem S can be represented by a diagonal matrix; since S is non-negative all the entries are non-negative and since S has trace 1, the diagonal entries must add up to 1. Now if it happens that the diagonal matrix has more than one non-zero entry it is clear that we can express it as a convex combination of other density operators.

The extreme points of the set of density operators are called pure states. If S is the projection on the 1-dimensional space generated by a vector ψ of norm 1 then

$$\text{Tr}(SE) = \langle E\psi | \psi \rangle$$

for any E in Q . In physics jargon, if

$$S = |\psi\rangle\langle\psi|,$$

where ψ has norm 1, then

$$\text{Tr}(SE) = \langle\psi|E|\psi\rangle.$$

Thus pure states can be identified with *rays* in the Hilbert space H .

The measurement process

Consider a quantum mechanical system with lattice Q which is in some statistical state given by a density operator S . This essentially means an ensemble of systems specified by a repeatable lab preparation process. The result of a cluster of measurements intended to determine the truth value of proposition E , is just as in the classical case, a probability distribution of truth values **T** and **F**. Say the probabilities are p for **T** and $q = 1 - p$ for **F**. By the previous section $p = \text{Tr}(SE)$ and $q = \text{Tr}(S(I - E))$.

Perhaps the most fundamental difference between classical and quantum systems is the following: regardless of what process is used to determine E immediately after the measurement the system will be in one of two statistical states:

- If the result of the measurement is **T**

$$\frac{1}{\text{Tr}(ES)}ESE.$$

- If the result of the measurement is **F**

$$\frac{1}{\text{Tr}((I - E)S)}(I - E)S(I - E).$$

(We leave to the reader the handling of the degenerate cases in which the denominators may be 0.) We now form the convex combination of these two ensembles using the relative frequencies p and q . We thus obtain the result that the measurement process applied to a statistical ensemble in state S yields another ensemble in statistical state:

$$M_E(S) = ESE + (I - E)S(I - E).$$

We see that a pure ensemble becomes a mixed ensemble after measurement. Measurement, as described above, is a special case of quantum operations.

Limitations

Quantum logic derived from propositional logic provides a satisfactory foundation for a theory of reversible quantum processes. Examples of such processes are the covariance transformations relating two frames of reference, such as change of time parameter or the transformations of special relativity. Quantum logic also provides a satisfactory understanding of density matrices. Quantum logic can be stretched to account for some kinds of measurement processes corresponding to answering yes-no questions about the state of a quantum system. However, for more general kinds of measurement operations (that is quantum operations), a more complete theory of filtering processes is necessary. Such an approach is provided by the consistent histories formalism. On the other hand, quantum logics derived from MV-logic extend its range of applicability to irreversible quantum processes and/or 'open' quantum systems.

In any case, these quantum logic formalisms must be generalized in order to deal with super-geometry (which is needed to handle Fermi-fields) and non-commutative geometry (which is needed in string theory and quantum gravity theory). Both of these theories use a partial algebra with an "integral" or "trace". The elements of the partial algebra are not observables; instead the "trace" yields "greens functions" which generate scattering amplitudes. One thus obtains a local S-matrix theory (see D. Edwards).

Since around 1978 the Flato school (see F. Bayen) has been developing an alternative to the quantum logics approach called deformation quantization (see Weyl quantization).

In 2004, Prakash Panangaden described how to capture the kinematics of quantum causal evolution using System BV, a deep inference logic originally developed for use in structural proof theory.^[6] Alessio Guglielmi, Lutz Straßburger, and Richard Blute have also done work in this area.^[7]

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See also

- Mathematical formulation of quantum mechanics
- Multi-valued logic
- Quasi-set theory
- HPO formalism (An approach to temporal quantum logic)
- Quantum field theory

Further reading

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External links

- Stanford Encyclopedia of Philosophy entry on Quantum Logic and Probability Theory (<http://plato.stanford.edu/entries/qt-quantlog/>)

Quantum computer

A **quantum computer** is a device for computation that makes direct use of quantum mechanical phenomena, such as superposition and entanglement, to perform operations on data. Quantum computers are different from traditional computers based on transistors. The basic principle behind quantum computation is that quantum properties can be used to represent data and perform operations on these data.^[1] A theoretical model is the quantum Turing machine, also known as the universal quantum computer.

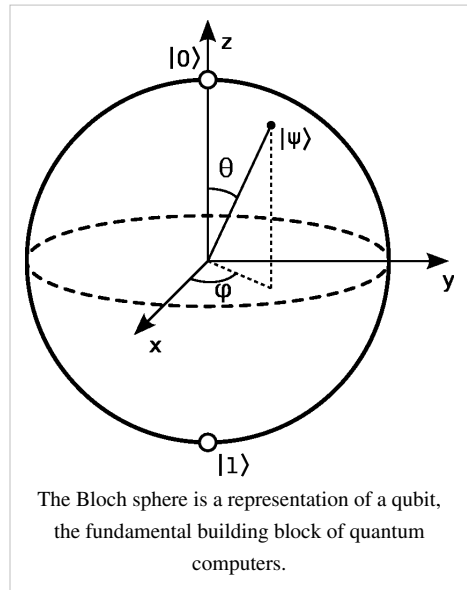
Although quantum computing is still in its infancy, experiments have been carried out in which quantum computational operations were executed on a very small number of qubits (quantum bit). Both practical and theoretical research continues, and many national government and military funding agencies support quantum computing research to develop quantum computers for both civilian and national security purposes, such as cryptanalysis.^[2]

If large-scale quantum computers can be built, they will be able to solve certain problems much faster than any current classical computers (for example Shor's algorithm). Quantum computers do not allow the computation of functions that are not theoretically computable by classical computers, i.e. they do not alter the Church–Turing thesis. The gain is only in efficiency.

Basis

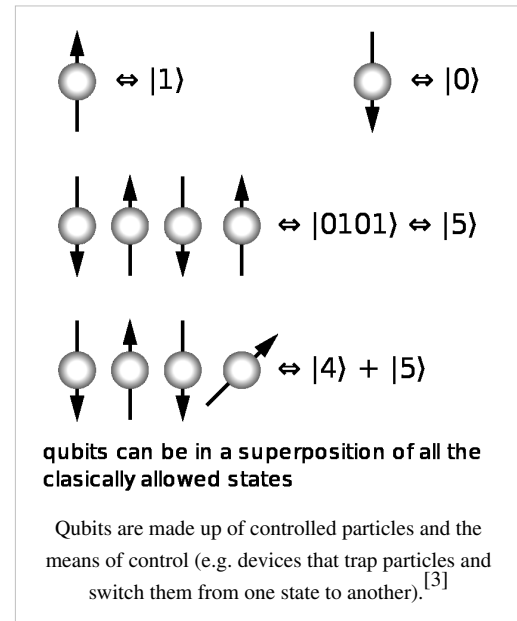
A classical computer has a memory made up of bits, where each bit represents either a one or a zero. A quantum computer maintains a sequence of qubits. A single qubit can represent a one, a zero, or, crucially, any quantum superposition of these; moreover, a pair of qubits can be in any quantum superposition of 4 states, and three qubits in any superposition of 8. In general a quantum computer with n qubits can be in an arbitrary superposition of up to 2^n different states simultaneously (this compares to a normal computer that can only be in *one* of these 2^n states at any one time). A quantum computer operates by manipulating those qubits with a fixed sequence of quantum logic gates. The sequence of gates to be applied is called a quantum algorithm.

An example of an implementation of qubits for a quantum computer could start with the use of particles with two spin states: "down" and "up" (typically written $|\downarrow\rangle$ and $|\uparrow\rangle$, or $|0\rangle$ and $|1\rangle$). But in fact any system possessing an observable quantity A which is *conserved* under time evolution and such that A has at least two discrete and sufficiently spaced consecutive eigenvalues, is a suitable candidate for implementing a qubit. This is true because any such system can be mapped onto an effective spin-1/2 system.



Bits vs. qubits

Consider first a classical computer that operates on a three-bit register. The state of the computer at any time is a probability distribution over the $2^3 = 8$ different three-bit strings 000, 001, 010, 011, 100, 101, 110, 111. If it is a deterministic computer, then it is in exactly one of these states with probability 1. However, if it is a probabilistic computer, then there is a possibility of it being in any *one* of a number of different states. We can describe this probabilistic state by eight nonnegative numbers a, b, c, d, e, f, g, h (where a = probability computer is in state 000, b = probability computer is in state 001, etc.). There is a restriction that these probabilities sum to 1.



The state of a three-qubit quantum computer is similarly described by an eight-dimensional vector (a, b, c, d, e, f, g, h) , called a ket. However, instead of adding to one, the sum of the *squares* of the coefficient magnitudes, $|a|^2 + |b|^2 + \dots + |h|^2$, must equal one. Moreover, the coefficients are complex numbers. Since states are represented by complex wavefunctions, two states being added together will undergo interference. This is a key difference between quantum computing and probabilistic classical computing.^[4]

If you measure the three qubits, then you will observe a three-bit string. The probability of measuring a string will equal the squared magnitude of that string's coefficients (using our example, probability that we read state as 000 = $|a|^2$, probability that we read state as 001 = $|b|^2$, etc.). Thus a measurement of the quantum state with coefficients $(a, b, \dots, h)$ gives the classical probability distribution $(|a|^2, |b|^2, \dots, |h|^2)$. We say that the quantum state "collapses" to a classical state.

Note that an eight-dimensional vector can be specified in many different ways, depending on what basis you choose for the space. The basis of three-bit strings 000, 001, ..., 111 is known as the computational basis, and is often convenient, but other bases of unit-length, orthogonal vectors can also be used. Ket notation is often used to make explicit the choice of basis. For example, the state (a, b, c, d, e, f, g, h) in the computational basis can be written as

$$a |000\rangle + b |001\rangle + c |010\rangle + d |011\rangle + e |100\rangle + f |101\rangle + g |110\rangle + h |111\rangle, \quad \text{where, e.g., } |010\rangle = (0, 0, 1, 0, 0, 0, 0, 0).$$

The computational basis for a single qubit (two dimensions) is $|0\rangle = (1, 0)$, $|1\rangle = (0, 1)$, but another common basis are the eigenvectors of the Pauli-x operator: $|+\rangle = \frac{1}{\sqrt{2}}(1, 1)$ and $|-\rangle = \frac{1}{\sqrt{2}}(1, -1)$.

Note that although recording a classical state of n bits, a 2^n -dimensional probability distribution, requires an exponential number of real numbers, practically we can always think of the system as being exactly one of the n -bit strings—we just don't know which one. Quantum mechanically, this is not the case, and all 2^n complex coefficients need to be kept track of to see how the quantum system evolves. For example, a 300-qubit quantum computer has a state described by 2^{300} (approximately 10^{90}) complex numbers, more than the number of atoms in the observable universe.

Operation

While a classical three-bit state and a quantum three-qubit state are both eight-dimensional vectors, they are manipulated quite differently for classical or quantum computation. For computing in either case, the system must be initialized, for example into the all-zeros string, $|000\rangle$, corresponding to the vector $(1,0,0,0,0,0,0,0)$. In classical randomized computation, the system evolves according to the application of stochastic matrices, which preserve that the probabilities add up to one (i.e., preserve the L1 norm). In quantum computation, on the other hand, allowed operations are unitary matrices, which are effectively rotations (they preserve that the sum of the squares add up to one, the Euclidean or L2 norm). (Exactly what unitaries can be applied depend on the physics of the quantum device.) Consequently, since rotations can be undone by rotating backward, quantum computations are reversible. (Technically, quantum operations can be probabilistic combinations of unitaries, so quantum computation really does generalize classical computation. See quantum circuit for a more precise formulation.)

Finally, upon termination of the algorithm, the result needs to be read off. In the case of a classical computer, we *sample* from the probability distribution on the three-bit register to obtain one definite three-bit string, say 000. Quantum mechanically, we *measure* the three-qubit state, which is equivalent to collapsing the quantum state down to a classical distribution (with the coefficients in the classical state being the squared magnitudes of the coefficients for the quantum state, as described above) followed by sampling from that distribution. Note that this destroys the original quantum state. Many algorithms will only give the correct answer with a certain probability, however by repeatedly initializing, running and measuring the quantum computer, the probability of getting the correct answer can be increased.

For more details on the sequences of operations used for various quantum algorithms, see universal quantum computer, Shor's algorithm, Grover's algorithm, Deutsch-Jozsa algorithm, amplitude amplification, quantum Fourier transform, quantum gate, quantum adiabatic algorithm and quantum error correction.

Potential

Integer factorization is believed to be computationally infeasible with an ordinary computer for large integers if they are the product of few prime numbers (e.g., products of two 300-digit primes).^[5] By comparison, a quantum computer could efficiently solve this problem using Shor's algorithm to find its factors. This ability would allow a quantum computer to decrypt many of the cryptographic systems in use today, in the sense that there would be a polynomial time (in the number of digits of the integer) algorithm for solving the problem. In particular, most of the popular public key ciphers are based on the difficulty of factoring integers (or the related discrete logarithm problem which can also be solved by Shor's algorithm), including forms of RSA. These are used to protect secure Web pages, encrypted email, and many other types of data. Breaking these would have significant ramifications for electronic privacy and security.

However, other existing cryptographic algorithms don't appear to be broken by these algorithms.^[6] ^[7] Some public-key algorithms are based on problems other than the integer factorization and discrete logarithm problems to which Shor's algorithm applies, like the McEliece cryptosystem based on a problem in coding theory.^[6] ^[8] Lattice based cryptosystems are also not known to be broken by quantum computers, and finding a polynomial time algorithm for solving the dihedral hidden subgroup problem, which would break many lattice based cryptosystems, is a well-studied open problem.^[9] It has been proven that applying Grover's algorithm to break a symmetric (secret key) algorithm by brute force requires roughly $2^{n/2}$ invocations of the underlying cryptographic algorithm, compared with roughly 2^n in the classical case,^[10] meaning that symmetric key lengths are effectively halved: AES-256 would have the same security against an attack using Grover's algorithm that AES-128 has against classical brute-force search (see Key size). Quantum cryptography could potentially fulfill some of the functions of public key cryptography.

Besides factorization and discrete logarithms, quantum algorithms offering a more than polynomial speedup over the best known classical algorithm have been found for several problems,^[11] including the simulation of quantum physical processes from chemistry and solid state physics, the approximation of Jones polynomials, and solving Pell's equation. No mathematical proof has been found that shows that an equally fast classical algorithm cannot be discovered, although this is considered unlikely. For some problems, quantum computers offer a polynomial speedup. The most well-known example of this is *quantum database search*, which can be solved by Grover's algorithm using quadratically fewer queries to the database than are required by classical algorithms. In this case the advantage is provable. Several other examples of provable quantum speedups for query problems have subsequently been discovered, such as for finding collisions in two-to-one functions and evaluating NAND trees.

Consider a problem that has these four properties:

1. The only way to solve it is to guess answers repeatedly and check them,
2. There are n possible answers to check,
3. Every possible answer takes the same amount of time to check, and
4. There are no clues about which answers might be better: generating possibilities randomly is just as good as checking them in some special order.

An example of this is a password cracker that attempts to guess the password for an encrypted file (assuming that the password has a maximum possible length).

For problems with all four properties, the time for a quantum computer to solve this will be proportional to the square root of n . That can be a very large speedup, reducing some problems from years to seconds. It can be used to attack symmetric ciphers such as Triple DES and AES by attempting to guess the secret key.

Grover's algorithm can also be used to obtain a quadratic speed-up [over a brute-force search] for a class of problems known as NP-complete.

Since chemistry and nanotechnology rely on understanding quantum systems, and such systems are impossible to simulate in an efficient manner classically, many believe quantum simulation will be one of the most important applications of quantum computing.^[12]

There are a number of practical difficulties in building a quantum computer, and thus far quantum computers have only solved trivial problems. David DiVincenzo, of IBM, listed the following requirements for a practical quantum computer:^[13]

- scalable physically to increase the number of qubits;
- qubits can be initialized to arbitrary values;
- quantum gates faster than decoherence time;
- universal gate set;
- qubits can be read easily.

Quantum decoherence

One of the greatest challenges is controlling or removing quantum decoherence. This usually means isolating the system from its environment as the slightest interaction with the external world would cause the system to decohere. This effect is irreversible, as it is non-unitary, and is usually something that should be highly controlled, if not avoided. Decoherence times for candidate systems, in particular the transverse relaxation time T_2 (for NMR and MRI technology, also called the *dephasing time*), typically range between nanoseconds and seconds at low temperature.^[4]

These issues are more difficult for optical approaches as the timescales are orders of magnitude shorter and an often-cited approach to overcoming them is optical pulse shaping. Error rates are typically proportional to the ratio of operating time to decoherence time, hence any operation must be completed much more quickly than the decoherence time.

If the error rate is small enough, it is thought to be possible to use quantum error correction, which corrects errors due to decoherence, thereby allowing the total calculation time to be longer than the decoherence time. An often cited figure for required error rate in each gate is 10^{-4} . This implies that each gate must be able to perform its task in one 10,000th of the decoherence time of the system.

Meeting this scalability condition is possible for a wide range of systems. However, the use of error correction brings with it the cost of a greatly increased number of required qubits. The number required to factor integers using Shor's algorithm is still polynomial, and thought to be between L and L^2 , where L is the number of bits in the number to be factored; error correction algorithms would inflate this figure by an additional factor of L . For a 1000-bit number, this implies a need for about 10^4 qubits without error correction.^[14] With error correction, the figure would rise to about 10^7 qubits. Note that computation time is about L^2 or about 10^7 steps and on 1 MHz, about 10 seconds.

A very different approach to the stability-decoherence problem is to create a topological quantum computer with anyons, quasi-particles used as threads and relying on braid theory to form stable logic gates.^{[15] [16]}

Developments

There are a number of quantum computing candidates, among those:

- Superconductor-based quantum computers (including SQUID-based quantum computers)^[17]
- Trapped ion quantum computer
- Optical lattices
- Topological quantum computer^[18]
- Quantum dot on surface (e.g. the Loss-DiVincenzo quantum computer)
- Nuclear magnetic resonance on molecules in solution (liquid NMR)
- Solid state NMR Kane quantum computers
- Electrons on helium quantum computers
- Cavity quantum electrodynamics (CQED)
- Molecular magnet
- Fullerene-based ESR quantum computer
- Optic-based quantum computers (Quantum optics)
- Diamond-based quantum computer^{[19] [20] [21]}
- Bose–Einstein condensate-based quantum computer^[22]
- Transistor-based quantum computer - string quantum computers with entrainment of positive holes using an electrostatic trap
- Spin-based quantum computer
- Adiabatic quantum computation^[23]
- Rare-earth-metal-ion-doped inorganic crystal based quantum computers^{[24] [25]}

The large number of candidates demonstrates that the topic, in spite of rapid progress, is still in its infancy. But at the same time there is also a vast amount of flexibility.

In 2005, researchers at the University of Michigan built a semiconductor chip which functioned as an ion trap. Such devices, produced by standard lithography techniques, may point the way to scalable quantum computing tools.^[26] An improved version was made in 2006.

In 2009, researchers at Yale University created the first rudimentary solid-state quantum processor. The two-qubit superconducting chip was able to run elementary algorithms. Each of the two artificial atoms (or qubits) were made up of a billion aluminum atoms but they acted like a single one that could occupy two different energy states.^{[27] [28]}

Another team, working at the University of Bristol, also created a silicon-based quantum computing chip, based on quantum optics. The team was able to run Shor's algorithm on the chip.^[29] The latest developments [for 2010] can be found in ^[30].

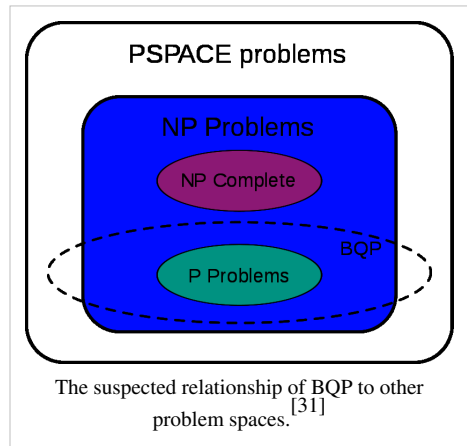
Relation to computational complexity theory

The class of problems that can be efficiently solved by quantum computers is called BQP, for "bounded error, quantum, polynomial time". Quantum computers only run probabilistic algorithms, so BQP on quantum computers is the counterpart of BPP ("bounded error, probabilistic, polynomial time") on classical computers. It is defined as the set of problems solvable with a polynomial-time algorithm, whose probability of error is bounded away from one half.^[32] A quantum computer is said to "solve" a problem if, for every instance, its answer will be right with high probability. If that solution runs in polynomial time, then that problem is in BQP.

BQP is contained in the complexity class $\#P$ (or more precisely in the associated class of decision problems $P^{\#P}$),^[33] which is a subclass of PSPACE.

BQP is suspected to be disjoint from NP-complete and a strict superset of P, but that is not known. Both integer factorization and discrete log are in BQP. Both of these problems are NP problems suspected to be outside BPP, and hence outside P. Both are suspected to not be NP-complete. There is a common misconception that quantum computers can solve NP-complete problems in polynomial time. That is not known to be true, and is generally suspected to be false.^[33]

Although quantum computers may be faster than classical computers, those described above can't solve any problems that classical computers can't solve, given enough time and memory (however, those amounts might be practically infeasible). A Turing machine can simulate these quantum computers, so such a quantum computer could never solve an undecidable problem like the halting problem. The existence of "standard" quantum computers does not disprove the Church–Turing thesis.^[34] It has been speculated that theories of quantum gravity, such as M-theory or loop quantum gravity, may allow even faster computers to be built. Currently, it's an open problem to even *define* computation in such theories due to the problem of time, i.e. there's no obvious way to describe what it means for an observer to submit input to a computer and later receive output.^[35]



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External links

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 - jQuantum: Java quantum circuit simulator (<http://jquantum.sourceforge.net/jQuantumApplet.html>)
 - QCAD: Quantum circuit emulator (<http://www.phys.cs.is.nagoya-u.ac.jp/~watanabe/qcad/index.html>)
 - C++ Quantum Library (<http://gna.org/projects/quantumlibrary>)
 - Haskell Library for Quantum computations (<http://hackage.haskell.org/cgi-bin/hackage-scripts/package/quantum-arrow>)
 - Video Lectures by David Deutsch (http://www.quiprocone.org/Protected/DD_lectures.htm)
 - Lectures at the Institut Henri Poincaré (slides and videos) (<http://www.quantware.ups-tlse.fr/IHP2006/>)
 - Online lecture on An Introduction to Quantum Computing, Edward Gerjuoy (2008) (<http://nanohub.org/resources/4778>)
 - Online Web-based Quantum Computer Simulator (University Of Patras, Wire Communications Laboratory) (<http://www.wcl.ece.upatras.gr/ai/resources/demo-quantum-simulation>)
-

Algebraic Logic

Boolean logic

Boolean logic is a complete system for logical operations, used in many systems. It was named after George Boole, who first defined an algebraic system of logic in the mid 19th century. Boolean logic has many applications in electronics, computer hardware and software, and is the basis of all modern digital electronics. In 1938, Claude Shannon showed how electric circuits with relays could be modeled with Boolean logic. This fact soon proved enormously consequential with the emergence of the electronic computer.

Using the algebra of sets, this article contains a basic introduction to sets, Boolean operations, Venn diagrams, truth tables, and Boolean applications. The Boolean algebra (structure) article discusses a type of algebraic structure that satisfies the axioms of Boolean logic. The binary arithmetic article discusses the use of binary numbers in computer systems.

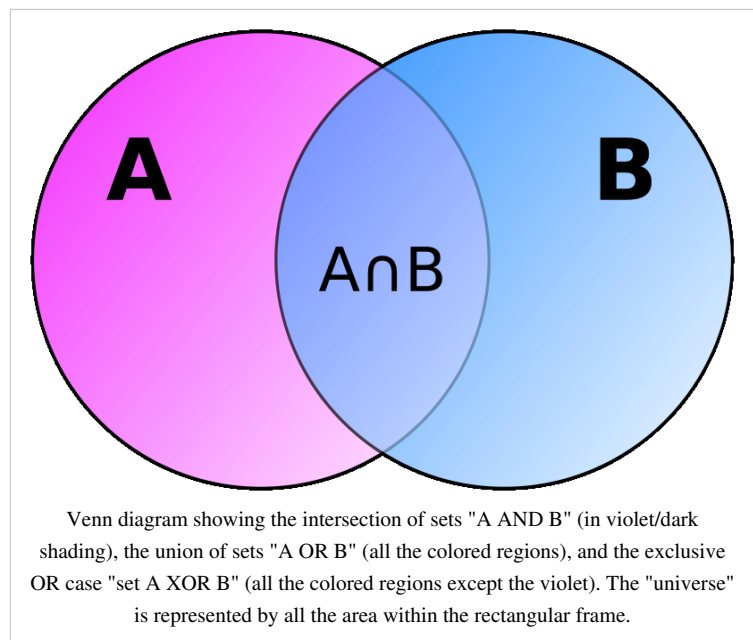
Set logic vs. Boolean logic

Sets can contain any elements. We will first start out by discussing general set logic, then restrict ourselves to Boolean logic, where elements (or "bits") each contain only two possible values, called various names, such as "true" and "false", "yes" and "no", "on" and "off", or "1" and "0".

Terms

Let X be a set:

- An **element** is one member of a set and is denoted by $\in$. If the element is not a member of a set it is denoted by $\notin$.
- The **universe** is the set X , sometimes denoted by 1. Note that this use of the word universe means "all elements being considered", which are not necessarily the same as "all elements there are".
- The **empty set** or **null set** is the set of no elements, denoted by $\emptyset$ and sometimes 0.
- A **unary operator** applies to a single set. There is only one unary operator, called logical **NOT**. It works by taking the complement with respect to the universe, i.e. the set of all elements under consideration.
- A **binary operator** applies to two sets. The basic binary operators are logical **OR** and logical **AND**. They perform the union and intersection of sets. There are also other derived binary operators, such as **XOR** (exclusive OR, i.e., "one or the other, but not both").
- A **subset** is denoted by $A \subseteq B$ and means every element in set A is also in set B .



- A **superset** is denoted by $A \supseteq B$ and means every element in set B is also in set A.
- The **identity** or **equivalence** of two sets is denoted by $A \equiv B$ and means that every element in set A is also in set B *and* every element in set B is also in set A.
- A **proper subset** is denoted by $A \subset B$ and means every element in set A is also in set B and the two sets are not identical.
- A **proper superset** is denoted by $A \supset B$ and means every element in set B is also in set A and the two sets are not identical.

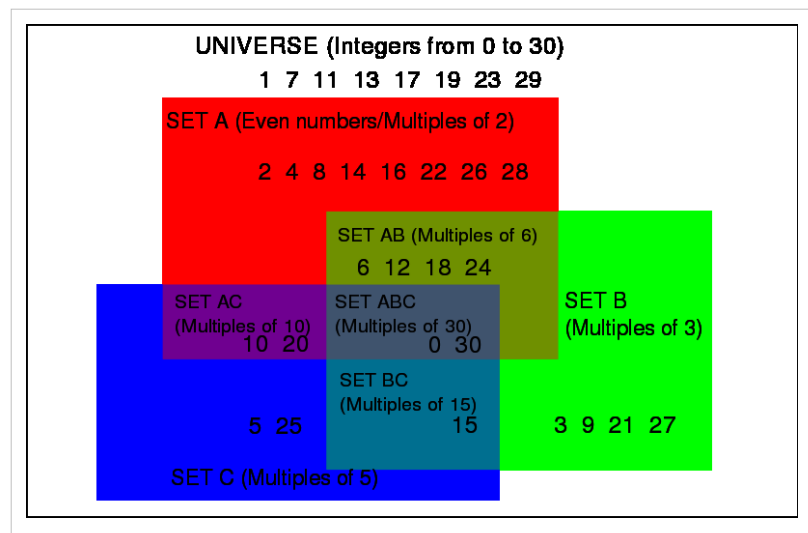
Example

Imagine that set A contains all even numbers (multiples of two) in "the universe" (defined in the example below as all integers between 0 and 30 inclusive) and set B contains all multiples of three in "the universe". Then the **intersection** of the two sets (all elements in sets A AND B) would be all multiples of six in "the universe". The complement of set A (all elements NOT in set A) would be all odd numbers in "the universe".

Chaining operations together

While at most two sets are joined in any Boolean operation, the new set formed by that operation can then be joined with other sets utilizing additional Boolean operations. Using the previous example, we can define a new set C as the set of all multiples of five in "the universe". Thus "sets A AND B AND C" would be all multiples of 30 in "the universe". If more convenient, we may consider set AB to be the intersection of sets A and B, or the set of all multiples of six in

"the universe". Then we can say "sets AB AND C" are the set of all multiples of 30 in "the universe". We could then take it a step further, and call this result set ABC.



Use of parentheses

While any number of logical ANDs (or any number of logical ORs) may be chained together without ambiguity, the combination of ANDs and ORs and NOTs can lead to ambiguous cases. In such cases, parentheses may be used to clarify the order of operations. As always, the operations within the innermost pair is performed first, followed by the next pair out, etc., until all operations within parentheses have been completed. Then any operations outside the parentheses are performed.

Application to binary values

In this example we have used natural numbers, while in Boolean logic binary numbers are used. The universe, for example, could contain just two elements, "0" and "1" (or "true" and "false", "yes" and "no", "on" or "off", etc.). We could also combine binary values together to get binary words, such as, in the case of two digits, "00", "01", "10", and "11". Applying set logic to those values, we could have a set of all values where the first digit is "0" ("00" and "01") and the set of all values where the first and second digits are different ("01" and "10"). The intersection of the two sets would then be the single element, "01". This could be shown by the following Boolean expression, where

"1st" is the first digit and "2nd" is the second digit:

(NOT 1st) AND (1st XOR 2nd)

Properties

We define symbols for the two primary binary operations as $\wedge/\cap$ (logical AND/set intersection) and $\vee/\cup$ (logical OR/set union), and for the single unary operation $\neg/\sim$ (logical NOT/set complement). We will also use the values 0 (logical FALSE/the empty set) and 1 (logical TRUE/the universe). The following properties apply to both Boolean logic and set logic (although only the notation for Boolean logic is displayed here):

$a \vee (b \vee c) = (a \vee b) \vee c$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$	associativity
$a \vee b = b \vee a$	$a \wedge b = b \wedge a$	commutativity
$a \vee (a \wedge b) = a$	$a \wedge (a \vee b) = a$	absorption
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$	$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	distributivity
$a \vee \neg a = 1$	$a \wedge \neg a = 0$	complements
$a \vee a = a$	$a \wedge a = a$	idempotency
$a \vee 0 = a$	$a \wedge 1 = a$	boundedness
$a \vee 1 = 1$	$a \wedge 0 = 0$	
$\neg 0 = 1$	$\neg 1 = 0$	0 and 1 are complements
$\neg(a \vee b) = \neg a \wedge \neg b$	$\neg(a \wedge b) = \neg a \vee \neg b$	de Morgan's laws
$\neg\neg a = a$		involution

The first three properties define a lattice; the first five define a Boolean algebra. The remaining five are a consequence of the first five.

Other notations

Mathematicians and engineers often use plus (+) for OR and a product sign ($\cdot$) for AND. OR and AND are somewhat analogous to addition and multiplication in other algebraic structures, and this notation makes it very easy to get sum of products form for normal algebra. NOT may be represented by a line drawn above the expression being negated ($\overline{x}$). It also commonly leads to giving $\cdot$ a higher precedence than +, removing the need for parenthesis in some cases.

Programmers will often use a pipe symbol (|) for OR, an ampersand (&) for AND, and a tilde (~) for NOT. In many programming languages, these symbols stand for bitwise operations. "||", "&&", and "!" are used for variants of these operations.

Another notation uses "meet" for AND and "join" for OR. However, this can lead to confusion, as the term "join" is also commonly used for any Boolean operation which combines sets together, which includes both AND and OR.

Basic mathematics use of Boolean terms

- In the case of simultaneous equations, they are connected with an implied logical AND:

$$x + y = 2$$

AND

$$x - y = 2$$

- The same applies to simultaneous inequalities:

$$x + y < 2$$

AND

$$x - y < 2$$

- The greater than or equals sign ($\geq$) and less than or equals sign ($\leq$) may be assumed to contain a logical OR:

$$X < 2$$

OR

$$X = 2$$

- The plus/minus sign ($\pm$), as in the case of the solution to a square root problem, may be taken as logical OR:

$$\text{WIDTH} = 3$$

OR

$$\text{WIDTH} = -3$$

English language use of Boolean terms

Care should be taken when converting an English sentence into a formal boolean statement. Many English sentences have imprecise meanings.

In certain cases, **AND** and **OR** can be used interchangeably in English:

- I always carry an umbrella for when it rains **and** snows.*
- I always carry an umbrella for when it rains **or** snows.*
- I never walk in the rain **or** snow.*

Sometimes the English words "and" and "or" have a meaning that is apparently opposite of its meaning in boolean logic:

- "Give me all the red **and** blue berries," usually means, "Give me all berries that are red **or** blue". (The former might have been interpreted as a request for berries that are each both red and blue.) An alternative phrasing for this request would be, "Give me all berries that are red and all berries that are blue."

Depending on the context, the word "or" may correspond with either logical **OR** or logical **XOR**:

- The waitress asked, "Would you like cream **or** sugar with your coffee?" (Logical **OR**.)*
- The waitress asked, "Would you like soup **or** salad with your meal?" (Logical **XOR**.)*

Logical **XOR** can be translated as "one, or the other, but not both". In most cases, this concept is most effectively communicated in English using "either/or".

The word combination "and/or" is sometimes used in English to specify a logical **OR**, when just using the word "or" alone might have been mistaken as meaning logical **XOR**:

- "I'm having chicken **and/or** beef for dinner." (Logical **OR**.) An alternative phrasing for standard written English would be, "For dinner, I'm having chicken or beef (or both)."

- The use of "and/or" is generally disfavored in formal writing.^[1] Its usage may introduce critical imprecision in legal agreements, research findings, and specifications for computer programs or electronic circuits.

This can be a significant challenge when providing precise specifications for a computer program or electronic circuit in English. The description of such functionality may be ambiguous. Take for example the statement, "The program should verify that the applicant has checked the male **or** female box." This should be interpreted as an **XOR** and a verification performed to ensure that one, and only one, box is selected. In other cases the proper interpretation of English may be less obvious; the author of the specification should be consulted to determine the original intent.

Applications

Digital electronic circuit design

Boolean logic is also used for circuit design in electrical engineering; here 0 and 1 may represent the two different states of one bit in a digital circuit, typically high and low voltage. Circuits are described by expressions containing variables, and two such expressions are equal for all values of the variables if, and only if, the corresponding circuits have the same input-output behavior. Furthermore, every possible input-output behavior can be modeled by a suitable Boolean expression.

Basic logic gates such as AND, OR, and NOT gates may be used alone, or in conjunction with NAND, NOR, and XOR gates, to control digital electronics and circuitry. Whether these gates are wired in series or parallel controls the precedence of the operations.

Database applications

Relational databases use SQL, or other database-specific languages, to perform queries, which may contain Boolean logic. For this application, each record in a table may be considered to be an "element" of a "set". For example, in SQL, these SELECT statements are used to retrieve data from tables in the database:

```
SELECT * FROM employees WHERE last_name = 'Dean' AND first_name =  
'James' ;
```

```
SELECT * FROM employees WHERE last_name = 'Dean' OR first_name =  
'James' ;
```

```
SELECT * FROM employees WHERE NOT last_name = 'Dean' ;
```

Parentheses may be used to explicitly specify the order in which Boolean operations occur, when multiple operations are present:

```
SELECT * FROM employees WHERE (NOT last_name = 'Smith') AND (first_name  
= 'John' OR first_name = 'Mary') ;
```

Multiple sets of nested parentheses may also be used, where needed.

Any Boolean operation (or operations) which combines two (or more) tables together is referred to as a **join**, in relational database terminology.

In the field of Electronic Medical Records, some software applications use Boolean logic to query their patient databases, in what has been named Concept Processing technology.

Search engine queries

Search engine queries also employ Boolean logic. For this application, each web page on the Internet may be considered to be an "element" of a "set". The following examples use a syntax supported by Google.^[2]

- Doublequotes are used to combine whitespace-separated words into a single search term.^[3]
- Whitespace is used to specify logical AND, as it is the default operator for joining search terms:

```
"Search term 1" "Search term 2"
```

- The OR keyword is used for logical OR:

```
"Search term 1" OR "Search term 2"
```

- The minus sign is used for logical NOT (AND NOT):

```
"Search term 1" -"Search term 2"
```

Notes and references

[1] Usage Guide (http://www.ntsc.navy.mil/Resources/Library/Acqguide/SpecWord.htm#and_or).

[2] Not all search engines support the same query syntax. Additionally, some organizations provide "specialized" search engines that support alternate or extended syntax. (See e.g., Syntax cheatsheet (<http://www.google.com/help/cheatsheet.html>), Google codesearch supports regular expressions (http://www.google.com/intl/en/help/faq_codesearch.html#regex)).

[3] Doublequote-delimited search terms are called "exact phrase" searches in the Google documentation.

External links

- The Calculus of Logic (<http://www.maths.tcd.ie/pub/HistMath/People/Boole/CalcLogic/CalcLogic.html>), by George Boole, Cambridge and Dublin Mathematical Journal Vol. III (1848), pp. 183–98.
- Logical Formula Evaluator (<http://sourceforge.net/projects/logicaleval/>) (for Windows), a software which calculates all possible values of a logical formula
- Maiki & Boaz BDD-PROJECT (<http://www.bdd-project.com>), a Web Application for BDD reduction and visualization.

Algebraic logic

In mathematical logic, **algebraic logic** is the study of logic presented in an algebraic style.

Algebras as models of logics

Algebraic logic treats algebraic structures, often bounded lattices, as models (interpretations) of certain logics, making logic a branch of order theory.

In algebraic logic:

- Variables are tacitly universally quantified over some universe of discourse. There are no existentially quantified variables or open formulas;
- Terms are built up from variables using primitive and defined operations. There are no connectives;
- Formulas, built from terms in the usual way, can be equated if they are logically equivalent. To express a tautology, equate a formula with a truth value;
- The rules of proof are the substitution of equals for equals, and uniform replacement. Modus ponens remains valid, but is seldom employed.

In the table below, the left column contains one or more logical or mathematical systems, and the algebraic structure which are its models are shown on the right in the same row. Some of these structures are either Boolean algebras or proper extensions thereof. Modal and other nonclassical logics are typically modeled by what are called "Boolean algebras with operators."

Algebraic formalisms going beyond first-order logic in at least some respects include:

- Combinatory logic, having the expressive power of set theory;
- Relation algebra, arguably the paradigmatic algebraic logic, can express Peano arithmetic and most axiomatic set theories, including the canonical ZFC.

logical system	its models
Classical sentential logic	Lindenbaum-Tarski algebra Two-element Boolean algebra
Intuitionistic propositional logic	Heyting algebra
Łukasiewicz logic	MV-algebra
Modal logic K	Modal algebra
Lewis's S4	Interior algebra
Lewis's S5; Monadic predicate logic	Monadic Boolean algebra
First-order logic	Cylindric algebra Polyadic algebra Predicate functor logic
Set theory	Combinatory logic Relation algebra

History

On the history of algebraic logic before World War II, see Brady (2000) and Grattan-Guinness (2000) and their ample references. On the postwar history, see Maddux (1991) and Quine (1976).

Algebraic logic has at least two meanings:

- The study of Boolean algebra, begun by George Boole, and of relation algebra, begun by Augustus DeMorgan, extended by Charles Sanders Peirce, and taking definitive form in the work of Ernst Schröder;
- Abstract algebraic logic, a branch of contemporary mathematical logic.

Perhaps surprisingly, algebraic logic is the oldest approach to formal logic, arguably beginning with a number of memoranda Leibniz wrote in the 1680s, some of which were published in the 19th century and translated into English by Clarence Lewis in 1918. But nearly all of Leibniz's known work on algebraic logic was published only in 1903, after Louis Couturat discovered it in Leibniz's Nachlass. Parkinson (1966) and Loemker (1969) translated selections from Couturat's volume into English.

Brady (2000) discusses the rich historical connections between algebraic logic and model theory. The founders of model theory, Ernst Schröder and Leopold Loewenheim, were logicians in the algebraic tradition. Alfred Tarski, the founder of set theoretic model theory as a major branch of contemporary mathematical logic, also:

- Co-discovered Lindenbaum-Tarski algebra;
- Invented cylindric algebra;
- Wrote the 1941 paper that revived relation algebra, and that can be seen as the starting point of abstract algebraic logic.

Modern mathematical logic began in 1847, with two pamphlets whose respective authors were Augustus DeMorgan and George Boole. They, and later C.S. Peirce, Hugh MacColl, Frege, Peano, Bertrand Russell, and A. N. Whitehead all shared Leibniz's dream of combining symbolic logic, mathematics, and philosophy. Relation algebra is arguably the culmination of Leibniz's approach to logic. With the exception of some writings by Leopold Loewenheim and Thoralf Skolem, algebraic logic went into eclipse soon after the 1910-13 publication of *Principia Mathematica*, not to revive until Tarski's 1940 reexposition of relation algebra.

Leibniz had no influence on the rise of algebraic logic because his logical writings were little studied before the Parkinson and Loemker translations. Our present understanding of Leibniz the logician stems mainly from the work of Wolfgang Lenzen, summarized in Lenzen (2004).^[1] To see how present-day work in logic and metaphysics can draw inspiration from, and shed light on, Leibniz's thought, see Zalta (2000).^[2]

References

- Brady, Geraldine, 2000. *From Peirce to Skolem: A neglected chapter in the history of logic*. North-Holland/Elsevier Science BV: catalog page^[3], Amsterdam, Netherlands, 625 pages.
- Burris, Stanley, 2009. The Algebra of Logic Tradition^[4]. Stanford Encyclopedia of Philosophy.
- Ivor Grattan-Guinness, 2000. *The Search for Mathematical Roots*. Princeton Univ. Press.
- Lenzen, Wolfgang, 2004, "Leibniz's Logic^[1]" in Gabbay, D., and Woods, J., eds., *Handbook of the History of Logic, Vol. 3: The Rise of Modern Logic from Leibniz to Frege*. North-Holland: 1-84.
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- Willard Quine, 1976, "Algebraic Logic and Predicate Functors" in *The Ways of Paradox*. Harvard Univ. Press: 283-307.
- Zalta, E. N., 2000, "A (Leibnizian) Theory of Concepts^[5]," *Philosophiegeschichte und logische Analyse / Logical Analysis and History of Philosophy* 3: 137-183.

External links

- Stanford Encyclopedia of Philosophy: "Propositional Consequence Relations and Algebraic Logic ^[6]" -- by Ramon Jansana.

References

- [1] <http://www.philosophie.uni-osnabrueck.de/Publikationen%20Lenzen/Lenzen%20Leibniz%20Logic.pdf>
- [2] <http://mally.stanford.edu/Papers/leibniz.pdf>
- [3] http://www.elsevier.com/wps/find/bookdescription.cws_home/621535/description
- [4] <http://plato.stanford.edu/entries/algebra-logic-tradition/>
- [5] <http://mally.stanford.edu/leibniz.pdf>
- [6] <http://plato.stanford.edu/entries/consequence-algebraic/>

Łukasiewicz logic

In mathematics, **Łukasiewicz logic** (English pronunciation: /luːkəˈʃeɪvɪtʃ/, Polish pronunciation: [wukaˈɕɛvʲitʂ]) is a non-classical, many valued logic. It was originally defined in the early 20th-century by Jan Łukasiewicz as a three-valued logic;^[1] it was later generalized to n -valued (for all finite n) as well as infinitely-many-valued variants, both propositional and first-order.^[2] It belongs to the classes of t-norm fuzzy logics^[3] and substructural logics.^[4]

Language

The propositional connectives of Łukasiewicz logic are *implication* $\rightarrow$, *negation* $\neg$, *equivalence* $\leftrightarrow$, *weak conjunction* $\wedge$, *strong conjunction* $\otimes$, *weak disjunction* $\vee$, *strong disjunction* $\oplus$, and propositional constants $\overline{0}$ and $\overline{1}$. The presence of weak and strong conjunction and disjunction is a common feature of substructural logics without the rule of contraction, among which Łukasiewicz logic belongs.

Axioms

The original system of axioms for propositional infinite-valued Łukasiewicz logic used implication and negation as the primitive connectives:

$$\begin{aligned} & A \rightarrow (B \rightarrow A) \\ & (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)) \\ & ((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A) \\ & (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B). \end{aligned}$$

Propositional infinite-valued Łukasiewicz logic can also be axiomatized by adding the following axioms to the axiomatic system of monoidal t-norm logic:

- *Divisibility*: $(A \wedge B) \rightarrow (A \otimes (A \rightarrow B))$
- *Double negation*: $\neg\neg A \rightarrow A$.

That is, infinite-valued Łukasiewicz logic arises by adding the axiom of double negation to basic t-norm logic BL, or by adding the axiom of divisibility to the logic IMTL.

Real-valued semantics

Infinite-valued Łukasiewicz logic is a real-valued logic in which sentences from sentential calculus may be assigned a truth value of not only zero or one but also any real number in between (eg. 0.25). Valuations have a recursive definition where:

- $w(\theta \circ \phi) = F_{\circ}(w(\theta), w(\phi))$ for a binary connective $\circ$,
- $w(\neg\theta) = F_{\neg}(w(\theta))$,
- $w(\bar{0}) = 0$ and $w(\bar{1}) = 1$,

and where the definitions of the operations hold as follows:

- **Implication:** $F_{\rightarrow}(x, y) = \min\{1, 1 - x + y\}$
- **Equivalence:** $F_{\leftrightarrow}(x, y) = 1 - |x - y|$
- **Negation:** $F_{\neg}(x) = 1 - x$
- **Weak Conjunction:** $F_{\wedge}(x, y) = \min\{x, y\}$
- **Weak Disjunction:** $F_{\vee}(x, y) = \max\{x, y\}$
- **Strong Conjunction:** $F_{\otimes}(x, y) = \max\{0, x + y - 1\}$
- **Strong Disjunction:** $F_{\oplus}(x, y) = \min\{1, x + y\}$.

The truth function $F_{\otimes}$ of strong conjunction is the Łukasiewicz t-norm and the truth function $F_{\oplus}$ of strong disjunction is its dual t-conorm. The truth function $F_{\rightarrow}$ is the residuum of the Łukasiewicz t-norm. All truth functions of the basic connectives are continuous.

By definition, a formula is a tautology of infinite-valued Łukasiewicz logic if it evaluates to 1 under any valuation of propositional variables by real numbers in the interval $[0, 1]$.

General algebraic semantics

The standard real-valued semantics determined by the Łukasiewicz t-norm is not the only possible semantics of Łukasiewicz logic. General algebraic semantics of propositional infinite-valued Łukasiewicz logic is formed by the class of all MV-algebras. The standard real-valued semantics is a special MV-algebra, called the *standard MV-algebra*.

Like other t-norm fuzzy logics, propositional infinite-valued Łukasiewicz logic enjoys completeness with respect to the class of all algebras for which the logic is sound (that is, MV-algebras) as well as with respect to only linear ones. This is expressed by the general, linear, and standard completeness theorems:

The following conditions are equivalent:

- $\mathcal{A}$ is provable in propositional infinite-valued Łukasiewicz logic
- $\mathcal{A}$ is valid in all MV-algebras (*general completeness*)
- $\mathcal{A}$ is valid in all linearly ordered MV-algebras (*linear completeness*)
- $\mathcal{A}$ is valid in the standard MV-algebra (*standard completeness*).

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- [3] Hájek P., 1998, *Metamathematics of Fuzzy Logic*. Dordrecht: Kluwer.
- [4] Ono, H., 2003, "Substructural logics and residuated lattices — an introduction". In F.V. Hendricks, J. Malinowski (eds.): *Trends in Logic: 50 Years of Studia Logica*, *Trends in Logic* **20**: 177–212.

Intuitionistic logic

Intuitionistic logic, or **constructive logic**, is a symbolic logic system that differs from classical logic in its definition of what it means for a statement to be true. In classical logic, all well-formed statements are assumed to be either true or false, even if we do not have a proof of either. In constructive logic, a statement is only true if there is a proof that it is true, and only false if there is a proof that it is false. Operations in constructive logic preserve justification, rather than truth. Syntactically, intuitionist logic differs from classical logic in that the law of excluded middle and double negation elimination are not axioms of the system, and cannot be proved in it.

Constructive logic is practically useful because its restrictions produce proofs that have the existence property, making it also suitable for other forms of mathematical constructivism. Informally, this means that if you have a constructive proof that an object exists, you can turn that constructive proof into an algorithm for generating an example of it.

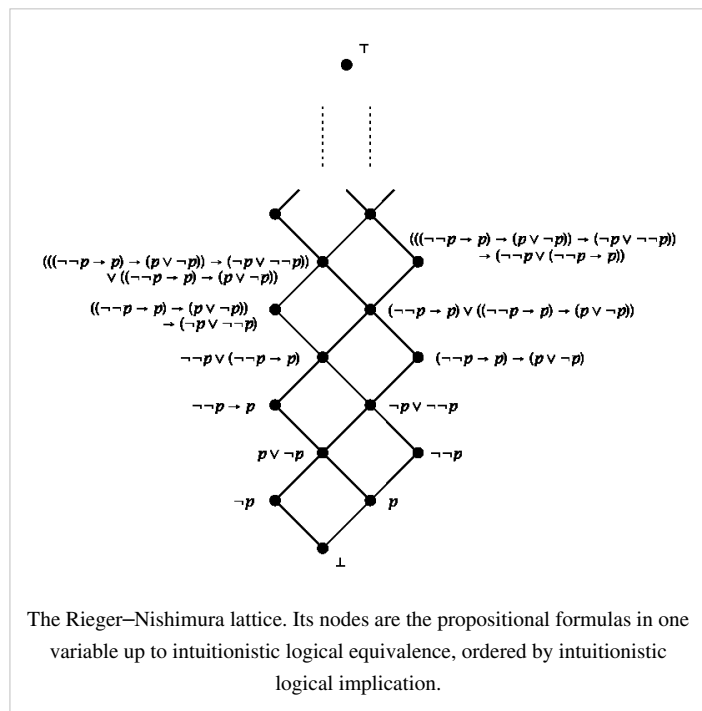
It was originally developed by Arend Heyting to provide a formal basis for Brouwer's programme of intuitionism.

Syntax

The syntax of formulas of intuitionistic logic is similar to propositional logic or first-order logic. However, intuitionistic connectives are not definable in terms of each other in the same way as in classical logic, hence their choice matters. In intuitionistic propositional logic it is customary to use $\rightarrow$, $\wedge$, $\vee$, $\perp$ as the basic connectives, treating $\neg A$ as an abbreviation for $(A \rightarrow \perp)$. In intuitionistic first-order logic both quantifiers $\exists$, $\forall$ are needed.

Many tautologies of classical logic can no longer be proven within intuitionistic logic. Examples include not only the law of excluded middle $p \vee \neg p$, but also Peirce's law $((p \rightarrow q) \rightarrow p) \rightarrow p$, and even double negation elimination. In classical logic, both $p \rightarrow \neg\neg p$ and also $\neg\neg p \rightarrow p$ are theorems. In intuitionistic logic, only the former is a theorem: double negation can be introduced, but it cannot be eliminated. Rejecting $p \vee \neg p$ may seem strange to those more familiar with classical logic, but proving this statement in constructive logic would require producing a proof for the truth or falsity of *all possible statements*, which is impossible for a variety of reasons.

Because many classically valid tautologies are not theorems of intuitionistic logic, but all theorems of intuitionist logic are valid classically, intuitionist logic can be viewed as a weakening of classical logic, albeit one with many useful properties.



Sequent calculus

Gentzen discovered that a simple restriction of his system LK (his sequent calculus for classical logic) results in a system which is sound and complete with respect to intuitionistic logic. He called this system LJ. In LK any number of formulas is allowed to appear on the conclusion side of a sequent; in contrast LJ allows at most one formula in this position.

Other derivatives of LK are limited to intuitionistic derivations but still allow multiple conclusions in a sequent. LJ^[1] is one example.

Hilbert-style calculus

Intuitionistic logic can be defined using the following Hilbert-style calculus. Compare with the deduction system at Propositional calculus#Alternative calculus.

In propositional logic, the inference rule is modus ponens

- MP: from ϕ and $\phi \rightarrow \psi$ infer ψ

and the axioms are

- THEN-1: $\phi \rightarrow (\chi \rightarrow \phi)$
- THEN-2: $(\phi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\phi \rightarrow \chi) \rightarrow (\phi \rightarrow \psi))$
- AND-1: $\phi \wedge \chi \rightarrow \phi$
- AND-2: $\phi \wedge \chi \rightarrow \chi$
- AND-3: $\phi \rightarrow (\chi \rightarrow (\phi \wedge \chi))$
- OR-1: $\phi \rightarrow \phi \vee \chi$
- OR-2: $\chi \rightarrow \phi \vee \chi$
- OR-3: $(\phi \rightarrow \psi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\phi \vee \chi \rightarrow \psi))$
- FALSE: $\perp \rightarrow \phi$

To make this a system of first-order predicate logic, the generalization rules

- $\forall$ -GEN: from $\psi \rightarrow \phi$ infer $\psi \rightarrow (\forall x \phi)$, if x is not free in ψ
- $\exists$ -GEN: from $\phi \rightarrow \psi$ infer $(\exists x \phi) \rightarrow \psi$, if x is not free in ψ

are added, along with the axioms

- PRED-1: $(\forall x \phi(x)) \rightarrow \phi(t)$, if the term t is free for substitution for the variable x in ϕ (i.e., if no occurrence of any variable in t becomes bound in $\phi(t)$)
- PRED-2: $\phi(t) \rightarrow (\exists x \phi(x))$, with the same restriction as for PRED-1

Optional connectives

Negation

If one wishes to include a connective $\neg$ for negation rather than consider it an abbreviation for $\phi \rightarrow \perp$, it is enough to add:

- NOT-1': $(\phi \rightarrow \perp) \rightarrow \neg\phi$
- NOT-2': $\neg\phi \rightarrow (\phi \rightarrow \perp)$

There are a number of alternatives available if one wishes to omit the connective $\perp$ (false). For example, one may replace the three axioms FALSE, NOT-1', and NOT-2' with the two axioms

- NOT-1: $(\phi \rightarrow \chi) \rightarrow ((\phi \rightarrow \neg\chi) \rightarrow \neg\phi)$
- NOT-2: $\phi \rightarrow (\neg\phi \rightarrow \chi)$

as at Propositional calculus#Axioms. Alternatives to NOT-1 are $(\phi \rightarrow \neg\chi) \rightarrow (\chi \rightarrow \neg\phi)$ or $(\phi \rightarrow \neg\phi) \rightarrow \neg\phi$.

Equivalence

The connective $\leftrightarrow$ for equivalence may be treated as an abbreviation, with $\phi \leftrightarrow \chi$ standing for $(\phi \rightarrow \chi) \wedge (\chi \rightarrow \phi)$. Alternatively, one may add the axioms

- IFF-1: $(\phi \leftrightarrow \chi) \rightarrow (\phi \rightarrow \chi)$
- IFF-2: $(\phi \leftrightarrow \chi) \rightarrow (\chi \rightarrow \phi)$
- IFF-3: $(\phi \rightarrow \chi) \rightarrow ((\chi \rightarrow \phi) \rightarrow (\phi \leftrightarrow \chi))$

IFF-1 and IFF-2 can, if desired, be combined into a single axiom $(\phi \leftrightarrow \chi) \rightarrow ((\phi \rightarrow \chi) \wedge (\chi \rightarrow \phi))$ using conjunction.

Relation to classical logic

The system of classical logic is obtained by adding any one of the following axioms:

- $\phi \vee \neg\phi$ (Law of the excluded middle. May also be formulated as $(\phi \rightarrow \chi) \rightarrow ((\neg\phi \rightarrow \chi) \rightarrow \chi)$.)
- $\neg\neg\phi \rightarrow \phi$ (Double negation elimination)
- $((\phi \rightarrow \chi) \rightarrow \phi) \rightarrow \phi$ (Peirce's law)

In general, one may take as the extra axiom any classical tautology that is not valid in the two-element Kripke frame $\circ \longrightarrow \circ$ (in other words, that is not included in Smetanich's logic).

Another relationship is given by the Gödel–Gentzen negative translation, which provides an embedding of classical first-order logic into intuitionistic logic: a first-order formula is provable in classical logic if and only if its Gödel–Gentzen translation is provable intuitionistically. Therefore intuitionistic logic can instead be seen as a means of extending classical logic with constructive semantics.

Non-interdefinability of operators

In classical propositional logic, it is possible to take one of conjunction, disjunction, or implication as primitive, and define the other two in terms of it together with negation, such as in Łukasiewicz's three axioms of propositional logic. It is even possible to define all four in terms of a sole sufficient operator such as the Peirce arrow (NOR) or Sheffer stroke (NAND). Similarly, in classical first-order logic, one of the quantifiers can be defined in terms of the other and negation.

These are fundamentally consequences of the law of bivalence, which makes all such connectives merely Boolean functions. The law of bivalence does not hold in intuitionistic logic, only the law of non-contradiction. As a result none of the basic connectives can be dispensed with, and the above axioms are all necessary. Most of the classical identities are only theorems of intuitionistic logic in one direction, although some are theorems in both directions.

They are as follows:

Conjunction versus disjunction:

- $(\phi \wedge \psi) \rightarrow \neg(\neg\phi \vee \neg\psi)$
- $(\phi \vee \psi) \rightarrow \neg(\neg\phi \wedge \neg\psi)$
- $(\neg\phi \vee \neg\psi) \rightarrow \neg(\phi \wedge \psi)$
- $(\neg\phi \wedge \neg\psi) \leftrightarrow \neg(\phi \vee \psi)$

Conjunction versus implication:

- $(\phi \wedge \psi) \rightarrow \neg(\phi \rightarrow \neg\psi)$
- $(\phi \rightarrow \psi) \rightarrow \neg(\phi \wedge \neg\psi)$
- $(\phi \wedge \neg\psi) \rightarrow \neg(\phi \rightarrow \psi)$
- $(\phi \rightarrow \neg\psi) \leftrightarrow \neg(\phi \wedge \psi)$

Disjunction versus implication:

- $(\phi \vee \psi) \rightarrow (\neg\phi \rightarrow \psi)$

- $(\neg\phi \vee \psi) \rightarrow (\phi \rightarrow \psi)$
- $\neg(\phi \rightarrow \psi) \rightarrow \neg(\neg\phi \vee \psi)$
- $\neg(\phi \vee \psi) \leftrightarrow \neg(\neg\phi \rightarrow \psi)$

Universal versus existential quantification:

- $(\forall x \phi(x)) \rightarrow \neg(\exists x \neg\phi(x))$
- $(\exists x \phi(x)) \rightarrow \neg(\forall x \neg\phi(x))$
- $(\exists x \neg\phi(x)) \rightarrow \neg(\forall x \phi(x))$
- $(\forall x \neg\phi(x)) \leftrightarrow \neg(\exists x \phi(x))$

So, for example, "a or b" is a stronger statement than "if not a, then b", whereas these are classically interchangeable. On the other hand, "neither a nor b" is equivalent to "not a, and also not b".

If we include equivalence in the list of connectives, some of the connectives become definable from others:

- $(\phi \leftrightarrow \psi) \leftrightarrow ((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$
- $(\phi \rightarrow \psi) \leftrightarrow ((\phi \vee \psi) \leftrightarrow \psi)$
- $(\phi \rightarrow \psi) \leftrightarrow ((\phi \wedge \psi) \leftrightarrow \phi)$
- $(\phi \wedge \psi) \leftrightarrow ((\phi \rightarrow \psi) \leftrightarrow \phi)$
- $(\phi \wedge \psi) \leftrightarrow (((\phi \vee \psi) \leftrightarrow \psi) \leftrightarrow \phi)$

In particular, $\{\vee, \leftrightarrow, \perp\}$ and $\{\vee, \leftrightarrow, \neg\}$ are complete bases of intuitionistic connectives.

As shown by Alexander Kuznetsov, either of the following defined connectives can serve the role of a sole sufficient operator for intuitionistic logic:^[2]

- $((p \vee q) \wedge \neg r) \vee (\neg p \wedge (q \leftrightarrow r)),$
- $p \rightarrow (q \wedge \neg r \wedge (s \vee t)).$

Semantics

The semantics are rather more complicated than for the classical case. A model theory can be given by Heyting algebras or, equivalently, by Kripke semantics.

Heyting algebra semantics

In classical logic, we often discuss the truth values that a formula can take. The values are usually chosen as the members of a Boolean algebra. The meet and join operations in the Boolean algebra are identified with the $\wedge$ and $\vee$ logical connectives, so that the value of a formula of the form $A \wedge B$ is the meet of the value of A and the value of B in the Boolean algebra. Then we have the useful theorem that a formula is a valid sentence of classical logic if and only if its value is 1 for every valuation—that is, for any assignment of values to its variables.

A corresponding theorem is true for intuitionistic logic, but instead of assigning each formula a value from a Boolean algebra, one uses values from a Heyting algebra, of which Boolean algebras are a special case. A formula is valid in intuitionistic logic if and only if it receives the value of the top element for any valuation on any Heyting algebra.

It can be shown that to recognize valid formulas, it is sufficient to consider a single Heyting algebra whose elements are the open subsets of the real line $\mathbf{R}$.^[3] In this algebra, the $\wedge$ and $\vee$ operations correspond to set intersection and union, and the value assigned to a formula $A \rightarrow B$ is $\text{int}(A^C \cup B)$, the interior of the union of the value of B and the complement of the value of A . The bottom element is the empty set $\emptyset$, and the top element is the entire line $\mathbf{R}$. The negation $\neg A$ of a formula A is (as usual) defined to be $A \rightarrow \emptyset$. The value of $\neg A$ then reduces to $\text{int}(A^C)$, the interior of the complement of the value of A , also known as the exterior of A . With these assignments, intuitionistically valid formulas are precisely those that are assigned the value of the entire line.^[3]

For example, the formula $\neg(A \wedge \neg A)$ is valid, because no matter what set X is chosen as the value of the formula A , the value of $\neg(A \wedge \neg A)$ can be shown to be the entire line:

$$\begin{aligned}
&\text{Value}(\neg(A \wedge \neg A)) = \\
&\text{int}((\text{Value}(A \wedge \neg A))^C) = \\
&\text{int}((\text{Value}(A) \cap \text{Value}(\neg A))^C) = \\
&\text{int}((X \cap \text{int}((\text{Value}(A))^C))^C) = \\
&\text{int}((X \cap \text{int}(X^C))^C)
\end{aligned}$$

A theorem of topology tells us that $\text{int}(X^C)$ is a subset of X^C , so the intersection is empty, leaving:

$$\text{int}(\emptyset^C) = \text{int}(\mathbf{R}) = \mathbf{R}$$

So the valuation of this formula is true, and indeed the formula is valid.

But the law of the excluded middle, $A \vee \neg A$, can be shown to be *invalid* by letting the value of A be $\{y : y > 0\}$. Then the value of $\neg A$ is the interior of $\{y : y \leq 0\}$, which is $\{y : y < 0\}$, and the value of the formula is the union of $\{y : y > 0\}$ and $\{y : y < 0\}$, which is $\{y : y \neq 0\}$, *not* the entire line.

The interpretation of any intuitionistically valid formula in the infinite Heyting algebra described above results in the top element, representing true, as the valuation of the formula, regardless of what values from the algebra are assigned to the variables of the formula.^[3] Conversely, for every invalid formula, there is an assignment of values to the variables that yields a valuation that differs from the top element.^{[4] [5]} No finite Heyting algebra has both these properties.^[3]

Kripke semantics

Building upon his work on semantics of modal logic, Saul Kripke created another semantics for intuitionistic logic, known as **Kripke semantics** or **relational semantics**^[6].

Relation to other logics

Intuitionistic logic is related by duality to a paraconsistent logic known as *Brazilian*, *anti-intuitionistic* or *dual-intuitionistic logic*^[7].

The subsystem of intuitionistic logic with the FALSE axiom removed is known as minimal logic.

Notes

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[2] Alexander Chagrov, Michael Zakharyashev, *Modal Logic*, vol. 35 of Oxford Logic Guides, Oxford University Press, 1997, pp. 58–59.

[3] Sørensen, Morten Heine B; Paweł Urzyczyn (2006). *Lectures on the Curry-Howard Isomorphism*. Studies in Logic and the Foundations of Mathematics. Elsevier. p. 42. ISBN 0444520775.

[4] Alfred Tarski, *Der Aussagenkalkül und die Topologie*, Fundamenta Mathematicae 31 (1938), 103–134. (<http://matwbn.icm.edu.pl/tresc.php?wyd=1&tom=31>)

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[7] Aoyama, Hiroshi (2004). "LK, LJ, Dual Intuitionistic Logic, and Quantum Logic". *Notre Dame Journal of Formal Logic* **45** (4): 193–213. doi:10.1305/ndjfl/1099238445.

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External links

- Stanford Encyclopedia of Philosophy: "Intuitionistic Logic" (<http://plato.stanford.edu/entries/logic-intuitionistic/>) -- by Joan Moschovakis.

Mathematical logic

Mathematical logic (also known as **symbolic logic**) is a subfield of mathematics with close connections to computer science and philosophical logic.^[1] The field includes both the mathematical study of logic and the applications of formal logic to other areas of mathematics. The unifying themes in mathematical logic include the study of the expressive power of formal systems and the deductive power of formal proof systems.

Mathematical logic is often divided into the fields of set theory, model theory, recursion theory, and proof theory. These areas share basic results on logic, particularly first-order logic, and definability. In computer science (particularly in the ACM Classification) mathematical logic encompasses additional topics not detailed in this article; see logic in computer science for those.

Since its inception, mathematical logic has contributed to, and has been motivated by, the study of foundations of mathematics. This study began in the late 19th century with the development of axiomatic frameworks for geometry, arithmetic, and analysis. In the early 20th century it was shaped by David Hilbert's program to prove the consistency of foundational theories. Results of Kurt Gödel, Gerhard Gentzen, and others provided partial resolution to the program, and clarified the issues involved in proving consistency. Work in set theory showed that almost all ordinary mathematics can be formalized in terms of sets, although there are some theorems that cannot be proven in common axiom systems for set theory. Contemporary work in the foundations of mathematics often focuses on establishing which parts of mathematics can be formalized in particular formal systems, rather than trying to find theories in which all of mathematics can be developed.

History

Mathematical logic emerged in the mid-19th century as a subfield of mathematics independent of the traditional study of logic (Ferreirós 2001, p. 443). Before this emergence, logic was studied with rhetoric, through the syllogism, and with philosophy. The first half of the 20th century saw an explosion of fundamental results, accompanied by vigorous debate over the foundations of mathematics.

Early history

Sophisticated theories of logic were developed in many cultures, including China, India, Greece and the Islamic world. In the 18th century, attempts to treat the operations of formal logic in a symbolic or algebraic way had been made by philosophical mathematicians including Leibniz and Lambert, but their labors remained isolated and little known.

19th century

In the middle of the nineteenth century, George Boole and then Augustus De Morgan presented systematic mathematical treatments of logic. Their work, building on work by algebraists such as George Peacock, extended the traditional Aristotelian doctrine of logic into a sufficient framework for the study of foundations of mathematics (Katz 1998, p. 686).

Charles Sanders Peirce built upon the work of Boole to develop a logical system for relations and quantifiers, which he published in several papers from 1870 to 1885. Gottlob Frege presented an independent development of logic with quantifiers in his *Begriffsschrift*, published in 1879, a work generally considered as marking a turning point in the history of logic. Frege's work remained obscure, however, until Bertrand Russell began to promote it near the turn of the century. The two-dimensional notation Frege developed was never widely adopted and is unused in contemporary texts.

From 1890 to 1905, Ernst Schröder published *Vorlesungen über die Algebra der Logik* in three volumes. This work summarized and extended the work of Boole, De Morgan, and Peirce, and was a comprehensive reference to symbolic logic as it was understood at the end of the 19th century.

Foundational theories

Some concerns that mathematics had not been built on a proper foundation led to the development of axiomatic systems for fundamental areas of mathematics such as arithmetic, analysis, and geometry.

In logic, the term *arithmetic* refers to the theory of the natural numbers. Giuseppe Peano (1888) published a set of axioms for arithmetic that came to bear his name (Peano axioms), using a variation of the logical system of Boole and Schröder but adding quantifiers. Peano was unaware of Frege's work at the time. Around the same time Richard Dedekind showed that the natural numbers are uniquely characterized by their induction properties. Dedekind (1888) proposed a different characterization, which lacked the formal logical character of Peano's axioms. Dedekind's work, however, proved theorems inaccessible in Peano's system, including the uniqueness of the set of natural numbers (up to isomorphism) and the recursive definitions of addition and multiplication from the successor function and mathematical induction.

In the mid-19th century, flaws in Euclid's axioms for geometry became known (Katz 1998, p. 774). In addition to the independence of the parallel postulate, established by Nikolai Lobachevsky in 1826 (Lobachevsky 1840), mathematicians discovered that certain theorems taken for granted by Euclid were not in fact provable from his axioms. Among these is the theorem that a line contains at least two points, or that circles of the same radius whose centers are separated by that radius must intersect. Hilbert (1899) developed a complete set of axioms for geometry, building on previous work by Pasch (1882). The success in axiomatizing geometry motivated Hilbert to seek complete axiomatizations of other areas of mathematics, such as the natural numbers and the real line. This would prove to be a major area of research in the first half of the 20th century.

The 19th century saw great advances in the theory of real analysis, including theories of convergence of functions and Fourier series. Mathematicians such as Karl Weierstrass began to construct functions that stretched intuition, such as nowhere-differentiable continuous functions. Previous conceptions of a function as a rule for computation, or a smooth graph, were no longer adequate. Weierstrass began to advocate the arithmetization of analysis, which sought to axiomatize analysis using properties of the natural numbers. The modern (ϵ, δ) -definition of limit and continuous functions was already developed by Bolzano in 1817 (Felscher 2000), but remained relatively unknown. Cauchy in 1821 defined continuity in terms of infinitesimals (see *Cours d'Analyse*, page 34). In 1858, Dedekind proposed a definition of the real numbers in terms of Dedekind cuts of rational numbers (Dedekind 1872), a definition still employed in contemporary texts.

Georg Cantor developed the fundamental concepts of infinite set theory. His early results developed the theory of cardinality and proved that the reals and the natural numbers have different cardinalities (Cantor 1874). Over the next twenty years, Cantor developed a theory of transfinite numbers in a series of publications. In 1891, he published

a new proof of the uncountability of the real numbers that introduced the diagonal argument, and used this method to prove Cantor's theorem that no set can have the same cardinality as its powerset. Cantor believed that every set could be well-ordered, but was unable to produce a proof for this result, leaving it as an open problem in 1895 (Katz 1998, p. 807).

20th century

In the early decades of the 20th century, the main areas of study were set theory and formal logic. The discovery of paradoxes in informal set theory caused some to wonder whether mathematics itself is inconsistent, and to look for proofs of consistency.

In 1900, Hilbert posed a famous list of 23 problems for the next century. The first two of these were to resolve the continuum hypothesis and prove the consistency of elementary arithmetic, respectively; the tenth was to produce a method that could decide whether a multivariate polynomial equation over the integers has a solution. Subsequent work to resolve these problems shaped the direction of mathematical logic, as did the effort to resolve Hilbert's *Entscheidungsproblem*, posed in 1928. This problem asked for a procedure that would decide, given a formalized mathematical statement, whether the statement is true or false.

Set theory and paradoxes

Ernst Zermelo (1904) gave a proof that every set could be well-ordered, a result Georg Cantor had been unable to obtain. To achieve the proof, Zermelo introduced the axiom of choice, which drew heated debate and research among mathematicians and the pioneers of set theory. The immediate criticism of the method led Zermelo to publish a second exposition of his result, directly addressing criticisms of his proof (Zermelo 1908a). This paper led to the general acceptance of the axiom of choice in the mathematics community.

Skepticism about the axiom of choice was reinforced by recently discovered paradoxes in naive set theory. Cesare Burali-Forti (1897) was the first to state a paradox: the Burali-Forti paradox shows that the collection of all ordinal numbers cannot form a set. Very soon thereafter, Bertrand Russell discovered Russell's paradox in 1901, and Jules Richard (1905) discovered Richard's paradox.

Zermelo (1908b) provided the first set of axioms for set theory. These axioms, together with the additional axiom of replacement proposed by Abraham Fraenkel, are now called Zermelo–Fraenkel set theory (ZF). Zermelo's axioms incorporated the principle of limitation of size to avoid Russell's paradox.

In 1910, the first volume of *Principia Mathematica* by Russell and Alfred North Whitehead was published. This seminal work developed the theory of functions and cardinality in a completely formal framework of type theory, which Russell and Whitehead developed in an effort to avoid the paradoxes. *Principia Mathematica* is considered one of the most influential works of the 20th century, although the framework of type theory did not prove popular as a foundational theory for mathematics (Ferreirós 2001, p. 445).

Fraenkel (1922) proved that the axiom of choice cannot be proved from the remaining axioms of Zermelo's set theory with urelements. Later work by Paul Cohen (1966) showed that the addition of urelements is not needed, and the axiom of choice is unprovable in ZF. Cohen's proof developed the method of forcing, which is now an important tool for establishing independence results in set theory.

Symbolic logic

Leopold Löwenheim (1915) and Thoralf Skolem (1920) obtained the Löwenheim–Skolem theorem, which says that first-order logic cannot control the cardinalities of infinite structures. Skolem realized that this theorem would apply to first-order formalizations of set theory, and that it implies any such formalization has a countable model. This counterintuitive fact became known as Skolem's paradox.

In his doctoral thesis, Kurt Gödel (1929) proved the completeness theorem, which establishes a correspondence between syntax and semantics in first-order logic. Gödel used the completeness theorem to prove the compactness theorem, demonstrating the finitary nature of first-order logical consequence. These results helped establish first-order logic as the dominant logic used by mathematicians.

In 1931, Gödel published *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*, which proved the incompleteness (in a different meaning of the word) of all sufficiently strong, effective first-order theories. This result, known as Gödel's incompleteness theorem, establishes severe limitations on axiomatic foundations for mathematics, striking a strong blow to Hilbert's program. It showed the impossibility of providing a consistency proof of arithmetic within any formal theory of arithmetic. Hilbert, however, did not acknowledge the importance of the incompleteness theorem for some time.

Gödel's theorem shows that a consistency proof of any sufficiently strong, effective axiom system cannot be obtained in the system itself, if the system is consistent, nor in any weaker system. This leaves open the possibility of consistency proofs that cannot be formalized within the system they consider. Gentzen (1936) proved the consistency of arithmetic using a finitistic system together with a principle of transfinite induction. Gentzen's result introduced the ideas of cut elimination and proof-theoretic ordinals, which became key tools in proof theory. Gödel (1958) gave a different consistency proof, which reduces the consistency of classical arithmetic to that of intuitionistic arithmetic in higher types.

Beginnings of the other branches

Alfred Tarski developed the basics of model theory.

Beginning in 1935, a group of prominent mathematicians collaborated under the pseudonym Nicolas Bourbaki to publish a series of encyclopedic mathematics texts. These texts, written in an austere and axiomatic style, emphasized rigorous presentation and set-theoretic foundations. Terminology coined by these texts, such as the words *bijection*, *injection*, and *surjection*, and the set-theoretic foundations the texts employed, were widely adopted throughout mathematics.

The study of computability came to be known as recursion theory, because early formalizations by Gödel and Kleene relied on recursive definitions of functions.^[2] When these definitions were shown equivalent to Turing's formalization involving Turing machines, it became clear that a new concept – the computable function – had been discovered, and that this definition was robust enough to admit numerous independent characterizations. In his work on the incompleteness theorems in 1931, Gödel lacked a rigorous concept of an effective formal system; he immediately realized that the new definitions of computability could be used for this purpose, allowing him to state the incompleteness theorems in generality that could only be implied in the original paper.

Numerous results in recursion theory were obtained in the 1940s by Stephen Cole Kleene and Emil Leon Post. Kleene (1943) introduced the concepts of relative computability, foreshadowed by Turing (1939), and the arithmetical hierarchy. Kleene later generalized recursion theory to higher-order functionals. Kleene and Kreisel studied formal versions of intuitionistic mathematics, particularly in the context of proof theory.

Subfields and scope

The *Handbook of Mathematical Logic* makes a rough division of contemporary mathematical logic into four areas:

1. set theory
2. model theory
3. recursion theory, and
4. proof theory and constructive mathematics (considered as parts of a single area).

Each area has a distinct focus, although many techniques and results are shared between multiple areas. The border lines between these fields, and the lines between mathematical logic and other fields of mathematics, are not always sharp. Gödel's incompleteness theorem marks not only a milestone in recursion theory and proof theory, but has also led to Löb's theorem in modal logic. The method of forcing is employed in set theory, model theory, and recursion theory, as well as in the study of intuitionistic mathematics.

The mathematical field of category theory uses many formal axiomatic methods, and includes the study of categorical logic, but category theory is not ordinarily considered a subfield of mathematical logic. Because of its applicability in diverse fields of mathematics, mathematicians including Saunders Mac Lane have proposed category theory as a foundational system for mathematics, independent of set theory. These foundations use toposes, which resemble generalized models of set theory that may employ classical or nonclassical logic.

Formal logical systems

At its core, mathematical logic deals with mathematical concepts expressed using formal logical systems. These systems, though they differ in many details, share the common property of considering only expressions in a fixed formal language, or signature. The system of first-order logic is the most widely studied today, because of its applicability to foundations of mathematics and because of its desirable proof-theoretic properties.^[3] Stronger classical logics such as second-order logic or infinitary logic are also studied, along with nonclassical logics such as intuitionistic logic.

First-order logic

First-order logic is a particular formal system of logic. Its syntax involves only finite expressions as well-formed formulas, while its semantics are characterized by the limitation of all quantifiers to a fixed domain of discourse.

Early results about formal logic established limitations of first-order logic. The Löwenheim–Skolem theorem (1919) showed that if a set of sentences in a countable first-order language has an infinite model then it has at least one model of each infinite cardinality. This shows that it is impossible for a set of first-order axioms to characterize the natural numbers, the real numbers, or any other infinite structure up to isomorphism. As the goal of early foundational studies was to produce axiomatic theories for all parts of mathematics, this limitation was particularly stark.

Gödel's completeness theorem (Gödel 1929) established the equivalence between semantic and syntactic definitions of logical consequence in first-order logic. It shows that if a particular sentence is true in every model that satisfies a particular set of axioms, then there must be a finite deduction of the sentence from the axioms. The compactness theorem first appeared as a lemma in Gödel's proof of the completeness theorem, and it took many years before logicians grasped its significance and began to apply it routinely. It says that a set of sentences has a model if and only if every finite subset has a model, or in other words that an inconsistent set of formulas must have a finite inconsistent subset. The completeness and compactness theorems allow for sophisticated analysis of logical consequence in first-order logic and the development of model theory, and they are a key reason for the prominence of first-order logic in mathematics.

Gödel's incompleteness theorems (Gödel 1931) establish additional limits on first-order axiomatizations. The **first incompleteness theorem** states that for any sufficiently strong, effectively given logical system there exists a

statement which is true but not provable within that system. Here a logical system is effectively given if it is possible to decide, given any formula in the language of the system, whether the formula is an axiom. A logical system is sufficiently strong if it can express the Peano axioms. When applied to first-order logic, the first incompleteness theorem implies that any sufficiently strong, consistent, effective first-order theory has models that are not elementarily equivalent, a stronger limitation than the one established by the Löwenheim–Skolem theorem. The **second incompleteness theorem** states that no sufficiently strong, consistent, effective axiom system for arithmetic can prove its own consistency, which has been interpreted to show that Hilbert's program cannot be completed.

Other classical logics

Many logics besides first-order logic are studied. These include infinitary logics, which allow for formulas to provide an infinite amount of information, and higher-order logics, which include a portion of set theory directly in their semantics.

The most well studied infinitary logic is $L_{\omega_1, \omega}$. In this logic, quantifiers may only be nested to finite depths, as in first order logic, but formulas may have finite or countably infinite conjunctions and disjunctions within them. Thus, for example, it is possible to say that an object is a whole number using a formula of $L_{\omega_1, \omega}$ such as

$$(x = 0) \vee (x = 1) \vee (x = 2) \vee \dots$$

Higher-order logics allow for quantification not only of elements of the domain of discourse, but subsets of the domain of discourse, sets of such subsets, and other objects of higher type. The semantics are defined so that, rather than having a separate domain for each higher-type quantifier to range over, the quantifiers instead range over all objects of the appropriate type. The logics studied before the development of first-order logic, for example Frege's logic, had similar set-theoretic aspects. Although higher-order logics are more expressive, allowing complete axiomatizations of structures such as the natural numbers, they do not satisfy analogues of the completeness and compactness theorems from first-order logic, and are thus less amenable to proof-theoretic analysis.

Another type of logics are fixed-point logics that allow inductive definitions, like one writes for primitive recursive functions.

One can formally define an extension of first-order logic — a notion which encompasses all logics in this section because they behave like first-order logic in certain fundamental ways, but does not encompass all logics in general, e.g. it does not encompass intuitionistic, modal or fuzzy logic. Lindström's theorem implies that the only extension of first-order logic satisfying both the Compactness theorem and the Downward Löwenheim–Skolem theorem is first-order logic.

Nonclassical and modal logic

Modal logics include additional modal operators, such as an operator which states that a particular formula is not only true, but necessarily true. Although modal logic is not often used to axiomatize mathematics, it has been used to study the properties of first-order provability (Solovay 1976) and set-theoretic forcing (Hamkins and Löwe 2007).

Intuitionistic logic was developed by Heyting to study Brouwer's program of intuitionism, in which Brouwer himself avoided formalization. Intuitionistic logic specifically does not include the law of the excluded middle, which states that each sentence is either true or its negation is true. Kleene's work with the proof theory of intuitionistic logic showed that constructive information can be recovered from intuitionistic proofs. For example, any provably total function in intuitionistic arithmetic is computable; this is not true in classical theories of arithmetic such as Peano arithmetic.

Algebraic logic

Algebraic logic uses the methods of abstract algebra to study the semantics of formal logics. A fundamental example is the use of Boolean algebras to represent truth values in classical propositional logic, and the use of Heyting algebras to represent truth values in intuitionistic propositional logic. Stronger logics, such as first-order logic and higher-order logic, are studied using more complicated algebraic structures such as cylindric algebras.

Set theory

Set theory is the study of sets, which are abstract collections of objects. Many of the basic notions, such as ordinal and cardinal numbers, were developed informally by Cantor before formal axiomatizations of set theory were developed. The first such axiomatization, due to Zermelo (1908b), was extended slightly to become Zermelo–Fraenkel set theory (ZF), which is now the most widely used foundational theory for mathematics.

Other formalizations of set theory have been proposed, including von Neumann–Bernays–Gödel set theory (NBG), Morse–Kelley set theory (MK), and New Foundations (NF). Of these, ZF, NBG, and MK are similar in describing a cumulative hierarchy of sets. New Foundations takes a different approach; it allows objects such as the set of all sets at the cost of restrictions on its set-existence axioms. The system of Kripke–Platek set theory is closely related to generalized recursion theory.

Two famous statements in set theory are the axiom of choice and the continuum hypothesis. The axiom of choice, first stated by Zermelo (1904), was proved independent of ZF by Fraenkel (1922), but has come to be widely accepted by mathematicians. It states that given a collection of nonempty sets there is a single set C that contains exactly one element from each set in the collection. The set C is said to "choose" one element from each set in the collection. While the ability to make such a choice is considered obvious by some, since each set in the collection is nonempty, the lack of a general, concrete rule by which the choice can be made renders the axiom nonconstructive. Stefan Banach and Alfred Tarski (1924) showed that the axiom of choice can be used to decompose a solid ball into a finite number of pieces which can then be rearranged, with no scaling, to make two solid balls of the original size. This theorem, known as the Banach–Tarski paradox, is one of many counterintuitive results of the axiom of choice.

The continuum hypothesis, first proposed as a conjecture by Cantor, was listed by David Hilbert as one of his 23 problems in 1900. Gödel showed that the continuum hypothesis cannot be disproven from the axioms of Zermelo–Fraenkel set theory (with or without the axiom of choice), by developing the constructible universe of set theory in which the continuum hypothesis must hold. In 1963, Paul Cohen showed that the continuum hypothesis cannot be proven from the axioms of Zermelo–Fraenkel set theory (Cohen 1966). This independence result did not completely settle Hilbert's question, however, as it is possible that new axioms for set theory could resolve the hypothesis. Recent work along these lines has been conducted by W. Hugh Woodin, although its importance is not yet clear (Woodin 2001).

Contemporary research in set theory includes the study of large cardinals and determinacy. Large cardinals are cardinal numbers with particular properties so strong that the existence of such cardinals cannot be proved in ZFC. The existence of the smallest large cardinal typically studied, an inaccessible cardinal, already implies the consistency of ZFC. Despite the fact that large cardinals have extremely high cardinality, their existence has many ramifications for the structure of the real line. *Determinacy* refers to the possible existence of winning strategies for certain two-player games (the games are said to be *determined*). The existence of these strategies implies structural properties of the real line and other Polish spaces.

Model theory

Model theory studies the models of various formal theories. Here a theory is a set of formulas in a particular formal logic and signature, while a model is a structure that gives a concrete interpretation of the theory. Model theory is closely related to universal algebra and algebraic geometry, although the methods of model theory focus more on logical considerations than those fields.

The set of all models of a particular theory is called an elementary class; classical model theory seeks to determine the properties of models in a particular elementary class, or determine whether certain classes of structures form elementary classes.

The method of quantifier elimination can be used to show that definable sets in particular theories cannot be too complicated. Tarski (1948) established quantifier elimination for real-closed fields, a result which also shows the theory of the field of real numbers is decidable. (He also noted that his methods were equally applicable to algebraically closed fields of arbitrary characteristic.) A modern subfield developing from this is concerned with o-minimal structures.

Morley's categoricity theorem, proved by Michael D. Morley (1965), states that if a first-order theory in a countable language is categorical in some uncountable cardinality, i.e. all models of this cardinality are isomorphic, then it is categorical in all uncountable cardinalities.

A trivial consequence of the continuum hypothesis is that a complete theory with less than continuum many nonisomorphic countable models can have only countably many. Vaught's conjecture, named after Robert Lawson Vaught, says that this is true even independently of the continuum hypothesis. Many special cases of this conjecture have been established.

Recursion theory

Recursion theory, also called **computability theory**, studies the properties of computable functions and the Turing degrees, which divide the uncomputable functions into sets which have the same level of uncomputability. Recursion theory also includes the study of generalized computability and definability. Recursion theory grew from the work of Alonzo Church and Alan Turing in the 1930s, which was greatly extended by Kleene and Post in the 1940s.

Classical recursion theory focuses on the computability of functions from the natural numbers to the natural numbers. The fundamental results establish a robust, canonical class of computable functions with numerous independent, equivalent characterizations using Turing machines, λ calculus, and other systems. More advanced results concern the structure of the Turing degrees and the lattice of recursively enumerable sets.

Generalized recursion theory extends the ideas of recursion theory to computations that are no longer necessarily finite. It includes the study of computability in higher types as well as areas such as hyperarithmetical theory and α -recursion theory.

Contemporary research in recursion theory includes the study of applications such as algorithmic randomness, computable model theory, and reverse mathematics, as well as new results in pure recursion theory.

Algorithmically unsolvable problems

An important subfield of recursion theory studies algorithmic unsolvability; a decision problem or function problem is **algorithmically unsolvable** if there is no possible computable algorithm which returns the correct answer for all legal inputs to the problem. The first results about unsolvability, obtained independently by Church and Turing in 1936, showed that the Entscheidungsproblem is algorithmically unsolvable. Turing proved this by establishing the unsolvability of the halting problem, a result with far-ranging implications in both recursion theory and computer science.

There are many known examples of undecidable problems from ordinary mathematics. The word problem for groups was proved algorithmically unsolvable by Pyotr Novikov in 1955 and independently by W. Boone in 1959. The busy beaver problem, developed by Tibor Radó in 1962, is another well-known example.

Hilbert's tenth problem asked for an algorithm to determine whether a multivariate polynomial equation with integer coefficients has a solution in the integers. Partial progress was made by Julia Robinson, Martin Davis and Hilary Putnam. The algorithmic unsolvability of the problem was proved by Yuri Matiyasevich in 1970 (Davis 1973).

Proof theory and constructive mathematics

Proof theory is the study of formal proofs in various logical deduction systems. These proofs are represented as formal mathematical objects, facilitating their analysis by mathematical techniques. Several deduction systems are commonly considered, including Hilbert-style deduction systems, systems of natural deduction, and the sequent calculus developed by Gentzen.

The study of **constructive mathematics**, in the context of mathematical logic, includes the study of systems in non-classical logic such as intuitionistic logic, as well as the study of predicative systems. An early proponent of predicativism was Hermann Weyl, who showed it is possible to develop a large part of real analysis using only predicative methods (Weyl 1918).

Because proofs are entirely finitary, whereas truth in a structure is not, it is common for work in constructive mathematics to emphasize provability. The relationship between provability in classical (or nonconstructive) systems and provability in intuitionistic (or constructive, respectively) systems is of particular interest. Results such as the Gödel–Gentzen negative translation show that it is possible to embed (or *translate*) classical logic into intuitionistic logic, allowing some properties about intuitionistic proofs to be transferred back to classical proofs.

Recent developments in proof theory include the study of proof mining by Ulrich Kohlenbach and the study of proof-theoretic ordinals by Michael Rathjen.

Connections with computer science

The study of computability theory in computer science is closely related to the study of computability in mathematical logic. There is a difference of emphasis, however. Computer scientists often focus on concrete programming languages and feasible computability, while researchers in mathematical logic often focus on computability as a theoretical concept and on noncomputability.

The theory of semantics of programming languages is related to model theory, as is program verification (in particular, model checking). The Curry–Howard isomorphism between proofs and programs relates to proof theory, especially intuitionistic logic. Formal calculi such as the lambda calculus and combinatory logic are now studied as idealized programming languages.

Computer science also contributes to mathematics by developing techniques for the automatic checking or even finding of proofs, such as automated theorem proving and logic programming.

Descriptive complexity theory relates logics to computational complexity. The first significant result in this area, Fagin's theorem (1974) established that NP is precisely the set of languages expressible by sentences of existential second-order logic.

Foundations of mathematics

In the 19th century, mathematicians became aware of logical gaps and inconsistencies in their field. It was shown that Euclid's axioms for geometry, which had been taught for centuries as an example of the axiomatic method, were incomplete. The use of infinitesimals, and the very definition of function, came into question in analysis, as pathological examples such as Weierstrass' nowhere-differentiable continuous function were discovered.

Cantor's study of arbitrary infinite sets also drew criticism. Leopold Kronecker famously stated "God made the integers; all else is the work of man," endorsing a return to the study of finite, concrete objects in mathematics. Although Kronecker's argument was carried forward by constructivists in the 20th century, the mathematical community as a whole rejected them. David Hilbert argued in favor of the study of the infinite, saying "No one shall expel us from the Paradise that Cantor has created."

Mathematicians began to search for axiom systems that could be used to formalize large parts of mathematics. In addition to removing ambiguity from previously-naïve terms such as function, it was hoped that this axiomatization would allow for consistency proofs. In the 19th century, the main method of proving the consistency of a set of axioms was to provide a model for it. Thus, for example, non-Euclidean geometry can be proved consistent by defining *point* to mean a point on a fixed sphere and *line* to mean a great circle on the sphere. The resulting structure, a model of elliptic geometry, satisfies the axioms of plane geometry except the parallel postulate.

With the development of formal logic, Hilbert asked whether it would be possible to prove that an axiom system is consistent by analyzing the structure of possible proofs in the system, and showing through this analysis that it is impossible to prove a contradiction. This idea led to the study of proof theory. Moreover, Hilbert proposed that the analysis should be entirely concrete, using the term *finitary* to refer to the methods he would allow but not precisely defining them. This project, known as Hilbert's program, was seriously affected by Gödel's incompleteness theorems, which show that the consistency of formal theories of arithmetic cannot be established using methods formalizable in those theories. Gentzen showed that it is possible to produce a proof of the consistency of arithmetic in a finitary system augmented with axioms of transfinite induction, and the techniques he developed to so do were seminal in proof theory.

A second thread in the history of foundations of mathematics involves nonclassical logics and constructive mathematics. The study of constructive mathematics includes many different programs with various definitions of *constructive*. At the most accommodating end, proofs in ZF set theory that do not use the axiom of choice are called constructive by many mathematicians. More limited versions of constructivism limit themselves to natural numbers, number-theoretic functions, and sets of natural numbers (which can be used to represent real numbers, facilitating the study of mathematical analysis). A common idea is that a concrete means of computing the values of the function must be known before the function itself can be said to exist.

In the early 20th century, Luitzen Egbertus Jan Brouwer founded intuitionism as a philosophy of mathematics. This philosophy, poorly understood at first, stated that in order for a mathematical statement to be true to a mathematician, that person must be able to *intuit* the statement, to not only believe its truth but understand the reason for its truth. A consequence of this definition of truth was the rejection of the law of the excluded middle, for there are statements that, according to Brouwer, could not be claimed to be true while their negations also could not be claimed true. Brouwer's philosophy was influential, and the cause of bitter disputes among prominent mathematicians. Later, Kleene and Kreisel would study formalized versions of intuitionistic logic (Brouwer rejected formalization, and presented his work in unformalized natural language). With the advent of the BHK interpretation and Kripke models, intuitionism became easier to reconcile with classical mathematics.

See also

- List of mathematical logic topics
- List of computability and complexity topics
- List of set theory topics
- List of first-order theories
- Knowledge representation
- Metalogic
- Logic symbols

Notes

- [1] Undergraduate texts include Boolos, Burgess, and Jeffrey (2002), Enderton (2001), and Mendelson (1997). A classic graduate text by Shoenfield (2001) first appeared in 1967.
- [2] A detailed study of this terminology is given by Soare (1996).
- [3] Ferreirós (2001) surveys the rise of first-order logic over other formal logics in the early 20th century.

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- Mathematical Logic around the world (<http://world.logic.at/>)
- Polyvalued logic (<http://home.swipnet.se/~w-33552/logic/home/index.htm>)
- *forall x: an introduction to formal logic* (<http://www.fecundity.com/logic/>), by P.D. Magnus, is a free textbook.
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- Stanford Encyclopedia of Philosophy: Classical Logic (<http://plato.stanford.edu/entries/logic-classical/>) – by Stewart Shapiro.
- Stanford Encyclopedia of Philosophy: First-order Model Theory (<http://plato.stanford.edu/entries/modeltheory-fo/>) – by Wilfrid Hodges.
- The London Philosophy Study Guide (<http://www.ucl.ac.uk/philosophy/LPSG/>) offers many suggestions on what to read, depending on the student's familiarity with the subject:
 - Mathematical Logic (<http://www.ucl.ac.uk/philosophy/LPSG/MathLogic.htm>)
 - Set Theory & Further Logic (<http://www.ucl.ac.uk/philosophy/LPSG/SetTheory.htm>)
 - Philosophy of Mathematics (<http://www.ucl.ac.uk/philosophy/LPSG/PhilMath.htm>)

Heyting arithmetic

In mathematical logic, **Heyting arithmetic** (sometimes abbreviated HA) is an axiomatization of arithmetic in accordance with the philosophy of intuitionism. It is named after Arend Heyting, who first proposed it.

Heyting arithmetic adopts the axioms of Peano arithmetic (PA), but uses intuitionistic logic as its rules of inference. In particular, the law of the excluded middle does not hold in general, though the induction axiom can be used to prove many specific cases. For instance, one can prove that $\forall x, y \in \mathbf{N} : x = y \vee x \neq y$ is a theorem (any two natural numbers are either equal to each other, or not equal to each other). In fact, since "=" is the only predicate symbol in Heyting arithmetic, it then follows that, for any quantifier-free formula p , $\forall x, y, z, \dots \in \mathbf{N} : p \vee \neg p$ is a theorem (where $x, y, z, \dots$ are the free variables in p).

Kurt Gödel studied the relationship between Heyting arithmetic and Peano arithmetic. He used the Gödel–Gentzen negative translation to prove in 1933 that if HA is consistent, then PA is also consistent.

Heyting arithmetic should not be confused with Heyting algebras, which are the intuitionistic analogue of Boolean algebras.

External links

- Stanford Encyclopedia of Philosophy: "Intuitionistic Number Theory ^[1]" by Joan Moschovakis.

References

- [1] <http://plato.stanford.edu/entries/logic-intuitionistic/#IntNumTheHeyAri>

Symbolic logic

Symbolic logic may refer to:

- First-order logic, a system of formal logic
 - Mathematical logic, a field of mathematics
-

Metatheory

A **metatheory** or **meta-theory** is a theory whose subject matter is some other theory. In other words it is a theory about a theory. Statements made in the metatheory about the theory are called metatheorems.

According to the systemic TOGA meta-theory ^[1], a meta-theory may refer to the specific point of view on a theory and to its subjective meta-properties, but not to its application domain. In the above sense, a theory **T** of the domain **D** is a meta-theory if **D** is a theory or a set of theories. A general theory is not a meta-theory because its domain **D** are not theories.

The following is an example of a meta-theoretical statement:^[2]

“Any physical theory is always provisional, in the sense that it is only a hypothesis; you can never prove it. No matter how many times the results of experiments agree with some theory, you can never be sure that the next time the result will not contradict the theory. On the other hand, you can disprove a theory by finding even a single observation that disagrees with the predictions of the theory.”

Meta-theory belongs to the philosophical specialty of epistemology and metamathematics, as well as being an object of concern to the area in which the individual theory is conceived. An emerging domain of meta-theories is systemics.

Taxonomy

Examining groups of related theories, a first finding may be to identify classes of theories, thus specifying a taxonomy of theories. A proof engendered by a metatheory is called a *metatheorem*.

History

The concept burst upon the scene of twentieth-century philosophy as a result of the work of the German mathematician David Hilbert, who in 1905 published a proposal for proof of the consistency of mathematics, creating the field of metamathematics. His hopes for the success of this proof were dashed by the work of Kurt Gödel who in 1931 proved this to be unattainable by his incompleteness theorems. Nevertheless, his program of unsolved mathematical problems, out of which grew this metamathematical proposal, continued to influence the direction of mathematics for the rest of the twentieth century.

The study of metatheory became widespread during the rest of that century by its application in other fields, notably scientific linguistics and its concept of metalanguage.

See also

- meta-
- meta-knowledge
- Metalogic
- Metamathematics
- Philosophy of social science

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[2] Stephen Hawking in *A Brief History of Time*

External links

- Meta-theoretical Issues (2003), Lyle Flint (<http://www.bsu.edu/classes/flint/comm360/metatheo.html>)

Metalogic

Metalogic is the study of the metatheory of logic. While *logic* is the study of the manner in which logical systems can be used to decide the correctness of arguments, metalogic studies the properties of the logical systems themselves.^[1] According to Geoffrey Hunter, while logic concerns itself with the "truths of logic," metalogic concerns itself with the theory of "sentences used to express truths of logic."^[2]

The basic objects of study in metalogic are formal languages, formal systems, and their interpretations. The study of interpretation of formal systems is the branch of mathematical logic known as model theory, while the study of deductive apparatus is the branch known as proof theory.

History

Metalogical questions have been asked since the time of Aristotle. However, it was only with the rise of formal languages in the late 19th and early 20th century that investigations into the foundations of logic began to flourish. In 1904, David Hilbert observed that in investigating the foundations of mathematics that logical notions are presupposed, and therefore a simultaneous account of metalogical and metamathematical principles was required. Today, metalogic and metamathematics are largely synonymous with each other, and both have been substantially subsumed by mathematical logic in academia.

Important distinctions in metalogic

Metalinguage—Object language

In metalogic, formal languages are sometimes called *object languages*. The language used to make statements about an object language is called a *metalinguage*. This distinction is a key difference between logic and metalogic. While logic deals with *proofs in a formal system*, expressed in some formal language, metalogic deals with *proofs about a formal system* which are expressed in a metalanguage about some object language.

Syntax—semantics

In metalogic, 'syntax' has to do with formal languages or formal systems without regard to any interpretation of them, whereas, 'semantics' has to do with interpretations of formal languages. The term 'syntactic' has a slightly wider scope than 'proof-theoretic', since it may be applied to properties of formal languages without any deductive systems, as well as to formal systems. 'Semantic' is synonymous with 'model-theoretic'.

Use–mention

In metalogic, the words 'use' and 'mention', in both their noun and verb forms, take on a technical sense in order to identify an important distinction.^[2] The *use–mention distinction* (sometimes referred to as the *words-as-words distinction*) is the distinction between *using* a word (or phrase) and *mentioning* it. Usually it is indicated that an expression is being mentioned rather than used by enclosing it in quotation marks, printing it in italics, or setting the expression by itself on a line. The enclosing in quotes of an expression gives us the name of an expression, for example:

'Metalogic' is the name of this article.

This article is about metalogic.

Type–token

The *type-token distinction* is a distinction in metalogic, that separates an abstract concept from the objects which are particular instances of the concept. For example, the particular bicycle in your garage is a token of the type of thing known as "The bicycle." Whereas, the bicycle in your garage is in a particular place at a particular time, that is not true of "the bicycle" as used in the sentence: "*The bicycle* has become more popular recently." This distinction is used to clarify the meaning of symbols of formal languages.

Overview

Formal language

A *formal language* is an organized set of symbols the essential feature of which is that it can be precisely defined in terms of just the shapes and locations of those symbols. Such a language can be defined, then, without any reference to any meanings of any of its expressions; it can exist before any interpretation is assigned to it—that is, before it has any meaning. First order logic is expressed in some formal language. A formal grammar determines which symbols and sets of symbols are formulas in a formal language.

A formal language can be defined formally as a set A of strings (finite sequences) on a fixed alphabet α . Some authors, including Carnap, define the language as the ordered pair $\langle \alpha, A \rangle$.^[3] Carnap also requires that each element of α must occur in at least one string in A .

Formation rules

Formation rules (also called *formal grammar*) are a precise description of the well-formed formulas of a formal language. It is synonymous with the set of strings over the alphabet of the formal language which constitute well formed formulas. However, it does not describe their semantics (i.e. what they mean).

Formal systems

A *formal system* (also called a *logical calculus*, or a *logical system*) consists of a formal language together with a deductive apparatus (also called a *deductive system*). The deductive apparatus may consist of a set of transformation rules (also called *inference rules*) or a set of axioms, or have both. A formal system is used to derive one expression from one or more other expressions.

A *formal system* can be formally defined as an ordered triple $\langle \alpha, \mathcal{I}, \mathcal{D} \rangle$, where $\mathcal{D}$ is the relation of direct derivability. This relation is understood in a comprehensive sense such that the primitive sentences of the formal system are taken as directly derivable from the empty set of sentences. Direct derivability is a relation between a sentence and a finite, possibly empty set of sentences. Axioms are laid down in such a way that every first place member of $\mathcal{D}$ is a member of $\mathcal{I}$ and every second place member is a finite subset of $\mathcal{I}$.

It is also possible to define a *formal system* using only the relation $\mathcal{D}$. In this way we can omit $\mathcal{I}$, and α in the definitions of *interpreted formal language*, and *interpreted formal system*. However, this method can be more difficult to understand and work with.^[3]

Formal proofs

A *formal proof* is a sequence of well-formed formulas of a formal language, the last one of which is a theorem of a formal system. The theorem is a syntactic consequence of all the well formed formulae preceding it in the proof. For a well formed formula to qualify as part of a proof, it must be the result of applying a rule of the deductive apparatus of some formal system to the previous well formed formulae in the proof sequence.

Interpretations

An *interpretation* of a formal system is the assignment of meanings, to the symbols, and truth-values to the sentences of the formal system. The study of interpretations is called Formal semantics. *Giving an interpretation* is synonymous with *constructing a model*.

Results in metalogic

Results in metalogic consist of such things as formal proofs demonstrating the consistency, completeness, and decidability of particular formal systems.

Major results in metalogic include:

- Proof of the uncountability of the set of all subsets of the set of natural numbers (Cantor's theorem 1891)
- Löwenheim-Skolem theorem (Leopold Löwenheim 1915 and Thoralf Skolem 1919)
- Proof of the consistency of truth-functional propositional logic (Emil Post 1920)
- Proof of the semantic completeness of truth-functional propositional logic (Paul Bernays 1918),^[4] (Emil Post 1920)^[2]
- Proof of the syntactic completeness of truth-functional propositional logic (Emil Post 1920)^[2]
- Proof of the decidability of truth-functional propositional logic (Emil Post 1920)^[2]
- Proof of the consistency of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the semantic completeness of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the decidability of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the consistency of first order predicate logic (David Hilbert and Wilhelm Ackermann 1928)
- Proof of the semantic completeness of first order predicate logic (Gödel's completeness theorem 1930)
- Proof of the undecidability of first order predicate logic (Church's theorem 1936)
- Gödel's first incompleteness theorem 1931
- Gödel's second incompleteness theorem 1931

See also

- Metamathematics

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- [4] Hao Wang, Reflections on Kurt Gödel

Quantum Biographies

Niels Bohr

Niels Bohr	
	
Born	Niels Henrik David Bohr7 October 1885Copenhagen, Denmark
Died	18 November 1962 (aged 77)Copenhagen, Denmark
Nationality	Denmark
Fields	Physics
Institutions	University of Copenhagen University of Cambridge University of Manchester
Alma mater	University of Copenhagen
Doctoral advisor	Christian Christiansen
Other academic advisors	J. J. Thomson Ernest Rutherford
Doctoral students	Hendrik Anthony Kramers
Known for	Copenhagen interpretation Complementarity Bohr model Sommerfeld–Bohr theory BKS theory Bohr-Einstein debates Bohr magneton
Influences	Ernest Rutherford
Influenced	Werner Heisenberg Wolfgang Pauli Paul Dirac Lise Meitner Max Delbrück and many others
Notable awards	Nobel Prize in Physics (1922) Franklin Medal (1929)

Signature 
Notes Harald Bohr is his younger brother, and Aage Bohr is his son.

Niels Henrik David Bohr (Danish pronunciation: [nels 'bø̥ʁ?]; 7 October 1885 – 18 November 1962) was a Danish physicist who made fundamental contributions to understanding atomic structure and quantum mechanics, for which he received the Nobel Prize in Physics in 1922. Bohr mentored and collaborated with many of the top physicists of the century at his institute in Copenhagen. He was part of a team of physicists working on the Manhattan Project. Bohr married Margrethe Nørlund in 1912, and one of their sons, Aage Bohr, grew up to be an important physicist who in 1975 also received the Nobel prize. Bohr has been described as one of the most influential scientists of the 20th century.^[1]

Biography

Early years

Bohr was born in Copenhagen, Denmark, in 1885. His father, Christian Bohr, a devout Lutheran, was professor of physiology at the University of Copenhagen (it is his name which is given to the Bohr shift or Bohr effect), while his mother, Ellen Adler Bohr, came from a wealthy Jewish family prominent in Danish banking and parliamentary circles. His brother was Harald Bohr, a mathematician and Olympic footballer who played on the Danish national team. Niels Bohr was a passionate footballer as well, and the two brothers played a number of matches for the Copenhagen-based Akademisk Boldklub, with Niels in goal. There is, however, no truth in the oft-repeated claim that Niels Bohr emulated his brother Harald by playing for the Danish national team.^[2]

In 1903 Bohr enrolled as an undergraduate at Copenhagen University, initially studying philosophy and mathematics. In 1905, prompted by a gold medal competition sponsored by the Royal Danish Academy of Sciences and Letters, he conducted a series of experiments to examine the properties of surface tension, using his father's laboratory in the university, familiar to him from assisting there since childhood. His essay won the prize, and it was this success that decided Bohr to abandon philosophy and adopt physics.^[3] As a student under Christian Christiansen he received his doctorate in 1911. As a post-doctoral student, Bohr first conducted experiments under J. J. Thomson at Trinity College, Cambridge. In 1912 he joined Ernest Rutherford at Manchester University and he adapted Rutherford's nuclear structure to Max Planck's quantum theory and so obtained a theory of atomic structure which, with later improvements, mainly as a result of Heisenberg's concepts, remains valid to this day. On the basis of Rutherford's theories, Bohr published his model of atomic structure in 1913, introducing the theory of electrons traveling in orbits around the atom's nucleus, the chemical properties of the element being largely determined by the number of electrons in the outer orbits. Bohr introduced the idea that an electron could drop from a higher-energy orbit to a lower one, emitting a photon (light quantum) of discrete energy. This became a basis for quantum theory. After four productive years with Ernest Rutherford in Manchester, Bohr returned to Denmark becoming in 1918 director of the newly created Institute of Theoretical Physics.

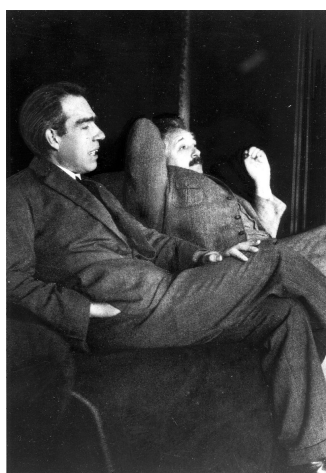
Niels Bohr and his wife Margrethe Nørlund Bohr had six sons. Their oldest died in a tragic boating accident and another died from childhood meningitis. The others went on to lead successful lives, including Aage Bohr, who became a very successful physicist and, like his father, won a Nobel Prize in physics, in 1975.

Physics

In 1916, Niels Bohr became a professor at the University of Copenhagen. With the assistance of the Danish government and the Carlsberg Foundation, he succeeded in founding the Institute of Theoretical Physics in 1921, of which he became director.^[4] In 1922, Bohr was awarded the Nobel Prize in physics "for his services in the investigation of the structure of atoms and of the radiation emanating from them." Bohr's institute served as a focal point for theoretical physicists in the 1920s and '30s, and most of the world's best known theoretical physicists of that period spent some time there.



Niels Bohr as a young man. Exact date of photo unknown.



Niels Bohr and Albert Einstein debating quantum theory at Paul Ehrenfest's home in Leiden (December 1925).

Bohr also conceived the principle of complementarity: that items could be separately analyzed as having several contradictory properties. For example, physicists currently conclude that light behaves either as a wave or a stream of particles depending on the experimental framework — two apparently mutually exclusive properties — on the basis of this principle. Bohr found philosophical applications for this daringly original principle. Albert Einstein much preferred the determinism of classical physics over the probabilistic new quantum physics (to which Max Planck and Einstein himself had contributed). He and Bohr had good-natured arguments over the truth of this principle throughout their lives (see Bohr–Einstein debates).

Werner Heisenberg worked as an assistant to Bohr and university lecturer in Copenhagen from 1926 to 1927. It was in Copenhagen, in 1927, that Heisenberg developed his uncertainty principle, while working on the mathematical foundations of quantum mechanics. Heisenberg was later to be head of the German atomic bomb project. In 1941, during the German occupation of Denmark in World War II, Bohr was visited by Heisenberg in Copenhagen (see section below). In 1943, shortly before he was to be arrested by the German police, Bohr escaped to Sweden, and then traveled to London.

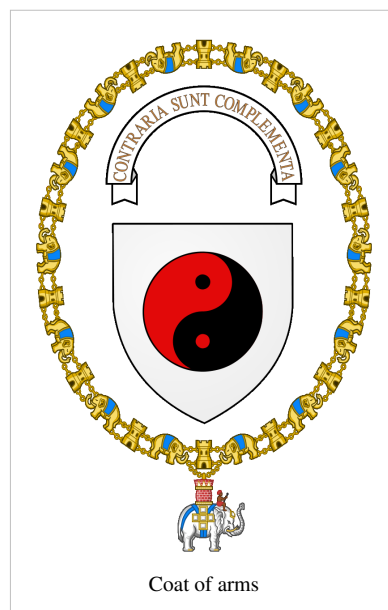
Atomic research

Niels Bohr worked at the top-secret Los Alamos laboratory in New Mexico, U.S., on the Manhattan Project, where he was known by the assumed name of *Nicholas Baker* for security reasons.^[5] His role in the project was important as he was a knowledgeable consultant or "father confessor" on the project. He was concerned about a nuclear arms race, and is quoted as saying, "That is why I went to America. They didn't need my help in making the atom bomb."^[6]

Bohr believed that atomic secrets should be shared by the international scientific community. After meeting with Bohr, J. Robert Oppenheimer suggested Bohr visit President Franklin D. Roosevelt to convince him that the Manhattan Project should be shared with the Russians in the hope of speeding up its results.

Roosevelt suggested Bohr return to the United Kingdom to try to win British approval. Winston Churchill disagreed with the idea of openness towards the Russians to the point that he wrote in a letter: "It seems to me Bohr ought to be confined or at any rate made to see that he is very near the edge of mortal crimes."^[7]

After the war Bohr returned to Copenhagen, advocating the peaceful use of nuclear energy. When awarded the Order of the Elephant by the Danish government, he designed his own coat of arms which featured a *taijitu* (symbol of yin and yang) and the Latin motto *contraria sunt complementa*: opposites are complementary.^[8] He died in Copenhagen in 1962 of heart failure.^[9] He is buried in the Assistens Kirkegård in the Nørrebro section of Copenhagen.



Contributions to Physics and Chemistry

- The Bohr model of the atom, the theory that electrons travel in discrete orbits around the atom's nucleus.
- The shell model of the atom, where the chemical properties of an element are determined by the electrons in the outermost orbit.
- The correspondence principle, the basic tool of Old quantum theory.
- The liquid drop model of the atomic nucleus.
- Identified the isotope of uranium that was responsible for slow-neutron fission - ^{235}U .^[10]
- Much work on the Copenhagen interpretation of quantum mechanics.
- The principle of complementarity: that items could be separately analyzed as having several contradictory properties.

Kierkegaard's influence on Bohr

It is generally accepted that Bohr read the 19th century Danish philosopher Søren Kierkegaard. Richard Rhodes argues in *The Making of the Atomic Bomb* that Bohr was influenced by Kierkegaard via the philosopher Harald Høffding, who was strongly influenced by Kierkegaard and who was an old friend of Bohr's father. In 1909, Bohr sent his brother Kierkegaard's *Stages on Life's Way* as a birthday gift. In the enclosed letter, Bohr wrote, "It is the only thing I have to send home; but I do not believe that it would be very easy to find anything better.... I even think it is one of the most delightful things I have ever read." Bohr enjoyed Kierkegaard's language and literary style, but mentioned that he had some "disagreement with [Kierkegaard's ideas]."^[11]

Given this, there has been some dispute over whether Kierkegaard influenced Bohr's philosophy and science. David Favrholdt^[12] argues that Kierkegaard had minimal influence over Bohr's work; taking Bohr's statement about disagreeing with Kierkegaard at face value, while Jan Faye^[13] endorses the opposing point of view by arguing that one can disagree with the content of a theory while accepting its general premises and structure.^[14]

Relationship with Heisenberg

Bohr and Werner Heisenberg enjoyed a strong mentor/protégé relationship up to the onset of World War II. Heisenberg had made Bohr aware of his talent during a lecture in 1922 in Göttingen. During the mid-1920s, Heisenberg worked with Bohr at the institute in Copenhagen. Heisenberg, like most of Bohr's assistants, learned Danish. Heisenberg's uncertainty principle was developed during this period, as was Bohr's complementarity principle.

By the time of World War II, the relationship became strained; this was in part because Bohr, with his partially Jewish heritage, remained in occupied Denmark, while Heisenberg remained in Germany and became head of the German nuclear effort. Heisenberg made a famous visit to Bohr in September 1941 and during a private moment it seems that he began to address nuclear energy and morality as well as the war. Neither Bohr nor Heisenberg spoke about it in any detail or left written records of this part of the meeting and they were alone and outside.^[15] Bohr

seems to have reacted by terminating that conversation abruptly while not giving Heisenberg hints in any direction.

While some suggest that the relationship became strained at this meeting, other evidence shows that the level of contact had been reduced considerably for some time already. Heisenberg suggested that the fracture occurred later. In correspondence to his wife, Heisenberg described the final visit of the trip: "Today I was once more, with Weizsaecker, at Bohr's. In many ways this was especially nice, the conversation revolved for a large part of the evening around purely human concerns, Bohr was reading aloud, I played a Mozart Sonata (A-Major)."^[16] Ivan Supek, one of Heisenberg's students and friends, claimed that the main figure of the meeting was actually Weizsäcker who tried to persuade Bohr to mediate peace between Great Britain and Germany.^[17]

Tube Alloys

"Tube Alloys" was the code-name for the British nuclear weapon program. British intelligence inquired about Bohr's availability for work or insights of particular value. Bohr's reply made it clear that he could not help. This reply, like his reaction to Heisenberg, made sure that if Gestapo intercepted anything attributed to Bohr it would point to no knowledge regarding nuclear energy as it stood in 1941. This does not exclude the possibility that Bohr privately made calculations going further than his work in 1939 with Wheeler.

After leaving Denmark in the dramatic day and night (October 1943) when most Jews were able to escape to Sweden due to exceptional circumstances (see Rescue of the Danish Jews), Bohr was quickly asked again to join the British effort and he was flown to the UK. He was evacuated from Stockholm in 1943 in an unarmed De Havilland Mosquito operated by British Overseas Airways Corporation (BOAC). Passengers on BOAC's Mosquitos were carried in an improvised cabin in the bomb bay. The flight almost ended in tragedy as Bohr did not don his oxygen equipment as instructed and passed out at high altitude. He would have died had not the pilot surmising from Bohr's lack of response to intercom communication that he had lost consciousness, descended to a lower altitude for the remainder of the flight. Bohr's comment was that he had slept like a baby for the entire flight.

As part of the UK team on "Tube Alloys" Bohr went to Los Alamos. Oppenheimer credited Bohr warmly for his guiding help during certain discussions among scientists there. Discreetly, he met President Franklin D. Roosevelt and later Winston Churchill to warn against the perilous perspectives that would follow from separate development of nuclear weapons by several powers rather than some form of controlled sharing of the knowledge, which would spread quickly in any case. Only in the 1950s, after the Soviet Union's first nuclear weapon test, was it possible to create the International Atomic Energy Agency along the lines of Bohr's suggestion.

Speculation

In 1957, while the author Robert Jungk was working on the book *Brighter Than a Thousand Suns*, Heisenberg wrote to Jungk explaining that he had visited Copenhagen to communicate to Bohr his view that scientists on either side should help prevent development of the atomic bomb, that the German attempts were entirely focused on energy production and that Heisenberg's circle of colleagues tried to keep it that way.^[18] Heisenberg acknowledged that his cryptic approach of the subject had so alarmed Bohr that the discussion failed. Heisenberg nuanced his claims and avoided the implication that he and his colleagues had sabotaged the bomb effort; this nuance was lost in Jungk's original publication of the book, which implied that the German atomic bomb project was obstructed by Heisenberg.

When Bohr saw Jungk's erroneous depiction in the Danish translation of the book, he disagreed. He drafted (but never sent) a letter to Heisenberg, stating that while Heisenberg had indeed discussed the subject of nuclear weapons in Copenhagen, Heisenberg had never alluded to the fact that he might be resisting efforts to build such weapons. Bohr dismissed the idea of any pact as hindsight.^[19]

Michael Frayn's play *Copenhagen*, which was performed in London (for five years), Copenhagen, Gothenburg, Rome, Athens, Geneva and on Broadway in New York, explores what might have happened at the 1941 meeting between Heisenberg and Bohr. Frayn points in particular to the onus of being one of the few to understand what it would mean to create a nuclear weapon.

Open World

Bohr advocated informing the Soviet authorities that the atomic bomb would soon be in use. In 1944 he obtained an audience with Winston Churchill, who became worried about whether Bohr was a security risk.^[20] In 1950 he addressed an 'Open Letter' to the United Nations.^[21]

Legacy

- He was one of the founding fathers of CERN in 1954.^[22]
- Received the first ever Atoms for Peace Award in 1957.
- In 1965, three years after Bohr's death, the Institute of Physics at the University of Copenhagen changed its name to the Niels Bohr Institute.
- The Bohr models semicentennial was commemorated in Denmark on 21 November 1963 with a postage stamp depicting Bohr, the hydrogen atom and the formula for the difference of any two hydrogen energy levels:

$$h\nu = \epsilon_2 - \epsilon_1.$$
- Bohrium (a chemical element, atomic number 107) is named in honour of Niels Bohr.
- Hafnium, another chemical element, whose properties were predicted by Niels Bohr, was named by him after Hafnia, Copenhagen's Latin name.
- Asteroid 3948 Bohr is named after him.
- The Centennial of Bohr's birth was commemorated in Denmark on 3 October 1985 with a postage stamp depicting Bohr with his wife Margrethe.
- In 1997 the Danish National Bank started circulating the 500-krone banknote with the portrait of Niels Bohr smoking a pipe.^[23] ^[24]
- Bohr has been a common name in Europe since the Middle Ages.^[25] It remains fairly common in Europe and spread to the U.S. with pilgrims named Bohr settling there. There was an notable increase in the middle name Bohr throughout Europe and America following Niels Bohr's death.

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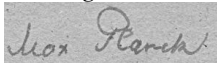
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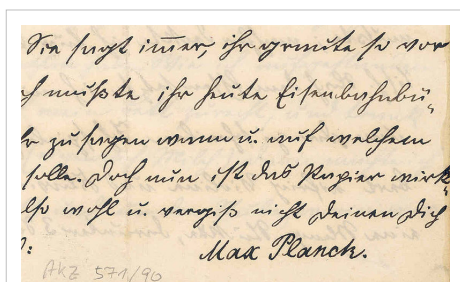
Max Planck

Max Planck	
	
Born	April 23, 1858Kiel, Duchy of Holstein
Died	October 4, 1947 (aged 89)Göttingen, Lower Saxony, Germany
Nationality	German
Fields	Physics
Institutions	University of Kiel University of Berlin University of Göttingen Kaiser-Wilhelm-Gesellschaft
Alma mater	Ludwig Maximilian University of Munich
Doctoral advisor	Alexander von Brill
Doctoral students	Gustav Ludwig Hertz Erich Kretschmann Walther Meißner Walter Schottky Max von Laue Max Abraham Moritz Schlick Walther Bothe Julius Edgar Lilienfeld
Known for	Planck constant Planck postulate Planck's law of black body radiation
Notable awards	Nobel Prize in Physics (1918)
Signature 	
Notes	He is the father of Erwin Planck who was executed in 1945 by the Gestapo for his part in the July 20 plot.

Max Planck (April 23, 1858 – October 4, 1947) was a German physicist. He is considered to be the founder of the quantum theory, and thus one of the most important physicists of the twentieth century. Planck was awarded the Nobel Prize in Physics in 1918.

Life and career

Planck came from a traditional, intellectual family. His paternal great-grandfather and grandfather were both theology professors in Göttingen; his father was a law professor in Kiel and Munich; and his paternal uncle was a judge.

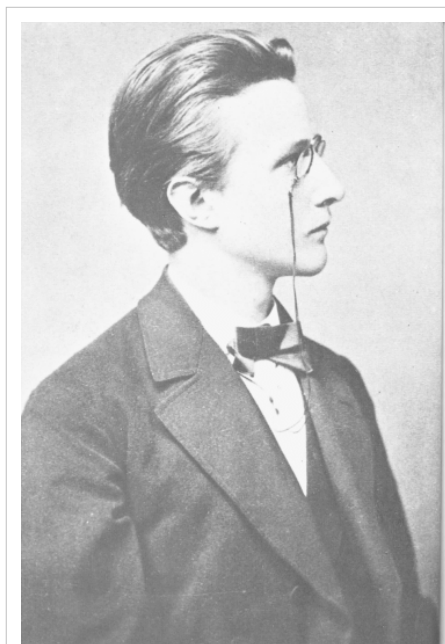


Max Planck's signature at ten years of age.

Planck was born in Kiel, Holstein, to Johann Julius Wilhelm Planck and his second wife, Emma Patzig. He was baptised with the name of *Karl Ernst Ludwig Marx Planck*; of his given names, *Marx* (a now obsolete variant of *Markus* or maybe simply an error for *Max*, which is actually short for *Maximilian*) was indicated as the primary name.^[1] However, by the age of ten he signed with the name *Max* and used this for the rest of his life.^[2]

He was the sixth child in the family, though two of his siblings were from his father's first marriage. Among his earliest memories was the marching of Prussian and Austrian troops into Kiel during the Danish-Prussian war of 1864. In 1867 the family moved to Munich, and Planck enrolled in the Maximilians gymnasium school, where he came under the tutelage of Hermann Müller, a mathematician who took an interest in the youth, and taught him astronomy and mechanics as well as mathematics. It was from Müller that Planck first learned the principle of conservation of energy. Planck graduated early, at age 17.^[3] This is how Planck first came in contact with the field of physics.

Planck was gifted when it came to music. He took singing lessons and played piano, organ and cello, and composed songs and operas. However, instead of music he chose to study physics.



Planck as a young man, 1878

The Munich physics professor Philipp von Jolly advised Planck against going into physics, saying, "in this field, almost everything is already discovered, and all that remains is to fill a few holes." Planck replied that he did not wish to discover new things, only to understand the known fundamentals of the field, and began his studies in 1874 at the University of Munich. Under Jolly's supervision, Planck performed the only experiments of his scientific career, studying the diffusion of hydrogen through heated platinum, but transferred to theoretical physics.

In 1877 he went to Berlin for a year of study with physicists Hermann von Helmholtz and Gustav Kirchhoff and the mathematician Karl Weierstrass. He wrote that Helmholtz was never quite prepared, spoke slowly, miscalculated endlessly, and bored his listeners, while Kirchhoff spoke in carefully prepared lectures which were dry and monotonous. He soon became close friends with Helmholtz. While there he undertook a program of mostly self-study of Clausius's writings, which led him to choose heat theory as his field.

In October 1878 Planck passed his qualifying exams and in February 1879 defended his dissertation, *Über den zweiten Hauptsatz der mechanischen Wärmetheorie* (*On the second law of thermodynamics*). He briefly taught mathematics and physics at his former school in Munich.

In June 1880 he presented his habilitation thesis, *Gleichgewichtszustände isotroper Körper in verschiedenen Temperaturen* (*Equilibrium states of isotropic bodies at different temperatures*).

Academic career

With the completion of his habilitation thesis, Planck became an unpaid private lecturer in Munich, waiting until he was offered an academic position. Although he was initially ignored by the academic community, he furthered his work on the field of heat theory and discovered one after another the same thermodynamical formalism as Gibbs without realizing it. Clausius's ideas on entropy occupied a central role in his work.

In April 1885 the University of Kiel appointed Planck as associate professor of theoretical physics. Further work on entropy and its treatment, especially as applied in physical chemistry, followed. He proposed a thermodynamic basis for Svante Arrhenius's theory of electrolytic dissociation.

Within four years he was named the successor to Kirchhoff's position at the University of Berlin — presumably thanks to Helmholtz's intercession — and by 1892 became a full professor. In 1907 Planck was offered Boltzmann's position in Vienna, but turned it down to stay in Berlin. During 1909, as University of Berlin professor, eight of his lectures were used by the Ernest Kempton Adams Fund for Physical Research in Theoretical Physics at Columbia University in New York City for a series of lectures translated by Columbia University professor A. P. Wills.^[4] He retired from Berlin on January 10, 1926, and was succeeded by Erwin Schrödinger.

Family

In March 1887 Planck married Marie Merck (1861–1909), sister of a school fellow, and moved with her into a sublet apartment in Kiel. They had four children: Karl (1888–1916), the twins Emma (1889–1919) and Grete (1889–1917), and Erwin (1893–1945).

After the apartment in Berlin, the Planck family lived in a villa in Berlin-Grunewald, Wangenheimstraße 21. Several other professors of Berlin University lived nearby, among them theologian Adolf von Harnack, who became a close friend of Planck. Soon the Planck home became a social and cultural centre. Numerous well-known scientists, such as Albert Einstein, Otto Hahn and Lise Meitner were frequent visitors. The tradition of jointly performing music had already been established in the home of Helmholtz.

After several happy years, in July 1909 Marie Planck died, possibly from tuberculosis. In March 1911 Planck married his second wife, Marga von Hoesslin (1882–1948); in December his third son Hermann was born.

During the First World War Planck's second son Erwin was taken prisoner by the French in 1914, while his oldest son Karl was killed in action at Verdun. Grete died in 1917 while giving birth to her first child. Her sister died the same way two years later, after having married Grete's widower. Both granddaughters survived and were named after their mothers. Planck endured these losses stoically.

In January 1945, Erwin, to whom he had been particularly close, was sentenced to death by the Nazi Volksgerichtshof because of his participation in the failed attempt to assassinate Hitler in July 1944. Erwin was executed on 23 January 1945.^[5]

- Wives: Marie Merck (m. 1887), Marga von Hoesslin (m. 1910)
- Children: Karl (1888–1916), twins Emma (1889–1919) and Grete (1889–1917), Erwin (1893–1945), Hermann (1911–1954)

Professor at Berlin University

In Berlin, Planck joined the local Physical Society. He later wrote about this time: "In those days I was essentially the only theoretical physicist there, whence things were not so easy for me, because I started mentioning entropy, but this was not quite fashionable, since it was regarded as a mathematical spook". Thanks to his initiative, the various local Physical Societies of Germany merged in 1898 to form the German Physical Society (Deutsche Physikalische Gesellschaft, DPG); from 1905 to 1909 Planck was the president.

Planck started a six-semester course of lectures on theoretical physics, "dry, somewhat impersonal" according to Lise Meitner, "using no notes, never making mistakes, never faltering; the best lecturer I ever heard" according to an

English participant, James R. Partington, who continues: "There were always many standing around the room. As the lecture-room was well heated and rather close, some of the listeners would from time to time drop to the floor, but this did not disturb the lecture". Planck did not establish an actual "school"; the number of his graduate students was only about 20, among them:

1897 Max Abraham (1875–1922)

1904 Moritz Schlick (1882–1936)

1906 Walther Meißner (1882–1974)

1906 Max von Laue (1879–1960)

1907 Fritz Reiche (1883–1960)

1912 Walter Schottky (1886–1976)

1914 Walther Bothe (1891–1957)

Black-body radiation

In 1894 Planck turned his attention to the problem of black-body radiation. He had been commissioned by electric companies to create maximum light from lightbulbs with minimum energy. The problem had been stated by Kirchhoff in 1859: how does the intensity of the electromagnetic radiation emitted by a black body (a perfect absorber, also known as a cavity radiator) depend on the frequency of the radiation (i.e., the color of the light) and the temperature of the body? The question had been explored experimentally, but no theoretical treatment agreed with experimental values. Wilhelm Wien proposed Wien's law, which correctly predicted the behaviour at high frequencies, but failed at low frequencies. The Rayleigh-Jeans law, another approach to the problem, created what was later known as the "ultraviolet catastrophe", but contrary to many textbooks this was not a motivation for Planck.^[6]

Planck's first proposed solution to the problem in 1899 followed from what Planck called the "principle of elementary disorder", which allowed him to derive Wien's law from a number of assumptions about the entropy of an ideal oscillator, creating what was referred-to as the Wien-Planck law. Soon it was found that experimental evidence did not confirm the new law at all, to Planck's frustration. Planck revised his approach, deriving the first version of the famous Planck black-body radiation law, which described the experimentally observed black-body spectrum well. It was first proposed in a meeting of the DPG on October 19, 1900 and published in 1901. This first derivation did not include energy quantisation, and did not use statistical mechanics, to which he held an aversion. In November 1900, Planck revised this first approach, relying on Boltzmann's statistical interpretation of the second law of thermodynamics as a way of gaining a more fundamental understanding of the principles behind his radiation law. As Planck was deeply suspicious of the philosophical and physical implications of such an interpretation of Boltzmann's approach, his recourse to them was, as he later put it, "an act of despair ... I was ready to sacrifice any of my previous convictions about physics."^[6] The central assumption behind his new derivation, presented to the DPG on 14 December 1900, was the supposition, now known as the Planck postulate, that electromagnetic energy could be emitted only in quantized form, in other words, the energy could only be a multiple of an elementary unit $E = h\nu$, where h is Planck's constant, also known as Planck's action quantum (introduced already in 1899), and ν (the Greek letter *nu*, not the letter *v*) is the frequency of the radiation.



Planck in 1918, the year he received the Nobel Prize in Physics for his work on quantum theory

At first Planck considered that quantisation was only "a purely formal assumption ... actually I did not think much about it..."; nowadays this assumption, incompatible with classical physics, is regarded as the birth of quantum physics and the greatest intellectual accomplishment of Planck's career (Ludwig Boltzmann had been discussing in a theoretical paper in 1877 the possibility that the energy states of a physical system could be discrete). Further interpretation of the implications of Planck's work was advanced by Albert Einstein in 1905 in connection with his work on the photoelectric effect—for this reason, the philosopher and historian of science Thomas Kuhn argued that Einstein should be given credit for quantum theory more so than Planck, since Planck did not understand in a deep sense that he was "introducing the quantum" as a real physical entity.^[7] Be that as it may, it was in recognition of Planck's monumental accomplishment that he was awarded the Nobel Prize in Physics in 1918.

The discovery of Planck's constant enabled him to define a new universal set of physical units (such as the Planck length and the Planck mass), all based on fundamental physical constants.

Subsequently, Planck tried to grasp the meaning of energy quanta, but to no avail. "My unavailing attempts to somehow reintegrate the action quantum into classical theory extended over several years and caused me much trouble." Even several years later, other physicists like Rayleigh, Jeans, and Lorentz set Planck's constant to zero in order to align with classical physics, but Planck knew well that this constant had a precise nonzero value. "I am unable to understand Jeans' stubbornness — he is an example of a theoretician as should never be existing, the same as Hegel was for philosophy. So much the worse for the facts, if they are wrong."

Max Born wrote about Planck: "He was by nature and by the tradition of his family conservative, averse to revolutionary novelties and skeptical towards speculations. But his belief in the imperative power of logical thinking based on facts was so strong that he did not hesitate to express a claim contradicting to all tradition, because he had convinced himself that no other resort was possible."

Einstein and the theory of relativity

In 1905 the three epochal papers of the hitherto completely unknown Albert Einstein were published in the journal *Annalen der Physik*. Planck was among the few who immediately recognized the significance of the special theory of relativity. Thanks to his influence this theory was soon widely accepted in Germany. Planck also contributed considerably to extend the special theory of relativity.

Einstein's hypothesis of light *quanta* (photons), based on Philipp Lenard's 1902 discovery of the photoelectric effect, was initially rejected by Planck. He was unwilling to discard completely Maxwell's theory of electrodynamics. "The theory of light would be thrown back not by decades, but by centuries, into the age when Christian Huygens dared to fight against the mighty emission theory of Isaac Newton ..."

In 1910 Einstein pointed out the anomalous behavior of specific heat at low temperatures as another example of a phenomenon which defies explanation by classical physics. Planck and Nernst, seeking to clarify the increasing number of contradictions, organized the First Solvay Conference (Brussels 1911). At this meeting Einstein was able to convince Planck.

Meanwhile Planck had been appointed dean of Berlin University, whereby it was possible for him to call Einstein to Berlin and establish a new professorship for him (1914). Soon the two scientists became close friends and met frequently to play music together.

World War I

At the onset of the First World War Planck endorsed the general excitement of the public, writing that, "... besides of much horrible also much unexpectedly great and beautiful: the swift solution of the most difficult issues of domestic policy through arrangement of all parties... the higher esteem for all that is brave and truthful..."

Nonetheless, Planck refrained from the extremes of nationalism. In 1915, at a time when Italy was about to join the Allied Powers, he voted successfully for a scientific paper from Italy, which received a prize from the Prussian Academy of Sciences, where Planck was one of four permanent presidents.

Planck also signed the infamous "Manifesto of the 93 intellectuals", a pamphlet of polemic war propaganda (while Einstein retained a strictly pacifistic attitude which almost led to his imprisonment, being spared by his Swiss citizenship). But in 1915 Planck, after several meetings with Dutch physicist Lorentz, he revoked parts of the Manifesto. Then in 1916 he signed a declaration against German annexationism.

Post War and Weimar Republic

In the turbulent post-war years, Planck, now the highest authority of German physics, issued the slogan "persevere and continue working" to his colleagues.

In October 1920 he and Fritz Haber established the *Notgemeinschaft der Deutschen Wissenschaft* (Emergency Organization of German Science), aimed at providing financial support for scientific research. A considerable portion of the monies the organization would distribute were raised abroad.

Planck also held leading positions at Berlin University, the Prussian Academy of Sciences, the German Physical Society and the Kaiser-Wilhelm-Gesellschaft (which in 1948 became the Max-Planck-Gesellschaft). During this time economic conditions in Germany were such that he was hardly able to conduct research.

During the interwar period, Planck became a member of the Deutsche Volks-Partei (German People's Party), the party of Nobel Peace Prize laureate Gustav Stresemann, which aspired to liberal aims for domestic policy and rather revisionistic aims for international politics.

Planck disagreed with the introduction of universal suffrage and later expressed the view that the Nazi dictatorship resulted from "the ascent of the rule of the crowds".

Quantum mechanics

At the end of the 1920s Bohr, Heisenberg and Pauli had worked out the Copenhagen interpretation of quantum mechanics, but it was rejected by Planck, as well as Schrödinger, Laue, and Einstein. Planck expected that wave mechanics would soon render quantum theory—his own child—unnecessary. This was not to be the case, however. Further work only cemented quantum theory, even against his and Einstein's philosophical revulsions. Planck experienced the truth of his own earlier observation from his struggle with the older views in his younger years: "A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it."^[8]

Nazi dictatorship and The Second World War

When the Nazis seized power in 1933, Planck was 74. He witnessed many Jewish friends and colleagues expelled from their positions and humiliated, and hundreds of scientists emigrated from Germany. Again he tried the "persevere and continue working" slogan and asked scientists who were considering emigration to remain in Germany. He hoped the crisis would abate soon and the political situation would improve. There was also a deeper argument against emigration. Emigrating German non-Jewish scientists would need to look for academic positions abroad, but these positions better served Jewish scientists, who had no chance of continuing to work in Germany.

Hahn asked Planck to gather well-known German professors in order to issue a public proclamation against the treatment of Jewish professors, but Planck replied, "If you are able to gather today 30 such gentlemen, then

tomorrow 150 others will come and speak against it, because they are eager to take over the positions of the others."^[9] Under Planck's leadership, the Kaiser-Wilhelm-Gesellschaft (KWG) avoided open conflict with the Nazi regime, except concerning Fritz Haber. Planck tried to discuss the issue with Adolf Hitler but was unsuccessful. In the following year, 1934, Haber died in exile.

One year later, Planck, having been the president of the KWG since 1930, organized in a somewhat provocative style an official commemorative meeting for Haber. He also succeeded in secretly enabling a number of Jewish scientists to continue working in institutes of the KWG for several years. In 1936, his term as president of the KWG ended, and the Nazi government pressured him to refrain from seeking another term.

As the political climate in Germany gradually became more hostile, Johannes Stark, prominent exponent of Deutsche Physik ("German Physics", also called "Aryan Physics") attacked Planck, Sommerfeld and Heisenberg for continuing to teach the theories of Einstein, calling them "white Jews." The "Hauptamt Wissenschaft" (Nazi government office for science) started an investigation of Planck's ancestry, but all they could find out was that he was "1/16 Jewish."

In 1938 Planck celebrated his 80th birthday. The DPG held a celebration, during which the Max-Planck medal (founded as the highest medal by the DPG in 1928) was awarded to French physicist Louis de Broglie. At the end of 1938 the Prussian Academy lost its remaining independence and was taken over by Nazis (*Gleichschaltung*). Planck protested by resigning his presidency. He continued to travel frequently, giving numerous public talks, such as his talk on Religion and Science, and five years later he was sufficiently fit to climb 3,000-meter peaks in the Alps.

During the Second World War, the increasing number of Allied bombing campaigns against Berlin forced Planck and his wife to leave the city temporarily and live in the countryside. In 1942 he wrote: "In me an ardent desire has grown to persevere this crisis and live long enough to be able to witness the turning point, the beginning of a new rise." In February 1944 his home in Berlin was completely destroyed by an air raid, annihilating all his scientific records and correspondence. Finally, he got into a dangerous situation in his rural retreat because of the rapid advance of the Allied armies from both sides. After the end of the war he was brought to a relative in Göttingen.

Planck endured many personal tragedies after the age of 50. In 1909, his first wife died after 22 years of marriage, leaving him with two sons and twin daughters. Planck's oldest son, Karl, was killed in action in 1916. His daughter Margarete died in childbirth in 1917, and another daughter, Emma, married her late sister's husband and then also died in childbirth, in 1919. During World War II, Planck's house in Berlin was completely destroyed by bombs in 1944 and his youngest son, Erwin, was implicated in the attempt made on Hitler's life in the July 20 plot. Consequently, Erwin died at the hands of the Gestapo in 1945. Erwin's death destroyed Planck's will to live.^[10] By the end of the war, Planck, his second wife and his son by her, moved to Göttingen where he died on October 4, 1947.

Religious view

Planck was a devoted and persistent adherent of Christianity from early life to death, but he was very tolerant towards alternative views and religions, and so was discontented with the Nazi church organizations' demands for unquestioning belief.

The God in which Planck believed was an almighty, all-knowing, benevolent but unintelligible God that permeated everything, manifest through symbols, including physical laws. His view may have been motivated by an opposition like Einstein's and Schrödinger's against the positivist view. Planck was interested in truth and a Universe beyond observation, and objected to atheism as an obsession with symbols.

Planck regarded the scientist as a man of imagination and faith, "faith" interpreted as being similar to "having a working hypothesis". For example the causality principle isn't true or false, it is an act of faith. Thereby Planck may have indicated a view that points toward Imre Lakatos' research programs process descriptions, where falsification is mostly tolerable, in faith of its future removal.^[11]

On the other hand, Planck wrote, "...'to believe' means 'to recognize as a truth,' and the knowledge of nature, continually advancing on incontestably safe tracks, has made it utterly impossible for a person possessing some training in natural science to recognize as founded on truth the many reports of extraordinary contradicting the laws of nature, of miracles which are still commonly regarded as essential supports and confirmations of religious doctrines, and which formerly used to be accepted as facts pure and simple, without doubt or criticism." ^[12]

Honors and awards

- "Pour le Mérite" for Science and Arts 1915 (in 1930 he became chancellor of this order)
- Nobel Prize in Physics 1918 (awarded 1919)
- Lorentz Medal 1927
- Franklin Medal (1927)
- *Adlerschild des Deutschen Reiches* (1928), an award from the German Reich President
- Max Planck medal (1929, together with Einstein)
- Copley Medal (1929)
- Planck received honorary doctorates from the universities of Frankfurt, Munich (TH), Rostock, Berlin (TH), Graz, Athens, Cambridge, London, and Glasgow.
- The asteroid 1069 was named "Stella Planckia" by the International Astronomical Union (1938)



The Max Planck two Deutsche Mark coin

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- [9] In a slightly different translation, Hahn remembers Planck saying: "If you bring together 30 such men today, then tomorrow 150 will come to denounce them because they want to take their places." This translated quote is found in: Heilbron, 2000, p. 150. Heilbron, at the end of the paragraph, on p. 151, cites the following references to Hahn's writings: Otto Hahn *Einige persönliche Erinnerungen an Max Planck* MPG, *Mitteilungen* (1957) p. 244, and Otto Hahn *My Life* (Herder and Herder, 1970) p. 140.
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- Life–Work–Personality (<http://www.max-planck.mpg.de>) - Exhibition on the 50th anniversary of Planck's death
 - Max Planck, Planck's constant, and Schrodinger's Cat (<http://www.harrymaugans.com/2006/05/03/in-search-of-schrodingers-cat/>)
-

Louis de Broglie

Louis de Broglie	
	
Born	15 August 1892Dieppe, France
Died	19 March 1987 (aged 94)Louveciennes, France
Nationality	French
Fields	Physics
Institutions	Sorbonne University of Paris
Alma mater	Sorbonne
Doctoral advisor	Paul Langevin
Doctoral students	Jean-Pierre Vigier Alexandru Proca
Known for	Wave nature of electrons de Broglie wavelength
Notable awards	Nobel Prize in Physics (1929)

Louis-Victor-Pierre-Raymond, 7th duc de Broglie, FRS (English pronunciation: /dəˈbrɔɪ/; French: [də bʁɔɛj] (listen); 15 August 1892 – 19 March 1987) was a French physicist and a Nobel laureate. He was the sixteenth member elected to occupy seat 1 of the Académie française in 1944, and served as Perpetual Secretary of the Académie des sciences, France.

Biography

Louis de Broglie was born to a noble family in Dieppe, Seine-Maritime, younger son of Victor, 5th duc de Broglie. He became the 7th duc de Broglie upon the death without heir in 1960 of his older brother, Maurice, 6th duc de Broglie, also a physicist. He did not marry. When he died in Louveciennes, he was succeeded as duke by a distant cousin, Victor-François, 8th duc de Broglie.

De Broglie had originally intended a career in humanities, and received his first degree in history. Afterwards, though, he turned his attention toward mathematics and physics and received a degree in physics. With the outbreak of the First World War in 1914, he offered his services to the army in the development of radio communications.

His 1924 doctoral thesis, *Recherches sur la théorie des quanta* (Research on Quantum Theory), introduced his theory of electron waves. This included the wave-particle duality theory of matter, based on the work of Max Planck and Albert Einstein on light. The thesis examiners, unsure of the material, passed his thesis to Einstein for evaluation who endorsed his wave-particle duality proposal wholeheartedly; de Broglie was awarded his doctorate. This research culminated in the de Broglie hypothesis stating that *any moving particle or object had an associated wave*.

De Broglie thus created a new field in physics, the *mécanique ondulatoire*, or wave mechanics, uniting the physics of energy (wave) and matter (particle). For this he won the Nobel Prize in Physics in 1929.

In his later career, de Broglie worked to develop a causal explanation of wave mechanics, in opposition to the wholly probabilistic models which dominate quantum mechanical theory; it was refined by David Bohm in the 1950s.

In addition to strictly scientific work, de Broglie thought and wrote about the philosophy of science, including the value of modern scientific discoveries.

De Broglie became a member of the Académie des sciences in 1933, and was the academy's perpetual secretary from 1942. On 12 October 1944, he was elected to the Académie française, replacing mathematician Émile Picard. Because of the deaths and imprisonments of Académie members during the occupation and other effects of the war, the Académie was unable to meet the quorum of twenty members for his election; due to the exceptional circumstances, however, his unanimous election by the seventeen members present was accepted. In an event unique in the history of the Académie, he was received as a member by his own brother Maurice, who had been elected in 1934. UNESCO awarded him the first Kalinga Prize in 1952 for his work in popularizing scientific knowledge, and he was elected a Foreign Member of the Royal Society on 23 April 1953. In 1961 he received the title of Knight of the Grand Cross in the Légion d'honneur. De Broglie was awarded a post as counselor to the French High Commission of Atomic Energy in 1945 for his efforts to bring industry and science closer together. He established a center for applied mechanics at the Henri Poincaré Institute, where research into optics, cybernetics, and atomic energy were carried out. He inspired the formation of the International Academy of Quantum Molecular Science and was an early member.

Honours and awards

- 1929 Nobel Prize in Physics
- 1929 Henri Poincaré Medal
- 1932 Albert I of Monaco Prize
- 1938 Max Planck Medal
- 1938 Fellow, Royal Swedish Academy of Sciences
- 1944 Fellow, Académie française
- 1952 Kalinga Prize
- 1953 Fellow, Royal Society

Publications

- *Recherches sur la théorie des quanta* (Researches on the quantum theory), Thesis, Paris, 1924.
 - *Ondes et mouvements* (Waves and Motions). Paris: Gauthier-Villars, 1926.
 - *Rapport au 5e Conseil de Physique Solvay*. Brussels, 1927.
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- "History of International Academy of Quantum Molecular Science"^[2]. IAQMS. Retrieved 2010-03-08.

External links

- Les Immortels: Louis de BROGLIE^[3]" (Académie française, in French)
- Louis de Broglie – Biography^[4]" (Nobel Foundation)
- Paul Theroff (2005) An Online Gotha: Broglie Genealogy^[5]

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 - [5] <http://www.angelfire.com/realm/gotha/gotha/broglie.html>
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The Lord Rutherford of Nelson

The Lord Rutherford of Nelson	
 Lord Rutherford of Nelson	
Born	30 August 1871Brightwater, New Zealand
Died	19 October 1937 (aged 66)Cambridge, England
Residence	New Zealand, UK, Canada
Citizenship	United Kingdom
Nationality	British-New Zealander
Fields	Physicist-Chemist
Institutions	McGill University University of Manchester
Alma mater	University of Canterbury Cambridge University
Academic advisors	Alexander Bickerton J. J. Thomson
Doctoral students	Edward Victor Appleton Alexander MacAulay Ernest Walton Robert William Boyle Cecil Powell Nazir Ahmed Rafi Muhammad Chaudhry Charles Wynn-Williams

Other notable students	Mark Oliphant Patrick Blackett Hans Geiger Niels Bohr Otto Hahn Teddy Bullard Pyotr Kapitsa John Cockcroft Charles Drummond Ellis James Chadwick Ernest Marsden Edward Andrade Frederick Soddy Edward Victor Appleton Bertram Boltwood Kazimierz Fajans Charles Galton Darwin A. J. B. Robertson George Laurence Henry DeWolf Smyth Harriet Brooks Douglas Hartree Iven Mackay Norman Alexander
Known for	Father of nuclear physics Rutherford model Rutherford scattering Rutherford backscattering spectroscopy Discovery of proton Rutherford (unit) Coining the term 'artificial disintegration'
Influenced	Henry Moseley Hans Geiger Albert Beaumont Wood
Notable awards	Rumford Medal (1905) Nobel Prize in Chemistry (1908) Elliott Cresson Medal (1910) Matteucci Medal (1913) Copley Medal (1922) Franklin Medal (1924)
Signature 	

Ernest Rutherford, 1st Baron Rutherford of Nelson OM, FRS (30 August 1871 – 19 October 1937) was a New Zealand-born British chemist and physicist who became known as the father of nuclear physics.^[1] In early work he discovered the concept of radioactive half life, proved that radioactivity involved the transmutation of one chemical element to another, and also differentiated and named alpha and beta radiation. This work was done at McGill University in Canada. It is the basis for the Nobel Prize in Chemistry he was awarded in 1908 "for his investigations into the disintegration of the elements, and the chemistry of radioactive substances".^[2]

Rutherford performed his most famous work after he had moved to the U.K. in 1907 and was already a Nobel laureate. In 1911, he postulated that atoms have their positive charge concentrated in a very small nucleus,^[3] and thereby pioneered the Rutherford model, or planetary, model of the atom, through his discovery and interpretation of Rutherford scattering in his gold foil experiment. He is widely credited with first "splitting the atom" in 1917. This

led to the first experiment to split the nucleus in a controlled manner, performed by two students working under his direction, John Cockcroft and Ernest Walton, in 1932.

Rutherford died in 1937, and was honoured in death by being interred near the greatest scientists of the United Kingdom, near Sir Isaac Newton's tomb in Westminster Abbey. The chemical element rutherfordium (element 104) was named for him in 1997.

Biography

Ernest Rutherford was the son of James Rutherford, a farmer, and his wife Martha Thompson, originally from Hornchurch, Essex, England.^[4] James had emigrated to New Zealand from Perth, Scotland, "to raise a little flax and a lot of children". Ernest was born at Spring Grove (now Brightwater), near Nelson, New Zealand. His first name was mistakenly spelled *Earnest* when his birth was registered.^[5]

He studied at Havelock School and then Nelson College and won a scholarship to study at Canterbury College, University of New Zealand where he was president of the debating society, among other things. After gaining his BA, MA and BSc, and doing two years of research at the forefront of electrical technology, in 1895 Rutherford travelled to England for postgraduate study at the Cavendish Laboratory, University of Cambridge (1895–1898),^[6] and he briefly held the world record for the distance over which electromagnetic waves could be detected.

In 1898 Rutherford was appointed to succeed Hugh Longbourne Callendar in the chair of Macdonald Professor of physics at McGill University in Montreal, Canada, where he did the work that gained him the Nobel Prize in Chemistry in 1908. In 1900 he gained a DSc from the University of New Zealand. Also in 1900 he married Mary Georgina Newton (1876–1945); they had one daughter, Eileen Mary (1901–1930), who married Ralph Fowler. In 1907 Rutherford moved to Britain to take the chair of physics at the University of Manchester.

Later years and honours

He was knighted in 1914. In 1916 he was awarded the Hector Memorial Medal. In 1919 he returned to the Cavendish as Director. Under him, Nobel Prizes were awarded to Chadwick for discovering the neutron (in 1932), Cockcroft and Walton for an experiment which was to be known as *splitting the atom* using a particle accelerator, and Appleton for demonstrating the existence of the ionosphere. He was admitted to the Order of Merit in 1925 and raised to the peerage as **Baron Rutherford of Nelson**, of Cambridge in the County of Cambridge, in 1931,^[7] a title that became extinct upon his unexpected death in hospital following an operation for an umbilical hernia (1937). Since he was a peer, British protocol at that time required that he be operated on by a titled doctor, and the delay cost him his life.^[8] He is interred in Westminster Abbey, alongside J. J. Thomson, and near Sir Isaac Newton.

Scientific research

During the investigation of radioactivity he coined the terms alpha and beta in 1899 to describe the two distinct types of radiation emitted by thorium and uranium. These rays were differentiated on the basis of penetrating power. From 1900 to 1903 he was joined at McGill by the young Frederick Soddy (Nobel Prize in Chemistry, 1921) and they collaborated on research into the transmutation of elements. Rutherford had demonstrated that radioactivity was the spontaneous disintegration of atoms. He noticed that a sample of radioactive material invariably took the same amount of time for half the sample to decay—its "half-life"—and created a practical application using this constant rate of decay as a clock, which could then be used to help determine the age of the Earth, which turned out to be much older than most of the scientists at the time believed.

In 1903, Rutherford realised that a type of radiation from radium discovered (but not named) by French chemist Paul Villard in 1900, must represent something different from alpha rays and beta rays, due to its very much greater penetrating power. Rutherford gave this third type of radiation its name also: the gamma ray.

In Manchester he continued to work with alpha radiation, in conjunction with Hans Geiger he developed zinc sulphide scintillation screens and ionisation chambers to count alphas. By dividing the total charge they produced by the number counted, Rutherford decided that the charge on the alpha was two. In late 1907 Ernest Rutherford and Thomas Royds allowed alphas to penetrate a very thin window into an evacuated tube. As they sparked the tube into discharge, the spectrum obtained from it changed, as the alphas were trapped. Eventually, the clear spectrum of helium gas appeared, proving that alphas were at least ionised helium atoms, and probably helium nuclei.

Along with Hans Geiger and Ernest Marsden he carried out the Geiger–Marsden experiment in 1909, which demonstrated the nuclear nature of atoms. Rutherford was inspired to ask Geiger and Marsden in this experiment to look for alpha particles with very high deflection angles, of a type not expected from any theory of matter at that time. Such deflections, though rare, were found, and proved to be a smooth but high-order function of the deflection angle. It was Rutherford's interpretation of this data that led him to formulate the Rutherford model of the atom in 1911 — that a very small positively charged nucleus was orbited by electrons.

In Cambridge in 1919, after taking over the Cavendish laboratory, Rutherford became the first person to transmute one element into another when he converted nitrogen into oxygen through the nuclear reaction $^{14}\text{N} + \alpha \rightarrow ^{17}\text{O} + \text{p}$. In 1921, while working with Niels Bohr (who postulated that electrons moved in specific orbits), Rutherford theorised about the existence of neutrons, which could somehow compensate for the repelling effect of the positive charges of protons by causing an attractive nuclear force and thus keeping the nuclei from breaking apart. Rutherford's theory of neutrons was proved in 1932 by his associate James Chadwick, who in 1935 was awarded the Nobel Prize in Physics for this discovery.

Legacy

Nuclear physics

Rutherford's research, and work done under him as laboratory director, established the nuclear structure of the atom and the essential nature of radioactive decay. Rutherford's team also demonstrated artificially induced nuclear transmutation. He is known as the father of nuclear physics. Rutherford died too early to see Leo Szilard's idea of controlled nuclear chain reactions come into being. A speech of Rutherford's printed in the September 12, 1933 London paper *The Times* is reported by Szilard to have been his inspiration for thinking of the possibility of a controlled nuclear chain reaction, in London, on the same day. Rutherford's speech, in part, read:

We might in these processes obtain very much more energy than the proton supplied, but on the average we could not expect to obtain energy in this way. It was a very poor and inefficient way of producing energy, and anyone who looked for a source of power in the transformation of the atoms was talking moonshine. But the subject was scientifically interesting because it gave insight into the atoms.^[9]



A plaque commemorating Rutherford's presence at the Victoria University, Manchester

Items named in honour of Rutherford's life and work

Scientific discoveries

- the element rutherfordium, Rf, Z=104. (1997)^[10]

Institutions

- Rutherford Appleton Laboratory, a scientific research laboratory near Abingdon, Oxfordshire, UK.
- Rutherford College, a school in Auckland, New Zealand
- Rutherford College, a college at the University of Kent in Canterbury, UK
- the Rutherford Institute for Innovation at the University of Cambridge, UK
- Rutherford Intermediate School, Wanganui, New Zealand

Buildings

- a building of the modern Cavendish Laboratory at the University of Cambridge, UK
- The Ernest Rutherford Physics Building at McGill University, Montreal, Canada^[11]
- the physics and chemistry building at the University of Canterbury, New Zealand
- The Coupland Building at the University of Manchester where Rutherford worked was renamed The Rutherford Building in 2006.
- The Rutherford lecture theatre in the Schuster Laboratory at the University of Manchester

Major streets

- Rutherford Close, a residential street in Abingdon, Oxfordshire, UK.
- Lord Rutherford Road in Brightwater, New Zealand — his birthplace.
- Rutherford Road in the biotech district of Carlsbad, California, USA.
- Rutherford Street in Nelson, New Zealand.

Other

- The crater Rutherford on the Moon, and the crater Rutherford on Mars
- The Rutherford Award at Thomas Carr College for excellence in VCE Chemistry, Australia
- Image on New Zealand \$100 note.
- Rutherford was the subject of a play by Stuart Hoar.
- On the side of the Mond Laboratory on the site of the original Cavendish Laboratory in Cambridge, there is an engraving in Rutherford's memory in the form of a crocodile, this being the nickname given to him by its commissioner, his colleague Peter Kapitza. The initials of the engraver, Eric Gill, are visible within the mouth.
- The Rutherford Foundation, a charitable trust set up by the Royal Society of New Zealand to support research in science and technology.^[12]

Publications

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 - *Radiations from Radioactive Substances* (1919)
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Famous statements

- "The energy produced by the breaking down of the atom is a very poor kind of thing. Anyone who expects a source of power from the transformation of these atoms is talking moonshine." – 1933^[13]

"It was almost as if you fired a 15 inch shell into a piece of tissue paper and it came back and hit you." (describing the Geiger-Marsden experiment)

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Further reading

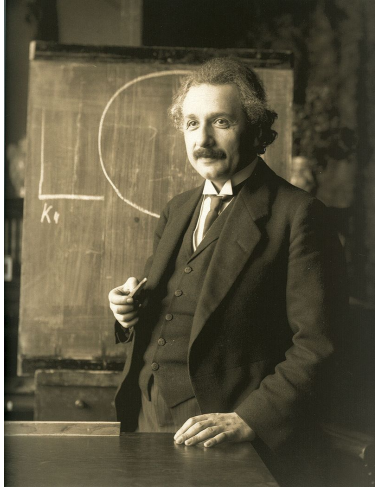
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External links

- Biography (<http://nobelprize.org/chemistry/laureates/1908/rutherford-bio.html>) from Nobel prize official website
- Nobel Lecture (http://nobelprize.org/nobel_prizes/chemistry/laureates/1908/rutherford-lecture.html) *The Chemical Nature of the Alpha Particles from Radioactive Substances*
- The Rutherford Museum (http://www.physics.mcgill.ca/museum/rutherford_museum.htm)
- Rutherford Scientist Supreme (<http://www.rutherford.org.nz>)
- Profile from American Public Broadcasting Service (<http://www.pbs.org/wgbh/aso/databank/entries/bpruth.html>)
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- Rutherford at Canterbury University College (<http://www.rutherfordjournal.com/article010112.html>) from The Rutherford Journal
- Rutherford's Timebomb (http://www.nzherald.co.nz/section/1/story.cfm?c_id=1&objectid=3566551) Article on Rutherford's contribution to dating the Age of the Earth
- BBC Radio 4: *In Our Time* — *Rutherford* (http://www.bbc.co.uk/radio4/history/inourtime/inourtime_20040219.shtml)
- The Rutherford Collection at his alma mater the University of Canterbury (http://digital-library.canterbury.ac.nz/awweb/login?un=mb&ps=mb&cl=collection5_lib&smd=1&field1=rutherford)
- Ernest Rutherford NZ Post stamp, 2008 (<http://www.nzhistory.net.nz/media/photo/ernest-rutherford-stamp>) — includes link to short biography and other sources (NZHistory.net.nz)

Authority control: PND: 118750488 (<http://d-nb.info/gnd/118750488>) | LCCN: n50024959 (<http://erol.oclc.org/laf/n50024959.html>) | VIAF: 66546175 (<http://viaf.org/viaf/66546175>)

Albert Einstein

Albert Einstein	
	
Albert Einstein in 1921	
Born	14 March 1879Ulm, Kingdom of Württemberg, German Empire
Died	18 April 1955 (aged 76)Princeton, New Jersey, United States
Resting place	Grounds of the Institute for Advanced Study, Princeton, New Jersey.
Residence	Germany, Italy, Switzerland, United States
Ethnicity	Jewish
Citizenship	- Württemberg/Germany (until 1896) - Stateless (1896–1901) - Switzerland (from 1901) - Austria (1911–12) - Germany (1914–33) - United States (from 1940)^[1]
Alma mater	- ETH Zurich - University of Zurich
Known for	- General relativity and special relativity - Photoelectric effect - Mass-energy equivalence - Quantification of the Brownian motion - Einstein field equations - Bose–Einstein statistics - Unified Field Theory
Spouse	- Mileva Marić (1903–1919) - Elsa Löwenthal, née Einstein, (1919–1936)
Awards	- Nobel Prize in Physics (1921) - Copley Medal (1925) - Max Planck Medal (1929) <i>Time</i> Person of the Century
Signature	



Albert Einstein (pronounced /ˈælbərt ˈaɪnʃtaɪn/; German: [ˈalbɐt ˈaɪnʃtaɪn] (listen); 14 March 1879 – 18 April 1955) was a theoretical physicist, philosopher and author who is widely regarded as one of the most influential and iconic scientists and intellectuals of all time. A German-Swiss Nobel laureate, Einstein is often regarded as the father of modern physics.^[2] He received the 1921 Nobel Prize in Physics "for his services to theoretical physics, and especially for his discovery of the law of the photoelectric effect".^[3]

Near the beginning of his career, Einstein thought that Newtonian mechanics was no longer enough to reconcile the laws of classical mechanics with the laws of the electromagnetic field. This led to the development of his special theory of relativity. He realized, however, that the principle of relativity could also be extended to gravitational fields, and with his subsequent theory of gravitation in 1916, he published a paper on the general theory of relativity. He continued to deal with problems of statistical mechanics and quantum theory, which led to his explanations of particle theory and the motion of molecules. He also investigated the thermal properties of light which laid the foundation of the photon theory of light. In 1917, Einstein applied the general theory of relativity to model the structure of the universe as a whole.^[4]

On the eve of World War II in 1939, he personally alerted President Franklin D. Roosevelt that Germany might be developing an atomic weapon, and recommended that the U.S. begin uranium procurement and nuclear research. As a result, Roosevelt advocated such research, leading to the creation of the top secret Manhattan Project, and the U.S. becoming the first and only country to possess nuclear weapons during the war. Days before his death, however, Einstein signed the Russell–Einstein Manifesto, that highlighted the dangers posed by the military usage of nuclear energy.

Einstein published more than 300 scientific along with over 150 non-scientific works, and received honorary doctorate degrees in science, medicine and philosophy from many European and American universities;^[4] he also wrote about various philosophical and political subjects such as socialism, international relations and the existence of God.^[5] His great intelligence and originality has made the word "Einstein" synonymous with genius.^[6]

Biography

Early life and education

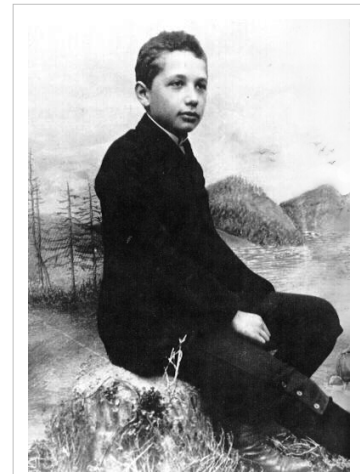


Einstein at the age of 4.

Albert Einstein was born in Ulm, in the Kingdom of Württemberg in the German Empire on 14 March 1879.^[7] His father was Hermann Einstein, a salesman and engineer. His mother was Pauline Einstein (née Koch). In 1880, the family moved to Munich, where his father and his uncle founded *Elektrotechnische Fabrik J. Einstein & Cie*, a company that manufactured electrical equipment based on direct current.^[7]

The Einsteins were non-observant Jews. Their son attended a Catholic elementary school from the age of five until ten.^[8] Although Einstein had early speech difficulties, he was a top student in elementary school.^{[9] [10]}

His father once showed him a pocket compass; Einstein realized that there must be something causing the needle to move, despite the apparent "empty space".^[11] As he grew, Einstein built models and mechanical devices for fun and began to show a talent for mathematics.^[7] In 1889, Max Talmud (later changed to Max Talmey) introduced the ten-year old Einstein to key texts in science, mathematics and philosophy, including Immanuel Kant's *Critique of Pure Reason* and *Euclid's Elements* (which Einstein called the "holy little geometry book").^[12] Talmud was a poor Jewish medical student from Poland. The Jewish community arranged for Talmud to take meals with the Einsteins each week on Thursdays for six years. During this time Talmud wholeheartedly guided Einstein through many secular educational interests.^{[13] [14]}



Albert Einstein in 1893 (age 14).

In 1894, his father's company failed: direct current (DC) lost the War of Currents to alternating current (AC). In search of business, the Einstein family moved to Italy, first to Milan and then, a few months later, to Pavia. When the family moved to Pavia, Einstein stayed in Munich to finish his studies at the Luitpold Gymnasium. His father intended for him to pursue electrical engineering, but Einstein clashed with authorities and resented the school's regimen and teaching method. He later wrote that the spirit of learning and creative thought were lost in strict rote learning. In the spring of 1895, he withdrew to join his family in Pavia, convincing the school to let him go by using a doctor's note.^[7] During this time, Einstein wrote his first scientific work, "The Investigation of the State of Aether in Magnetic Fields".^[15]

Einstein applied directly to the Eidgenössische Polytechnische Schule (ETH) in Zürich, Switzerland. Lacking the requisite Matura certificate, he took an entrance examination, which he failed, although he got exceptional marks in mathematics and physics.^[16] The Einsteins sent Albert to Aarau, in northern Switzerland to finish secondary school.^[7] While lodging with the family of Professor Jost Winteler, he fell in love with the family's daughter, Marie. (His sister Maja later married the Winteler's son Paul.)^[17] In Aarau, Einstein studied Maxwell's electromagnetic theory. At age 17, he graduated, and, with his father's approval, renounced his citizenship in the German Kingdom of Württemberg to avoid military service, and in 1896 he enrolled in the four year mathematics and physics teaching diploma program at the Polytechnic in Zurich. Marie Winteler moved to Olsberg, Switzerland for a teaching post.

Einstein's future wife, Mileva Marić, also enrolled at the Polytechnic that same year, the only woman among the six students in the mathematics and physics section of the teaching diploma course. Over the next few years, Einstein and Marić's friendship developed into romance, and they read books together on extra-curricular physics in which Einstein was taking an increasing interest. In 1900 Einstein was awarded the Zurich Polytechnic teaching diploma, but Marić failed the examination with a poor grade in the mathematics component, theory of functions.^[18] There have been claims that Marić collaborated with Einstein on his celebrated 1905 papers,^{[19] [20]} but historians of physics who have studied the issue find no evidence that she made any substantive contributions.^{[21] [22] [23] [24]}

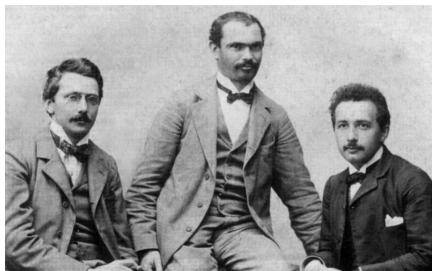
Marriages and children

In early 1902, Einstein and Mileva Marić had a daughter they named Lieserl in their correspondence, who was born in Novi Sad where Marić's parents lived.^[25] Her full name is not known, and her fate is uncertain after 1903.^[26]

Einstein and Marić married in January 1903. In May 1904, the couple's first son, Hans Albert Einstein, was born in Bern, Switzerland. Their second son, Eduard, was born in Zurich in July 1910. In 1914, Einstein moved to Berlin, while his wife remained in Zurich with their sons. Marić and Einstein divorced on 14 February 1919, having lived apart for five years.

Einstein married Elsa Löwenthal (née Einstein) on 2 June 1919, after having had a relationship with her since 1912. She was his first cousin maternally and his second cousin paternally. In 1933, they emigrated permanently to the United States. In 1935, Elsa Einstein was diagnosed with heart and kidney problems and died in December 1936.^[27]

Patent office



Left to right: Conrad Habicht, Maurice Solovine and Einstein, who founded the Olympia Academy

After graduating, Einstein spent almost two frustrating years searching for a teaching post, but a former classmate's father helped him secure a job in Bern, at the Federal Office for Intellectual Property, the patent office, as an assistant examiner.^[28] He evaluated patent applications for electromagnetic devices. In 1903, Einstein's position at the Swiss Patent Office became permanent, although he was passed over for promotion until he "fully mastered machine technology".^[29]

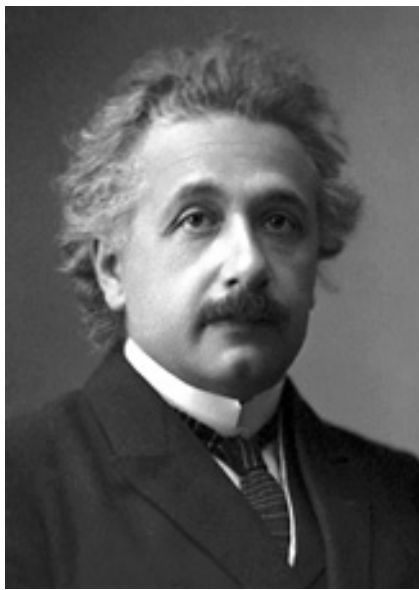
Much of his work at the patent office related to questions about transmission of electric signals and electrical-mechanical synchronization of time, two technical problems that show up conspicuously in the thought experiments that eventually led Einstein to his radical conclusions about the nature of light and the fundamental connection between space and time.^[30]

With a few friends he met in Bern, Einstein started a small discussion group, self-mockingly named "The Olympia Academy", which met regularly to discuss science and philosophy. Their readings included the works of Henri Poincaré, Ernst Mach, and David Hume, which influenced his scientific and philosophical outlook.



Einstein's home in Bern

Academic career



Einstein's official 1921 portrait after receiving the Nobel Prize in Physics.

In 1901, Einstein had a paper on the capillary forces of a straw published in the prestigious *Annalen der Physik*.^[31] On 30 April 1905, he completed his thesis, with Alfred Kleiner, Professor of Experimental Physics, serving as pro-forma advisor. Einstein was awarded a PhD by the University of Zurich. His dissertation was entitled "A New Determination of Molecular Dimensions".^[32] That same year, which has been called Einstein's *annus mirabilis* or "miracle year", he published four groundbreaking papers, on the photoelectric effect, Brownian motion, special relativity, and the equivalence of matter and energy, which were to bring him to the notice of the academic world.

By 1908, he was recognized as a leading scientist, and he was appointed lecturer at the University of Berne. The following year, he quit the patent office and the lectureship to take the position of physics docent^[33] at the University of Zurich. He became a full professor at Karl-Ferdinand University in Prague in 1911. In 1914, he returned to Germany after being appointed director of the Kaiser Wilhelm Institute for Physics (1914–1932)^[34] and a professor at the Humboldt University of Berlin, although with a special clause in his contract that freed him

from most teaching obligations. He became a member of the Prussian Academy of Sciences. In 1916, Einstein was appointed president of the German Physical Society (1916–1918).^{[35] [36]}

In 1911, he had calculated that, based on his new theory of general relativity, light from another star would be bent by the Sun's gravity. That prediction was claimed confirmed by observations made by a British expedition led by Sir Arthur Eddington during the solar eclipse of May 29, 1919. International media reports of this made Einstein world famous. On 7 November 1919, the leading British newspaper *The Times* printed a banner headline that read: "Revolution in Science – New Theory of the Universe – Newtonian Ideas Overthrown".^[37] (Much later, questions were raised whether the measurements were accurate enough to support Einstein's theory.)

In 1921, Einstein was awarded the Nobel Prize in Physics. Because relativity was still considered somewhat controversial, it was officially bestowed for his explanation of the photoelectric effect. He also received the Copley Medal from the Royal Society in 1925.

Travels abroad

Einstein visited New York City for the first time on 2 April 1921. When asked where he got his scientific ideas, Einstein explained that he believed scientific work best proceeds from an examination of physical reality and a search for underlying axioms, with consistent explanations that apply in all instances and avoid contradicting each other. He also recommended theories with visualizable results.(Einstein 1954)^[38]

In 1922, he traveled throughout Asia and later to Palestine, as part of a six-month excursion and speaking tour. His travels included Singapore, Ceylon, and Japan, where he gave a series of lectures to thousands of Japanese. His first lecture in Tokyo lasted four hours, after which he met the emperor and empress at the Imperial Palace where thousands came to watch. Einstein later gave his impressions of the Japanese in a letter to his sons:^[39] :307 "Of all the people I have met, I like the Japanese most, as they are modest, intelligent, considerate, and have a feel for art."^[39] :308

On his return voyage, he also visited Palestine for twelve days in what would become his only visit to that region. "He was greeted with great British pomp, as if he were a head of state rather than a theoretical physicist", writes Isaacson. This included a cannon salute upon his arrival at the residence of the British high commissioner, Sir

Herbert Samuel. During one reception given to him, the building was "stormed by throngs who wanted to hear him". In Einstein's talk to the audience, he expressed his happiness over the event:

I consider this the greatest day of my life. Before, I have always found something to regret in the Jewish soul, and that is the forgetfulness of its own people. Today, I have been made happy by the sight of the Jewish people learning to recognize themselves and to make themselves recognized as a force in the world.^[40] :308

Emigration from Germany

In 1933, Einstein was compelled to immigrate to the United States due to the rise to power of the Nazis under Germany's new chancellor, Adolf Hitler.^[41] While visiting American universities in April, 1933, he learned that the new German government had passed a law barring Jews from holding any official positions, including teaching at universities. A month later, the Nazi book burnings occurred, with Einstein's works being among those burnt, and Nazi propaganda minister Joseph Goebbels proclaimed, "Jewish intellectualism is dead."^[40] Einstein also learned that his name was on a list of assassination targets, with a "\$5,000 bounty on his head". One German magazine included him in a list of enemies of the German regime with the phrase, "not yet hanged".^[40] ^[42]

Among other German scientists forced to flee were fourteen Nobel laureates and twenty-six of the sixty professors of theoretical physics in the country. Among the other scientists who left Germany, or the other countries it came to dominate, were Edward Teller, Niels Bohr, Enrico Fermi, Otto Stern, Victor Weisskopf, Hans Bethe, and Lise Meitner, many of whom made certain that the Allies would develop nuclear weapons first, before the Nazis.^[40] With so many other Jewish scientists now forced by circumstances to live in America, often working side by side, Einstein wrote to a friend, "For me the most beautiful thing is to be in contact with a few fine Jews—a few millennia of a civilized past do mean something after all." In another letter he writes, "In my whole life I have never felt so Jewish as now."^[40]

He took up a position at the Institute for Advanced Study at Princeton, New Jersey,^[43] an affiliation that lasted until his death in 1955. There, he tried unsuccessfully to develop a unified field theory and to refute the accepted interpretation of quantum physics. He and Kurt Gödel, another Institute member, became close friends. They would take long walks together discussing their work. His last assistant was Bruria Kaufman, who later became a renowned physicist.

World War II and the Manhattan Project

In the summer of 1939, a few months before the beginning of World War II, Einstein was persuaded to write a letter to President Franklin D. Roosevelt and warn him that Nazi Germany might be developing an atomic bomb. The letter, written with the help of Hungarian emigre physicist Leo Szilard, gave the letter more prestige, with Einstein also recommending that the U.S. begin uranium enrichment and nuclear research. According to F.G. Gosling of the U.S. Department of Energy, Einstein, Szilard, and other refugees including Edward Teller and Eugene Wigner, "regarded it as their responsibility to alert Americans to the possibility that German scientists might win the race to build an atomic bomb, and to warn that Hitler would be more than willing to resort to such a weapon."^[44]

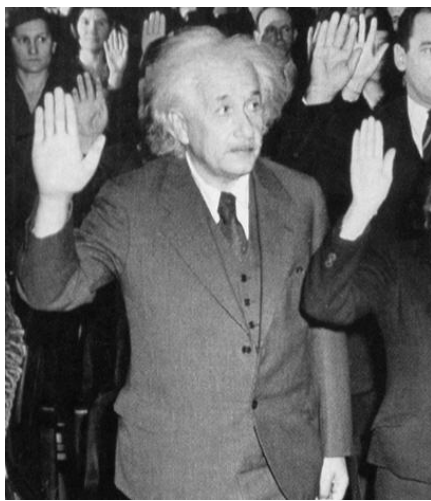
British columnist Ambrose Evans-Pritchard notes, however, that Washington at first "brushed off with disbelief" the fears they expressed. He then describes how quickly Roosevelt changed his mind:



Next to Oliver Locker-Lampson, being protected in Norfolk, England, after escaping Nazi Germany in 1933

Albert Einstein interceded through the Belgian queen mother, eventually getting a personal envoy into the Oval Office. Roosevelt initially fobbed him off. He listened more closely at a second meeting over breakfast the next day, then made up his mind within minutes. "This needs action," he told his military aide. It was the birth of the Manhattan Project.^[45]

Gosling adds that "the President was a man of considerable action once he had chosen a direction," and believed that the U.S. "could not take the risk of allowing Hitler" to possess nuclear bombs.^[44] Other weapons historians agree that the letter was "arguably the key stimulus for the U.S. adoption of serious investigations into nuclear weapons on the eve of the U.S. entry into World War II". As a result of Einstein's letter, and his meetings with Roosevelt, the U.S. entered the "race" to develop the bomb first, drawing on its "immense material, financial, and scientific resources". It became the only country to develop an atomic bomb during World War II as a result of its Manhattan Project.^[46] Einstein said to his old friend, Linus Pauling, in 1954, the last year of his life: "I made one great mistake in my life — when I signed the letter to President Roosevelt recommending that atom bombs be made; but there was some justification — the danger that the Germans would make them..."^[47]



Taking oath of allegiance for U.S. citizenship,
(1940)

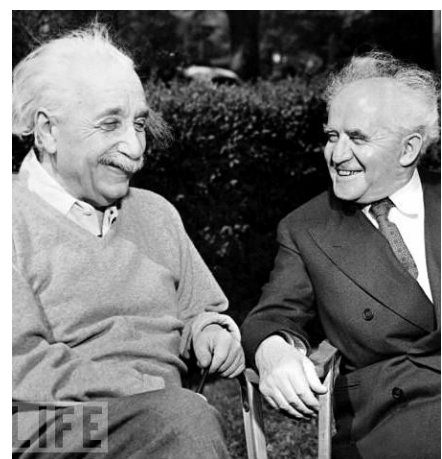
U.S. citizenship

Einstein became an American citizen in 1940. Not long after settling into his career at Princeton, he expressed his appreciation of the "meritocracy" in American culture when compared to Europe. According to Isaacson, he recognized the "right of individuals to say and think what they pleased", without social barriers, and as result, the individual was "encouraged" to be more creative, a trait he valued from his own early education. Einstein writes:

What makes the new arrival devoted to this country is the democratic trait among the people. No one humbles himself before another person or class. . . American youth has the good fortune not to have its outlook troubled by outworn traditions.^[40]
:432

As a member of the NAACP at Princeton who campaigned for the civil rights of African Americans, Einstein corresponded with civil rights activist W. E. B. Du Bois, and in 1946 Einstein called racism America's "worst disease".^[48] He later stated, "Race prejudice has unfortunately become an American tradition which is uncritically handed down from one generation to the next. The only remedies are enlightenment and education".^[49]

After the death of Israel's first president, Chaim Weizmann, in November 1952, Prime Minister David Ben-Gurion offered Einstein the position of President of Israel, a mostly ceremonial post.^[50] The offer was presented by Israel's ambassador in Washington, Abba Eban, who explained that the offer "embodies the deepest respect which the Jewish people can repose in any of its sons".^[39] :522 However, Einstein declined, and wrote in his response that he was "deeply moved", and "at once saddened and ashamed" that he could not accept it:

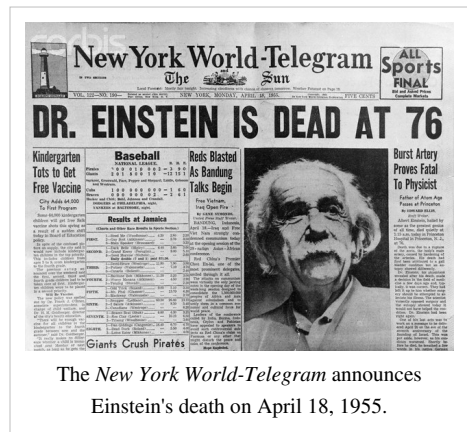


Einstein with David Ben Gurion, 1951

All my life I have dealt with objective matters, hence I lack both the natural aptitude and the experience to deal properly with people and to exercise official function. I am the more more distressed over these circumstances because my relationship with the Jewish people became my strongest human tie once I achieved complete clarity about our precarious position among the nations of the world.^[39] :522 [50] [51]

Death

On April 17, 1955, Albert Einstein experienced internal bleeding caused by the rupture of an abdominal aortic aneurysm, which had previously been reinforced surgically by Dr. Rudolph Nissen in 1948.^[52] He took the draft of a speech he was preparing for a television appearance commemorating the State of Israel's seventh anniversary with him to the hospital, but he did not live long enough to complete it.^[53] Einstein refused surgery, saying: "I want to go when I want. It is tasteless to prolong life artificially. I have done my share, it is time to go. I will do it elegantly."^[54] He died in Princeton Hospital early the next morning at the age of 76, having continued to work until near the end.



The New York World-Telegram announces Einstein's death on April 18, 1955.

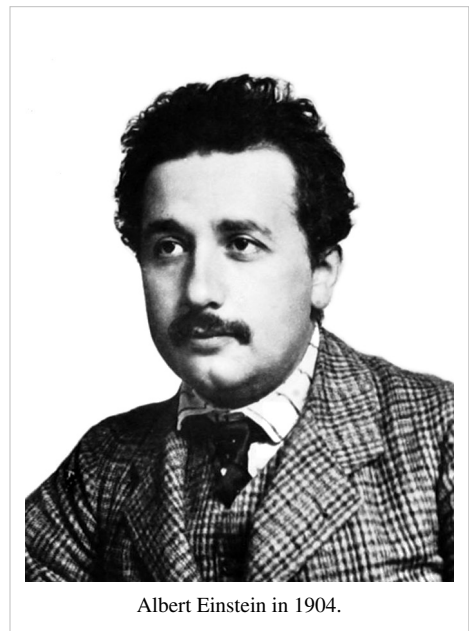
Einstein's remains were cremated and his ashes were scattered around the grounds of the Institute for Advanced Study.^[55] ^[56] During the autopsy, the pathologist of Princeton Hospital, Thomas Stoltz Harvey, removed Einstein's brain for preservation, without the permission of his family, in hope that the neuroscience of the future would be able to discover what made Einstein so intelligent.^[57]

Scientific career

Throughout his life, Einstein published hundreds of books and articles. Most were about physics, but a few expressed leftist political opinions about pacifism, socialism, and zionism.^[5] ^[7] In addition to the work he did by himself he also collaborated with other scientists on additional projects including the Bose–Einstein statistics, the Einstein refrigerator and others.^[58]

Physics in 1900

Einstein's early papers all come from attempts to demonstrate that atoms exist and have a finite nonzero size. At the time of his first paper in 1902, it was not yet completely accepted by physicists that atoms were real, even though chemists had good evidence ever since Antoine Lavoisier's work a century earlier. The reason physicists were skeptical was because no 19th century theory could fully explain the properties of matter from the properties of atoms.



Albert Einstein in 1904.

Ludwig Boltzmann was a leading 19th century atomist physicist, who had struggled for years to gain acceptance for atoms. Boltzmann had given an interpretation of the laws of thermodynamics, suggesting that the law of entropy increase is statistical. In Boltzmann's way of thinking, the entropy is the logarithm of the number of ways a system could be configured inside. The reason the entropy goes up is only because it is more likely for a system to go from a special state with only a few possible internal configurations to a more generic state with many. While Boltzmann's statistical interpretation of entropy is

universally accepted today, and Einstein believed it, at the turn of the 20th century it was a minority position.

The statistical idea was most successful in explaining the properties of gases. James Clerk Maxwell, another leading atomist, had found the distribution of velocities of atoms in a gas, and derived the surprising result that the viscosity of a gas should be independent of density. Intuitively, the friction in a gas would seem to go to zero as the density goes to zero, but this is not so, because the mean free path of atoms becomes large at low densities. A subsequent experiment by Maxwell and his wife confirmed this surprising prediction. Other experiments on gases and vacuum, using a rotating slitted drum, showed that atoms in a gas had velocities distributed according to Maxwell's distribution law.

In addition to these successes, there were also inconsistencies. Maxwell noted that at cold temperatures, atomic theory predicted specific heats that are too large. In classical statistical mechanics, every spring-like motion has thermal energy $k_B T$ on average at temperature T , so that the specific heat of every spring is Boltzmann's constant k_B . A monatomic solid with N atoms can be thought of as N little balls representing N atoms attached to each other in a box grid with $3N$ springs, so the specific heat of every solid is $3Nk_B$, a result which became known as the Dulong–Petit law. This law is true at room temperature, but not for colder temperatures. At temperatures near zero, the specific heat goes to zero.

Similarly, a gas made up of a molecule with two atoms can be thought of as two balls on a spring. This spring has energy $k_B T$ at high temperatures, and should contribute an extra k_B to the specific heat. It does at temperatures of about 1000 degrees, but at lower temperature, this contribution disappears. At zero temperature, all other contributions to the specific heat from rotations and vibrations also disappear. This behavior was inconsistent with classical physics.

The most glaring inconsistency was in the theory of light waves. Continuous waves in a box can be thought of as infinitely many spring-like motions, one for each possible standing wave. Each standing wave has a specific heat of k_B , so the total specific heat of a continuous wave like light should be infinite in classical mechanics. This is obviously wrong, because it would mean that all energy in the universe would be instantly sucked up into light waves, and everything would slow down and stop.

These inconsistencies led some people to say that atoms were not physical, but mathematical. Notable among the skeptics was Ernst Mach, whose positivist philosophy led him to demand that if atoms are real, it should be possible to see them directly.^[59] Mach believed that atoms were a useful fiction, that in reality they could be assumed to be infinitesimally small, that Avogadro's number was infinite, or so large that it might as well be infinite, and k_B was infinitesimally small. Certain experiments could then be explained by atomic theory, but other experiments could not, and this is the way it will always be.

Einstein opposed this position. Throughout his career, he was a realist. He believed that a single consistent theory should explain all observations, and that this theory would be a description of what was really going on, underneath it all. So he set out to show that the atomic point of view was correct. This led him first to thermodynamics, then to statistical physics, and to the theory of specific heats of solids.

In 1905, while he was working in the patent office, the leading German language physics journal *Annalen der Physik* published four of Einstein's papers. The four papers eventually were recognized as revolutionary, and 1905 became known as Einstein's "Miracle Year", and the papers as the *Annus Mirabilis Papers*.

Thermodynamic fluctuations and statistical physics

Einstein's earliest papers were concerned with thermodynamics. He wrote a paper establishing a thermodynamic identity in 1902, and a few other papers which attempted to interpret phenomena from a statistical atomic point of view.

His research in 1903 and 1904 was mainly concerned with the effect of finite atomic size on diffusion phenomena. As in Maxwell's work, the finite nonzero size of atoms leads to effects which can be observed. This research, and the thermodynamic identity, were well within the mainstream of physics in his time. They would eventually form the content of his PhD thesis.^[60]

His first major result in this field was the theory of thermodynamic fluctuations. When in equilibrium, a system has a maximum entropy and, according to the statistical interpretation, it can fluctuate a little bit. Einstein pointed out that the statistical fluctuations of a macroscopic object, like a mirror suspended on spring, would be completely determined by the second derivative of the entropy with respect to the position of the mirror.

Searching for ways to test this relation, his great breakthrough came in 1905. The theory of fluctuations, he realized, would have a visible effect for an object which could move around freely. Such an object would have a velocity which is random, and would move around randomly, just like an individual atom. The average kinetic energy of the object would be $k_B T$, and the time decay of the fluctuations would be entirely determined by the law of friction.

The law of friction for a small ball in a viscous fluid like water was discovered by George Stokes. He showed that for small velocities, the friction force would be proportional to the velocity, and to the radius of the particle (see Stokes' law). This relation could be used to calculate how far a small ball in water would travel due to its random thermal motion, and Einstein noted that such a ball, of size about a micrometre, would travel about a few micrometres per second. This motion could be easily detected with a microscope and indeed, as Brownian motion, had actually been observed by the botanist Robert Brown. Einstein was able to identify this motion with that predicted by his theory. Since the fluctuations which give rise to Brownian motion are just the same as the fluctuations of the velocities of atoms, measuring the precise amount of Brownian motion using Einstein's theory would show that Boltzmann's constant is non-zero and would measure Avogadro's number.

These experiments were carried out a few years later by Jean Baptiste Perrin, and gave a rough estimate of Avogadro's number consistent with the more accurate estimates due to Max Planck's theory of blackbody light and Robert Millikan's measurement of the charge of the electron.^[61] Unlike the other methods, Einstein's required very few theoretical assumptions or new physics, since it was directly measuring atomic motion on visible grains.

Einstein's theory of Brownian motion was the first paper in the field of statistical physics. It established that thermodynamic fluctuations were related to dissipation. This was shown by Einstein to be true for time-independent fluctuations, but in the Brownian motion paper he showed that dynamical relaxation rates calculated from classical mechanics could be used as statistical relaxation rates to derive dynamical diffusion laws. These relations are known as Einstein relations.

The theory of Brownian motion was the least revolutionary of Einstein's *Annus mirabilis* papers, but it is the most frequently cited, and had an important role in securing the acceptance of the atomic theory by physicists.

Thought experiments and a-priori physical principles

Einstein's thinking underwent a transformation in 1905. He had come to understand that quantum properties of light mean that Maxwell's equations were only an approximation. He knew that new laws would have to replace these, but he did not know how to go about finding those laws. He felt that guessing formal relations would not go anywhere.

So he decided to focus on a-priori principles instead, which are statements about physical laws which can be understood to hold in a very broad sense even in domains where they have not yet been shown to apply. A well accepted example of an a-priori principle is rotational invariance. If a new force is discovered in physics, it is assumed to be rotationally invariant almost automatically, without thought. Einstein sought new principles of this sort, to guide the production of physical ideas. Once enough principles are found, then the new physics will be the simplest theory consistent with the principles and with previously known laws.

The first general a-priori principle he found was the principle of relativity, that uniform motion is indistinguishable from rest. This was understood by Hermann Minkowski to be a generalization of rotational invariance from space to space-time. Other principles postulated by Einstein and later vindicated are the principle of equivalence and the principle of adiabatic invariance of the quantum number. Another of Einstein's general principles, Mach's principle, is fiercely debated, and whether it holds in our world or not is still not definitively established.

The use of a-priori principles is a distinctive unique signature of Einstein's early work, and has become a standard tool in modern theoretical physics.

Special relativity

His 1905 paper on the electrodynamics of moving bodies introduced his theory of special relativity, which showed that the observed independence of the speed of light on the observer's state of motion required fundamental changes to the notion of simultaneity. Consequences of this include the time-space frame of a moving body slowing down and contracting (in the direction of motion) relative to the frame of the observer. This paper also argued that the idea of a luminiferous aether – one of the leading theoretical entities in physics at the time – was superfluous.^[62] In his paper on *mass–energy equivalence*, which had previously been considered to be distinct concepts, Einstein deduced from his equations of special relativity what has been called the 20th century's best-known equation: $E = mc^2$.^[63] ^[64] This equation suggests that tiny amounts of mass could be converted into huge amounts of energy and presaged the development of nuclear power.^[65] Einstein's 1905 work on relativity remained controversial for many years, but was accepted by leading physicists, starting with Max Planck.^[66] ^[67]

Photons

In a 1905 paper,^[68] Einstein postulated that light itself consists of localized particles (*quanta*). Einstein's light quanta were nearly universally rejected by all physicists, including Max Planck and Niels Bohr. This idea only became universally accepted in 1919, with Robert Millikan's detailed experiments on the photoelectric effect, and with the measurement of Compton scattering.

Einstein's paper on the light particles was almost entirely motivated by thermodynamic considerations. He was not at all motivated by the detailed experiments on the photoelectric effect, which did not confirm his theory until fifteen years later. Einstein considers the entropy of light at temperature T , and decomposes it into a low-frequency part and a high-frequency part. The high-frequency part, where the light is described by Wien's law, has an entropy which looks exactly the same as the entropy of a gas of classical particles.

Since the entropy is the logarithm of the number of possible states, Einstein concludes that the number of states of short wavelength light waves in a box with volume V is equal to the number of states of a group of localizable particles in the same box. Since (unlike others) he was comfortable with the statistical interpretation, he confidently postulates that the light itself is made up of localized particles, as this is the only reasonable interpretation of the entropy.

This leads him to conclude that each wave of frequency f is associated with a collection of photons with energy hf each, where h is Planck's constant. He does not say much more, because he is not sure how the particles are related to the wave. But he does suggest that this idea would explain certain experimental results, notably the photoelectric effect.^[69]

Quantized atomic vibrations

Einstein continued his work on quantum mechanics in 1906, by explaining the specific heat anomaly in solids. This was the first application of quantum theory to a mechanical system. Since Planck's distribution for light oscillators had no problem with infinite specific heats, the same idea could be applied to solids to fix the specific heat problem there. Einstein showed in a simple model that the hypothesis that solid motion is quantized explains why the specific heat of a solid goes to zero at zero temperature.

Einstein's model treats each atom as connected to a single spring. Instead of connecting all the atoms to each other, which leads to standing waves with all sorts of different frequencies, Einstein imagined that each atom was attached to a fixed point in space by a spring. This is not physically correct, but it still predicts that the specific heat is $3Nk_B$, since the number of independent oscillations stays the same.

Einstein then assumes that the motion in this model is quantized, according to the Planck law, so that each independent spring motion has energy which is an integer multiple of hf , where f is the frequency of oscillation. With this assumption, he applied Boltzmann's statistical method to calculate the average energy of the spring. The result was the same as the one that Planck had derived for light: for temperatures where $k_B T$ is much smaller than hf , the motion is frozen, and the specific heat goes to zero.

So Einstein concluded that quantum mechanics would solve the main problem of classical physics, the specific heat anomaly. The particles of sound implied by this formulation are now called phonons. Because all of Einstein's springs have the same stiffness, they all freeze out at the same temperature, and this leads to a prediction that the specific heat should go to zero exponentially fast when the temperature is low. The solution to this problem is to solve for the independent normal modes individually, and to quantize those. Then each normal mode has a different frequency, and long wavelength vibration modes freeze out at colder temperatures than short wavelength ones. This was done by Peter Debye, and after this modification Einstein's quantization method reproduced quantitatively the behavior of the specific heats of solids at low temperatures.

This work was the foundation of condensed matter physics.

Adiabatic principle and action-angle variables

Throughout the 1910s, quantum mechanics expanded in scope to cover many different systems. After Ernest Rutherford discovered the nucleus and proposed that electrons orbit like planets, Niels Bohr was able to show that the same quantum mechanical postulates introduced by Planck and developed by Einstein would explain the discrete motion of electrons in atoms, and the periodic table of the elements.

Einstein contributed to these developments by linking them with the 1898 arguments Wilhelm Wien had made. Wien had shown that the hypothesis of adiabatic invariance of a thermal equilibrium state allows all the blackbody curves at different temperature to be derived from one another by a simple shifting process. Einstein noted in 1911 that the same adiabatic principle shows that the quantity which is quantized in any mechanical motion must be an adiabatic invariant. Arnold Sommerfeld identified this adiabatic invariant as the action variable of classical mechanics. The law that the action variable is quantized was the basic principle of the quantum theory as it was known between 1900 and 1925.

Wave-particle duality

Although the patent office promoted Einstein to Technical Examiner Second Class in 1906, he had not given up on academia. In 1908, he became a *privatdozent* at the University of Bern.^[70] In "über die Entwicklung unserer Anschauungen über das Wesen und die Konstitution der Strahlung" ("The Development of Our Views on the Composition and Essence of Radiation"), on the quantization of light, and in an earlier 1909 paper, Einstein showed that Max Planck's energy quanta must have well-defined momenta and act in some respects as independent, point-like particles. This paper introduced the *photon* concept (although the name *photon* was introduced later by Gilbert N. Lewis in 1926) and inspired the notion of wave-particle duality in quantum mechanics.

Theory of critical opalescence

Einstein returned to the problem of thermodynamic fluctuations, giving a treatment of the density variations in a fluid at its critical point. Ordinarily the density fluctuations are controlled by the second derivative of the free energy with respect to the density. At the critical point, this derivative is zero, leading to large fluctuations. The effect of density fluctuations is that light of all wavelengths is scattered, making the fluid look milky white. Einstein relates this to Raleigh scattering, which is what happens when the fluctuation size is much smaller than the wavelength, and which explains why the sky is blue.^[71]



Einstein at the Solvay Conference in 1911.

Zero-point energy

Einstein's physical intuition led him to note that Planck's oscillator energies had an incorrect zero point. He modified Planck's hypothesis by stating that the lowest energy state of an oscillator is equal to $\frac{1}{2}hf$, to half the energy spacing between levels. This argument, which was made in 1913 in collaboration with Otto Stern, was based on the thermodynamics of a diatomic molecule which can split apart into two free atoms.

Principle of equivalence

In 1907, while still working at the patent office, Einstein had what he would call his "happiest thought". He realized that the principle of relativity could be extended to gravitational fields. He thought about the case of a uniformly accelerated box not in a gravitational field, and noted that it would be indistinguishable from a box sitting still in an unchanging gravitational field.^[72] He used special relativity to see that the rate of clocks at the top of a box accelerating upward would be faster than the rate of clocks at the bottom. He concludes that the rates of clocks depend on their position in a gravitational field, and that the difference in rate is proportional to the gravitational potential to first approximation.

Although this approximation is crude, it allowed him to calculate the deflection of light by gravity, and show that it is nonzero. This gave him confidence that the scalar theory of gravity proposed by Gunnar Nordström was incorrect. But the actual value for the deflection that he calculated was too small by a factor of two, because the approximation he used doesn't work well for things moving at near the speed of light. When Einstein finished the full theory of general relativity, he would rectify this error and predict the correct amount of light deflection by the sun.

From Prague, Einstein published a paper about the effects of gravity on light, specifically the gravitational redshift and the gravitational deflection of light. The paper challenged astronomers to detect the deflection during a solar

eclipse.^[73] German astronomer Erwin Finlay-Freundlich publicized Einstein's challenge to scientists around the world.^[74]

Einstein thought about the nature of the gravitational field in the years 1909–1912, studying its properties by means of simple thought experiments. A notable one is the rotating disk. Einstein imagined an observer making experiments on a rotating turntable. He noted that such an observer would find a different value for the mathematical constant π than the one predicted by Euclidean geometry. The reason is that the radius of a circle would be measured with an uncontracted ruler, but, according to special relativity, the circumference would seem to be longer because the ruler would be contracted.

Since Einstein believed that the laws of physics were local, described by local fields, he concluded from this that spacetime could be locally curved. This led him to study Riemannian geometry, and to formulate general relativity in this language.

Hole argument and Entwurf theory

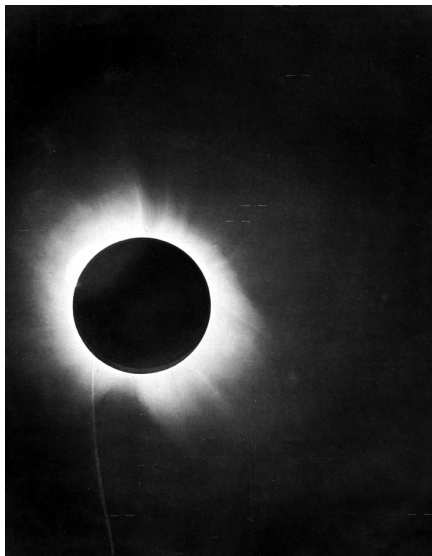
While developing general relativity, Einstein became confused about the gauge invariance in the theory. He formulated an argument that led him to conclude that a general relativistic field theory is impossible. He gave up looking for fully generally covariant tensor equations, and searched for equations that would be invariant under general linear transformations only.

In June, 1913 the Entwurf ("draft") theory was the result of these investigations. As its name suggests, it was a sketch of a theory, with the equations of motion supplemented by additional gauge fixing conditions. Simultaneously less elegant and more difficult than general relativity, after more than two years of intensive work Einstein abandoned the theory in November, 1915 after realizing that the hole argument was mistaken.^[75]

General relativity

In 1912, Einstein returned to Switzerland to accept a professorship at his *alma mater*, the ETH. Once back in Zurich, he immediately visited his old ETH classmate Marcel Grossmann, now a professor of mathematics, who introduced him to Riemannian geometry and, more generally, to differential geometry. On the recommendation of Italian mathematician Tullio Levi-Civita, Einstein began exploring the usefulness of general covariance (essentially the use of tensors) for his gravitational theory. For a while Einstein thought that there were problems with the approach, but he later returned to it and, by late 1915, had published his general theory of relativity in the form in which it is used today.^[76] This theory explains gravitation as distortion of the structure of spacetime by matter, affecting the inertial motion of other matter. During World War I, the work of Central Powers scientists was available only to Central Powers academics, for national security reasons. Some of Einstein's work did reach the United Kingdom and the United States through the efforts of the Austrian Paul Ehrenfest and physicists in the Netherlands, especially 1902 Nobel Prize-winner Hendrik Lorentz and Willem de Sitter of Leiden University. After the war ended, Einstein maintained his relationship with Leiden University, accepting a contract as an *Extraordinary Professor*; for ten years, from 1920 to 1930, he travelled to Holland regularly to lecture.^[77]

In 1917, several astronomers accepted Einstein's 1911 challenge from Prague. The Mount Wilson Observatory in California, U.S., published a solar spectroscopic analysis that showed no gravitational redshift.^[78] In 1918, the Lick Observatory, also in California, announced that it too had disproved Einstein's prediction, although its findings were not published.^[79]



Eddington's photograph of a solar eclipse, which confirmed Einstein's theory that light "bends".

However, in May 1919, a team led by the British astronomer Arthur Stanley Eddington claimed to have confirmed Einstein's prediction of gravitational deflection of starlight by the Sun while photographing a solar eclipse with dual expeditions in Sobral, northern Brazil, and Príncipe, a west African island.^[74] Nobel laureate Max Born praised general relativity as the "greatest feat of human thinking about nature";^[80] fellow laureate Paul Dirac was quoted saying it was "probably the greatest scientific discovery ever made".^[81] The international media guaranteed Einstein's global renown.

There have been claims that scrutiny of the specific photographs taken on the Eddington expedition showed the experimental uncertainty to be comparable to the same magnitude as the effect Eddington claimed to have demonstrated, and that a 1962 British expedition concluded that the method was inherently unreliable.^[37] The deflection of light during a solar eclipse was confirmed by later, more accurate observations.^[82] Some resented the newcomer's fame, notably among some German physicists, who later started the *Deutsche Physik* (German Physics)

movement.^{[83] [84]}

Cosmology

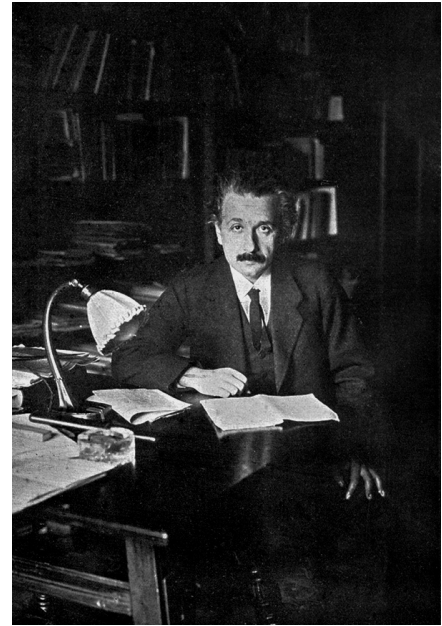
In 1917, Einstein applied the General theory of relativity to model the structure of the universe as a whole. He wanted the universe to be eternal and unchanging, but this type of universe is not consistent with relativity. To fix this, Einstein modified the general theory by introducing a new notion, the cosmological constant. With a positive cosmological constant, the universe could be an eternal static sphere^[85]

Einstein believed a spherical static universe is philosophically preferred, because it would obey Mach's principle. He had shown that general relativity incorporates Mach's principle to a certain extent in frame dragging by gravitomagnetic fields, but he knew that Mach's idea would not work if space goes on forever. In a closed universe, he believed that Mach's principle would hold.

Mach's principle has generated much controversy over the years.

Modern quantum theory

In 1917, at the height of his work on relativity, Einstein published an article in *Physikalische Zeitschrift* that proposed the possibility of stimulated emission, the physical process that makes possible the maser and the laser.^[86] This article showed that the statistics of absorption and emission of light would only be consistent with Planck's distribution law if the emission of light into a mode with n photons would be enhanced statistically compared to the emission of light into an empty mode. This paper was enormously influential in the later development of quantum mechanics, because it was the first paper to show that the statistics of atomic transitions had simple laws. Einstein discovered Louis de Broglie's work, and supported his ideas, which were received skeptically at first. In another major paper from this era, Einstein gave a wave equation for de Broglie waves, which Einstein suggested was the Hamilton–Jacobi equation of mechanics. This paper would inspire Schrödinger's work of 1926.



Einstein in his office at the University of Berlin.

Bose–Einstein statistics

In 1924, Einstein received a description of a statistical model from Indian physicist Satyendra Nath Bose, based on a counting method that assumed that light could be understood as a gas of indistinguishable particles. Einstein noted that Bose's statistics applied to some atoms as well as to the proposed light particles, and submitted his translation of Bose's paper to the *Zeitschrift für Physik*. Einstein also published his own articles describing the model and its implications, among them the Bose–Einstein condensate phenomenon that some particulates should appear at very low temperatures.^[87] It was not until 1995 that the first such condensate was produced experimentally by Eric Allin Cornell and Carl Wieman using ultra-cooling equipment built at the NIST–JILA laboratory at the University of Colorado at Boulder.^[88] Bose–Einstein statistics are now used to describe the behaviors of any assembly of bosons. Einstein's sketches for this project may be seen in the Einstein Archive in the library of the Leiden University.^[1]

Energy momentum pseudotensor

General relativity includes a dynamical spacetime, so it is difficult to see how to identify the conserved energy and momentum. Noether's theorem allows these quantities to be determined from a Lagrangian with translation invariance, but general covariance makes translation invariance into something of a gauge symmetry. The energy and momentum derived within general relativity by Noether's prescriptions do not make a real tensor for this reason.

Einstein argued that this is true for fundamental reasons, because the gravitational field could be made to vanish by a choice of coordinates. He maintained that the non-covariant energy momentum pseudotensor was in fact the best description of the energy momentum distribution in a gravitational field. This approach has been echoed by Lev Landau and Evgeny Lifshitz, and others, and has become standard.

The use of non-covariant objects like pseudotensors was heavily criticized in 1917 by Erwin Schrödinger and others.

Unified field theory

Following his research on general relativity, Einstein entered into a series of attempts to generalize his geometric theory of gravitation, which would allow the explanation of electromagnetism. In 1950, he described his "unified field theory" in a *Scientific American* article entitled "On the Generalized Theory of Gravitation".^[89] Although he continued to be lauded for his work, Einstein became increasingly isolated in his research, and his efforts were ultimately unsuccessful. In his pursuit of a unification of the fundamental forces, Einstein ignored some mainstream developments in physics, most notably the strong and weak nuclear forces, which were not well understood until many years after his death. Mainstream physics, in turn, largely ignored Einstein's approaches to unification. Einstein's dream of unifying other laws of physics with gravity motivates modern quests for a theory of everything and in particular string theory, where geometrical fields emerge in a unified quantum-mechanical setting.

Wormholes

Einstein collaborated with others to produce a model of a wormhole. His motivation was to model elementary particles with charge as a solution of gravitational field equations, in line with the program outlined in the paper "Do Gravitational Fields play an Important Role in the Constitution of the Elementary Particles?". These solutions cut and pasted Schwarzschild black holes to make a bridge between two patches.

If one end of a wormhole was positively charged, the other end would be negatively charged. These properties led Einstein to believe that pairs of particles and antiparticles could be described in this way.

Einstein–Cartan theory

In order to incorporate spinning point particles into general relativity, the affine connection needed to be generalized to include an antisymmetric part, called the torsion. This modification was made by Einstein and Cartan in the 1920s.

Equations of motion

The theory of general relativity has a fundamental law – the Einstein equations which describe how space curves, the geodesic equation which describes how particles move may be derived from the Einstein equations.

Since the equations of general relativity are non-linear, a lump of energy made out of pure gravitational fields, like a black hole, would move on a trajectory which is determined by the Einstein equations themselves, not by a new law. So Einstein proposed that the path of a singular solution, like a black hole, would be determined to be a geodesic from general relativity itself.

This was established by Einstein, Infeld and Hoffmann for pointlike objects without angular momentum, and by Roy Kerr for spinning objects.

Einstein's controversial beliefs in physics

In addition to his well-accepted results, some of Einstein's views are regarded as controversial:

- In the special relativity paper (in 1905), Einstein noted that, given a specific definition of the word "force" (a definition which he later agreed was not advantageous), and if we choose to maintain (by convention) the equation mass $\times$ acceleration = force, then one arrives at $m/(1-v^2/c^2)$ as the expression for the transverse mass of a fast moving particle. This differs from the accepted expression today, because, as noted in the footnotes to Einstein's paper added in the 1913 reprint, "it is more to the point to define force in such a way that the laws of energy and momentum assume the simplest form", as was done, for example, by Max Planck in 1906, who gave the now familiar expression $m/\sqrt{1-v^2/c^2}$ for the transverse mass. As Miller points out, this is equivalent to the transverse mass predictions of both Einstein and Lorentz. Einstein had commented already in the 1905 paper that "With a different definition of force and acceleration, we should naturally obtain other expressions for the masses. This shows that in comparing different theories... we must proceed very cautiously."^[90]

- Einstein published (in 1922) a qualitative theory of superconductivity based on the vague idea of electrons shared in orbits. This paper predated modern quantum mechanics, and today is regarded as being incorrect. The current theory of low temperature superconductivity was only worked out in 1957, thirty years after the establishing of modern quantum mechanics. However, even today, superconductivity is not well understood, and alternative theories continue to be put forward, especially to account for high-temperature superconductors.
- After introducing the concept of gravitational waves in 1917, Einstein subsequently entertained doubts about whether they could be physically realized. In 1937 he published a paper saying that the focusing properties of geodesics in general relativity would lead to an instability which causes plane gravitational waves to collapse in on themselves. While this is true to a certain extent in some limits, because gravitational instabilities can lead to a concentration of energy density into black holes, for plane waves of the type Einstein and Rosen considered in their paper, the instabilities are under control. Einstein retracted this position a short time later.
- Einstein denied several times that black holes could form. In 1939 he published a paper that argues that a star collapsing would spin faster and faster, spinning at the speed of light with infinite energy well before the point where it is about to collapse into a black hole. This paper received no citations, and the conclusions are well understood to be wrong. Einstein's argument itself is inconclusive, since he only shows that stable spinning objects have to spin faster and faster to stay stable before the point where they collapse. But it is well understood today (and was understood well by some even then) that collapse cannot happen through stationary states the way Einstein imagined. Nevertheless, the extent to which the models of black holes in classical general relativity correspond to physical reality remains unclear, and in particular the implications of the central singularity implicit in these models are still not understood. Efforts to conclusively prove the existence of event horizons have still not been successful.
- Closely related to his rejection of black holes, Einstein believed that the exclusion of singularities might restrict the class of solutions of the field equations so as to force solutions compatible with quantum mechanics, but no such theory has ever been found.
- In the early days of quantum mechanics, Einstein tried to show that the uncertainty principle was not valid, but by 1927 he had become convinced that it was valid.
- In the EPR paper, Einstein argued that quantum mechanics cannot be a complete realistic and local representation of phenomena, given specific definitions of "realism", "locality", and "completeness". The modern consensus is that Einstein's concept of realism is too restrictive.
- Einstein himself considered the introduction of the cosmological term in his 1917 paper founding cosmology as a "blunder".^[91] The theory of general relativity predicted an expanding or contracting universe, but Einstein wanted a universe which is an unchanging three dimensional sphere, like the surface of a three dimensional ball in four dimensions. He wanted this for philosophical reasons, so as to incorporate Mach's principle in a reasonable way. He stabilized his solution by introducing a cosmological constant, and when the universe was shown to be expanding, he retracted the constant as a blunder. This is not really much of a blunder – the cosmological constant is necessary within general relativity as it is currently understood, and it is widely believed to have a nonzero value today.
- Einstein did not immediately appreciate the value of Minkowski's four-dimensional formulation of special relativity, although within a few years he had adopted it as the basis for his theory of gravitation.
- Finding it too formal, Einstein believed that Heisenberg's matrix mechanics was incorrect. He changed his mind when Schrödinger and others demonstrated that the formulation in terms of the Schrödinger equation, based on Einstein's wave-particle duality was equivalent to Heisenberg's matrices.

Collaboration with other scientists

In addition to long time collaborators Leopold Infeld, Nathan Rosen, Peter Bergmann and others, Einstein also had some one-shot collaborations with various scientists.

Einstein-de Haas experiment

Einstein and De Haas demonstrated that magnetization is due to the motion of electrons, nowadays known to be the spin. In order to show this, they reversed the magnetization in an iron bar suspended on a torsion pendulum. They confirmed that this leads the bar to rotate, because the electron's angular momentum changes as the magnetization changes. This experiment needed to be sensitive, because the angular momentum associated with electrons is small, but it definitively established that electron motion of some kind is responsible for magnetization.

Schrödinger gas model

Einstein suggested to Erwin Schrödinger that he might be able to reproduce the statistics of a Bose–Einstein gas by considering a box. Then to each possible quantum motion of a particle in a box associate an independent harmonic oscillator. Quantizing these oscillators, each level will have an integer occupation number, which will be the number of particles in it.

This formulation is a form of second quantization, but it predates modern quantum mechanics. Erwin Schrödinger applied this to derive the thermodynamic properties of a semiclassical ideal gas. Schrödinger urged Einstein to add his name as co-author, although Einstein declined the invitation.^[92]

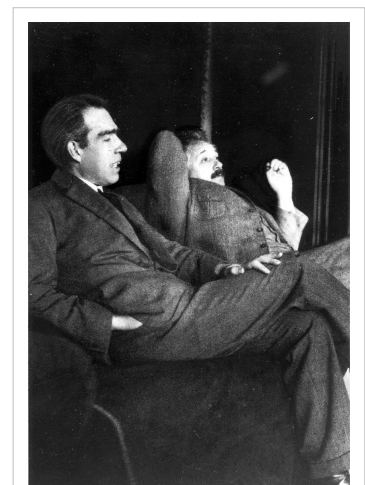
Einstein refrigerator

In 1926, Einstein and his former student Leó Szilárd co-invented (and in 1930, patented) the Einstein refrigerator. This absorption refrigerator was then revolutionary for having no moving parts and using only heat as an input.^[93] On 11 November 1930, U.S. Patent 1781541^[94] was awarded to Albert Einstein and Leó Szilárd for the refrigerator. Their invention was not immediately put into commercial production, as the most promising of their patents were quickly bought up by the Swedish company Electrolux to protect its refrigeration technology from competition.^[95]

Bohr versus Einstein

In the 1920s, quantum mechanics developed into a more complete theory. Einstein was unhappy with the Copenhagen interpretation of quantum theory developed by Niels Bohr and Werner Heisenberg, both in its outcomes and its instrumentalist methodology, Einstein being a scientific realist. In this interpretation, quantum phenomena are inherently probabilistic, with definite states resulting only upon interaction with classical systems. A public debate between Einstein and Bohr followed, lasting on and off for many years (including during the Solvay Conferences). Einstein formulated thought experiments against the Copenhagen interpretation, which were all rebutted by Bohr. In a 1926 letter to Max Born, Einstein wrote: "I, at any rate, am convinced that He [God] does not throw dice."^[96]

Einstein was never satisfied by what he perceived to be quantum theory's intrinsically incomplete description of nature, and in 1935 he further explored the issue in collaboration with Boris Podolsky and Nathan Rosen, noting that the theory seems to require non-local interactions; this is known as the EPR paradox.^[97] The EPR experiment has since been performed, with results confirming quantum theory's predictions.^[98] Repercussions of the Einstein–Bohr debate have found their way into philosophical discourse.



Einstein and Niels Bohr, 1925

Einstein–Podolsky–Rosen paradox

In 1935, Einstein returned to the question of quantum mechanics. He considered how a measurement on one of two entangled particles would affect the other. He noted, along with his collaborators, that by performing different measurements on the distant particle, either of position or momentum, different properties of the entangled partner could be discovered without disturbing it in any way.

He then used a hypothesis of local realism to conclude that the other particle had these properties already determined. The principle he proposed is that if it is possible to determine what the answer to a position or momentum measurement would be, without in any way disturbing the particle, then the particle actually has values of position or momentum.

This principle distilled the essence of Einstein's objection to quantum mechanics. As a physical principle, it has since been shown to be incompatible with experiments.

Political views

Einstein flouted the ascendant Nazi movement and later tried to be a voice of moderation in the tumultuous formation of the State of Israel.^[99] Fred Jerome in his *Einstein on Israel and Zionism: His Provocative Ideas About the Middle East* argues that Einstein was a Cultural Zionist who supported the idea of a Jewish homeland but opposed the establishment of a Jewish state in Palestine "with borders, an army, and a measure of temporal power." Instead, he preferred a bi-national state with "continuously functioning, mixed, administrative, economic, and social organizations."^[100] ^[101] However Ami Isseroff in his article *Was Einstein a Zionist*, argues that Einstein supported the recognition of the State of Israel and declared it "the fulfillment of our dream" when President Harry Truman recognize Israel in May 1948 and in presidential election 1948 Einstein supported Henry A. Wallace's Progressive Party which advocate pro-Soviet and pro-Israel foreign policy.^[102] ^[103]



Albert Einstein, seen here with his wife Elsa Einstein and Zionist leaders, including future President of Israel Chaim Weizmann, his wife Dr. Vera Weizmann, Menahem Ussishkin, and Ben-Zion Mossinson on arrival in New York City in 1921.

Throughout the November Revolution in Germany Einstein signed an appeal for the foundation of a nationwide liberal and democratic party,^[104] ^[105] which was published in the *Berliner Tageblatt* on 16 November 1918,^[106] and became a member of the German Democratic Party.^[107]

In his article *Why Socialism?*,^[108] published in 1949 in the *Monthly Review*, Einstein described a chaotic capitalist society, a source of evil to be overcome, as the "predatory phase of human development". He came to the following conclusion:

I am convinced there is only one way to eliminate these grave evils [capitalism], namely through the establishment of a socialist economy, accompanied by an educational system which would be oriented toward social goals. In such an economy, the means of production are owned by society itself and are utilized in a planned fashion. A planned economy, which adjusts production to the needs of the community, would distribute the work to be done among all those able to work and would guarantee a livelihood to every man, woman, and child. The education of the individual, in addition to promoting his own innate abilities, would attempt to develop in him a sense of responsibility for his fellow men in place of the glorification of power and success in our present society.^[108]

He braved anti-communist politics and resistance to the civil rights movement in the United States. On the floor of the US Congress, Einstein was accused by John E. Rankin of Mississippi of being a "foreign-born agitator" who

sought "to further the spread of Communism throughout the world".^[109] He also participated in the 1927 congress of the League against Imperialism in Brussels.^[110]

After World War II, as enmity between the former allies became a serious issue, Einstein wrote, "I do not know how the third World War will be fought, but I can tell you what they will use in the Fourth – rocks!"^[111] (Einstein 1949) With Albert Schweitzer and Bertrand Russell, Einstein lobbied to stop nuclear testing and future bombs. Days before his death, Einstein signed the Russell–Einstein Manifesto, which led to the Pugwash Conferences on Science and World Affairs.^[112]

Einstein was a member of several civil rights groups, including the Princeton chapter of the NAACP. When the aged W. E. B. Du Bois was accused of being a Communist spy, Einstein volunteered as a character witness, and the case was dismissed shortly afterward. Einstein's friendship with activist Paul Robeson, with whom he served as co-chair of the American Crusade to End Lynching, lasted twenty years.^[113]

Einstein said "Politics is for the moment, equation for the eternity."^[114] He declined the presidency of Israel in 1952.^[115]

Religious views

The question of scientific determinism gave rise to questions about Einstein's position on theological determinism, and whether or not he believed in God, or in a god. He once said:

You may call me an agnostic... I do not share the crusading spirit of the professional atheist whose fervor is mostly due to a painful act of liberation from the fetters of religious indoctrination received in youth. I prefer an attitude of humility corresponding to the weakness of our intellectual understanding of nature and of our own being.^[116]

Non-scientific legacy

While travelling, Einstein wrote daily to his wife Elsa and adopted stepdaughters Margot and Ilse. The letters were included in the papers bequeathed to The Hebrew University. Margot Einstein permitted the personal letters to be made available to the public, but requested that it not be done until twenty years after her death (she died in 1986^[117]). Barbara Wolff, of The Hebrew University's Albert Einstein Archives, told the BBC that there are about 3,500 pages of private correspondence written between 1912 and 1955.^[118]

Einstein bequeathed the royalties from use of his image to The Hebrew University of Jerusalem. Corbis, successor to The Roger Richman Agency, licenses the use of his name and associated imagery, as agent for the university.^[119]
[120]

In popular culture

In the period before World War II, Einstein was so well-known in America that he would be stopped on the street by people wanting him to explain "that theory". He finally figured out a way to handle the incessant inquiries. He told his inquirers "Pardon me, sorry! Always I am mistaken for Professor Einstein."^[121]

Einstein has been the subject of or inspiration for many novels, films, plays, and works of music.^[122] He is a favorite model for depictions of mad scientists and absent-minded professors; his expressive face and distinctive hairstyle have been widely copied and exaggerated. *Time* magazine's Frederic Golden wrote that Einstein was "a cartoonist's dream come true".^[123]

Awards and honors

In 1922, Einstein was awarded the 1921 Nobel Prize in Physics,^[124] "for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect". This refers to his 1905 paper on the photoelectric effect, "On a Heuristic Viewpoint Concerning the Production and Transformation of Light", which was well supported by the experimental evidence by that time. The presentation speech began by mentioning "his theory of relativity [which had] been the subject of lively debate in philosophical circles [and] also has astrophysical implications which are being rigorously examined at the present time". (Einstein 1923)

It was long reported that Einstein gave the Nobel prize money to his first wife, Mileva Marić, in compliance with their 1919 divorce settlement. However, personal correspondence made public in 2006^[125] shows that he invested much of it in the United States, and saw much of it wiped out in the Great Depression.

In 1929, Max Planck presented Einstein with the Max Planck medal of the German Physical Society in Berlin, for extraordinary achievements in theoretical physics.^[126]

In 1936, Einstein was awarded the Franklin Institute's Franklin Medal for his extensive work on relativity and the photo-electric effect.^[126]

The International Union of Pure and Applied Physics named 2005 the "World Year of Physics" in commemoration of the 100th anniversary of the publication of the annus mirabilis papers.^[127]

The *Albert Einstein Science Park* is located on the hill Telegrafenberg in Potsdam, Germany. The best known building in the park is the Einstein Tower which has a bronze bust of Einstein at the entrance. The Tower is an astrophysical observatory that was built to perform checks of Einstein's theory of General Relativity.^[128]

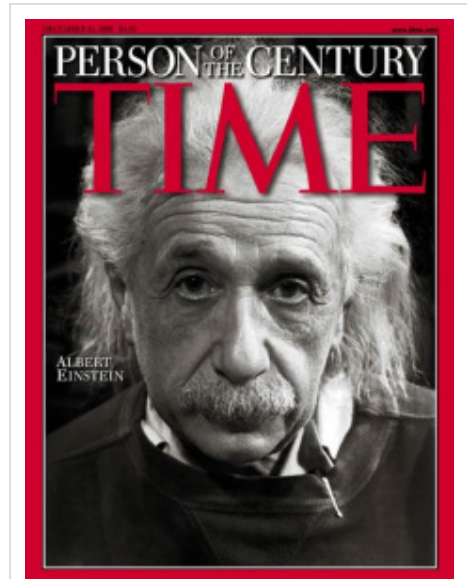
The *Albert Einstein Memorial* in central Washington, D.C. is a monumental bronze statue depicting Einstein seated with manuscript papers in hand. The statue, commissioned in 1979, is located in a grove of trees at the southwest corner of the grounds of the National Academy of Sciences on Constitution Avenue.

The chemical element 99, einsteinium, was named for him in August 1955, four months after Einstein's death.^[129]
^[130] 2001 Einstein is an inner main belt asteroid discovered on 5 March 1973.^[131]

In 1999 *Time* magazine named him the Person of the Century,^[123] ^[132] ahead of Mahatma Gandhi and Franklin Roosevelt, among others. In the words of a biographer, "to the scientifically literate and the public at large, Einstein is synonymous with genius".^[133] Also in 1999, an opinion poll of 100 leading physicists ranked Einstein the "greatest physicist ever".^[134] A Gallup poll recorded him as the fourth most admired person of the 20th century in the U.S.^[135]

In 1990, his name was added to the Walhalla temple for "laudable and distinguished Germans",^[136] which is located east of Regensburg, in Bavaria, Germany.^[137]

The United States Postal Service honored Einstein with a Prominent Americans series (1965–1978) 8¢ postage stamp.



In 1999 Albert Einstein was named the Person of the Century.

Awards named after him

The Albert Einstein Award (sometimes called the Albert Einstein Medal because it is accompanied with a gold medal) is an award in theoretical physics, established to recognize high achievement in the natural sciences. It was endowed by the Lewis and Rosa Strauss Memorial Fund in honor of Albert Einstein's 70th birthday. It was first awarded in 1951 and included a prize money of \$ 15,000,^{[138] [139]} which was later reduced to \$ 5,000.^{[140] [141]} The winner is selected by a committee (the first of which consisted of Einstein, Oppenheimer, von Neumann and Weyl^[142]) of the Institute for Advanced Study, which administers the award.^[139]

The Albert Einstein Medal is an award presented by the Albert Einstein Society in Bern, Switzerland. First given in 1979, the award is presented to people who have "rendered outstanding services" in connection with Einstein.^[143]

The Albert Einstein Peace Prize is given yearly by the Chicago, Illinois-based Albert Einstein Peace Prize Foundation. Winners of the prize receive \$50,000.^[144]

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- *The MacTutor History of Mathematics archive* (<http://www-history.mcs.st-andrews.ac.uk/Biographies/Einstein.html>), School of Mathematics and Statistics, University of St Andrews, Scotland, April 1997, retrieved 14 June 2009
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- Albert Einstein (<http://www.history.com/topics/albert-einstein>)--Watch Videos
- Science Odyssey People And Discoveries (<http://www.pbs.org/wgbh/aso/databank/entries/bpeins.html>)

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Erwin Schrödinger

Erwin Schrödinger	
	
Born	Erwin Rudolf Josef Alexander Schrödinger12 August 1887Erdberg, Vienna, Austria-Hungary
Died	4 January 1961 (aged 73)Vienna, Austria
Citizenship	Austria, Germany, Ireland
Nationality	Austria
Fields	Physics
Institutions	University of Breslau University of Zürich Humboldt University of Berlin University of Oxford University of Graz Dublin Institute for Advanced Studies Ghent University
Alma mater	University of Vienna
Doctoral advisor	Friedrich Hasenöhl
Other academic advisors	Franz S. Exner Friedrich Hasenöhl
Notable students	Linus Pauling Felix Bloch
Known for	Schrödinger equation Schrödinger's cat Schrödinger method Schrödinger functional Schrödinger picture Schrödinger-Newton equations Schrödinger field Rayleigh-Schrödinger perturbation Schrödinger logics Cat state
Notable awards	Nobel Prize in Physics (1933)
Signature 	

Erwin Rudolf Josef Alexander Schrödinger (German pronunciation: [ˈɛʁviːn ˈʃøːdɪŋɐ]; 12 August 1887 – 4 January 1961) was an Austrian theoretical physicist who was one of the fathers of quantum mechanics, and is famed for a number of important contributions to physics, especially the Schrödinger equation, for which he received the Nobel Prize in Physics in 1933. In 1935, after extensive correspondence with personal friend Albert Einstein, he proposed the Schrödinger's cat thought experiment.

Biography

Early years

In 1887 Schrödinger was born in Vienna, Austria to **Rudolf Schrödinger** (cereal producer, botanist) and **Georgine Emilia Brenda** (daughter of **Alexander Bauer**, Professor of Chemistry, k.u.k. Technische Hochschule Vienna).

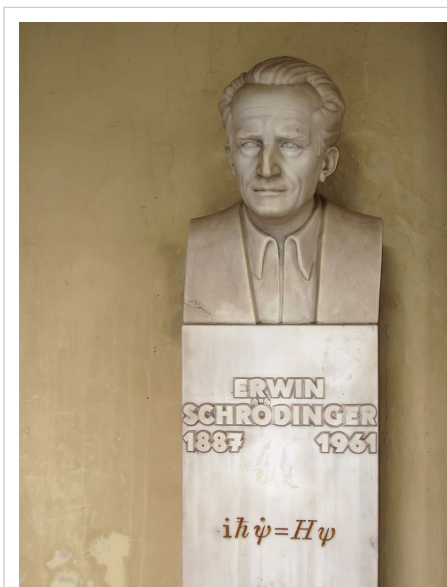
His mother was half Austrian and half English; the English side of her family came from Leamington Spa. Schrödinger learned English and German almost at the same time due to the fact that both were spoken in the family household. His father was a Catholic and his mother was a Lutheran.

In 1898 he attended the Akademisches Gymnasium. Between 1906 and 1910 Schrödinger studied in Vienna under Franz Serafin Exner (1849–1926) and Friedrich Hasenöhl (1874–1915). He also conducted experimental work with Karl Wilhelm Friedrich ("Fritz") Kohlrausch (1884–1953).^[1] ^[2] In 1911 Schrödinger became an assistant to Exner. At an early age, Schrödinger was strongly influenced by Schopenhauer.^[3] As a result of his extensive reading of Schopenhauer's works, he became deeply interested throughout his life in color theory, philosophy,^[4] perception, and eastern religion, especially Hindu Vedanta.

Middle years

In 1914 Erwin Schrödinger achieved Habilitation (*venia legendi*). Between 1914 and 1918 he participated in war work as a commissioned officer in the Austrian fortress artillery (Gorizia, Duino, Sistiana, Prosecco, Vienna). On 6 April 1920, Schrödinger married Annemarie Bertel. The same year, he became the assistant to Max Wien, in Jena, and in September 1920 he attained the position of ao. Prof. (*Ausserordentlicher Professor*), roughly equivalent to Reader (UK) or associate professor (US), in Stuttgart. In 1921, he became o. Prof. (*Ordentlicher Professor*, i.e. full professor), in Breslau (now Wrocław, Poland).

In 1921, he moved to the University of Zürich. In January 1926, Schrödinger published in *Annalen der Physik* the paper "*Quantisierung als Eigenwertproblem*" [tr. Quantization as an Eigenvalue Problem] on wave mechanics and what is now known as the Schrödinger equation. In this paper he gave a "derivation" of the wave equation for time independent systems, and showed that it gave the correct energy eigenvalues for the hydrogen-like atom. This paper has been universally celebrated as one of the most important achievements of the twentieth century, and created a revolution in quantum mechanics, and indeed of all physics and chemistry. A second paper was submitted just four weeks later that solved the quantum harmonic oscillator, the rigid rotor and the diatomic molecule, and gives a new derivation of the Schrödinger equation. A third paper in May showed the equivalence of his approach to that of Heisenberg and gave the treatment of the Stark effect. A fourth paper in this most remarkable series showed how to treat problems in which the system changes with time, as in scattering problems. These papers were the central achievement of his career and were at once recognized as having great significance by the physics community.



Bust of Schrödinger, in the courtyard arcade of the main building, University of Vienna, Austria.

In 1927, he succeeded Max Planck at the Friedrich Wilhelm University in Berlin. In 1933, however, Schrödinger decided to leave Germany; he disliked the Nazis' anti-semitism. He became a Fellow of Magdalen College at the University of Oxford. Soon after he arrived, he received the Nobel Prize together with Paul Adrien Maurice Dirac. His position at Oxford did not work out; his unconventional personal life (Schrödinger lived with two women)^[5] was not met with acceptance. In 1934, Schrödinger lectured at Princeton University; he was offered a permanent position there, but did not accept it. Again, his wish to set up house with his wife and his mistress may have posed a problem. He had the prospect of a position at the University of Edinburgh but visa delays occurred, and in the end he took up a position at the University of Graz in Austria in 1936.

In the midst of these tenure issues in 1935, after extensive correspondence with personal friend Albert Einstein, he proposed the Schrödinger's cat thought experiment.

Later years

In 1939, after the Anschluss, Schrödinger had problems because of his flight from Germany in 1933 and his known opposition to Nazism. He issued a statement recanting this opposition (he later regretted doing so, and he personally apologized to Einstein). However, this did not fully appease the new dispensation and the university dismissed him from his job for political unreliability. He suffered harassment and received instructions not to leave the country, but he and his wife fled to Italy. From there he went to visiting positions in Oxford and Ghent Universities.

In 1940 he received a personal invitation from Ireland's Taoiseach Éamon de Valera to reside in Ireland and agree to help establish an Institute for Advanced Studies in Dublin. He moved to Clontarf, Dublin and became the Director of the School for Theoretical Physics and remained there for 17 years, during which time he became a naturalized Irish citizen. He wrote about 50 further publications on various topics, including his explorations of unified field theory.

In 1944, he wrote *What is Life?*, which contains a discussion of negentropy and the concept of a complex molecule with the genetic code for living organisms. According to James D. Watson's memoir, *DNA, the Secret of Life*, Schrödinger's book gave Watson the inspiration to research the gene, which led to the discovery of the DNA double helix structure. Similarly, Francis Crick, in his autobiographical book *What Mad Pursuit*, described how he was influenced by Schrödinger's speculations about how genetic information might be stored in molecules. However, the geneticist and 1946 Nobel-prize winner H.J. Muller had in his 1922 article "Variation due to Change in the Individual Gene"^[6] already laid out all the basic properties of the heredity molecule that Schrödinger derives from first principles in *What is Life?*, properties which Muller refined in his 1929 article "The Gene As The Basis of Life"^[7] and further clarified during the 1930s, long before the publication of *What is Life?*.^[8]

Schrödinger stayed in Dublin until retiring in 1955. During this time he remained committed to his particular passion; involvements with students occurred and he fathered two children by two different Irish women. He had a life-long interest in the Vedanta philosophy of Hinduism, which influenced his speculations at the close of *What is Life?* about the possibility that individual consciousness is only a manifestation of a unitary consciousness pervading the universe.^[9]

In 1956, he returned to Vienna (chair *ad personam*). At an important lecture during the World Energy Conference he refused to speak on nuclear energy because of his skepticism about it and gave a philosophical lecture instead. During this period Schrödinger turned from mainstream quantum mechanics' definition of wave-particle duality and



Erwin Schrödinger in 1933

promoted the wave idea alone causing much controversy

Personal life

Schrödinger suffered from tuberculosis and several times in the 1920s stayed at a sanatorium in Arosa. It was there that he discovered his wave equation.^[10]

Schrödinger decided in 1933 that he could not live in a country in which persecution of Jews had become a national policy. Alexander Frederick Lindemann, the head of physics at Oxford University, visited Germany in the spring of 1933 to try to arrange positions in England for some young Jewish scientists from Germany. He spoke to Schrödinger about posts for one of his assistants and was surprised to discover that Schrödinger himself was interested in leaving Germany. Schrödinger asked for a colleague, Arthur March, to be offered a post as his assistant.

The request for March stemmed from Schrödinger's unconventional relationships with women: although his relations with his wife Anny were good, he had had many lovers with his wife's full knowledge (and in fact, Anny had her own lover, Hermann Weyl). Schrödinger asked for March to be his assistant because, at that time, he was in love with March's wife Hilde.

Many of the scientists who had left Germany spent mid-1933 in the Italian province of Bolzano. Here Hilde became pregnant with Schrödinger's child. On 4 November 1933 Schrödinger, his wife and Hilde March arrived in Oxford. Schrödinger had been elected a fellow of Magdalen College. Soon after they arrived in Oxford, Schrödinger heard that, for his work on wave mechanics, he had been awarded the Nobel prize.

In early 1934 Schrödinger was invited to lecture at Princeton University and while there he was made an offer of a permanent position. On his return to Oxford he negotiated about salary and pension conditions at Princeton but in the end he did not accept. It is thought that the fact that he wished to live at Princeton with Anny and Hilde both sharing the upbringing of his child was not found acceptable. The fact that Schrödinger openly had two wives, even if one of them was married to another man, was not well received in Oxford either. Nevertheless, his daughter Ruth Georgie Erica was born there on 30 May 1934.^[11]

On 4 January 1961, Schrödinger died in Vienna at the age of 73 of tuberculosis. He left a widow, Anny (born Annemarie Bertel on 3 December 1896, died 3 October 1965), and was buried in Alpbach, Austria.

Legacy

The philosophical issues raised by Schrödinger's cat are still debated today and remains his most enduring legacy in popular science, while Schrödinger's equation is his most enduring legacy at a more technical level. The huge crater Schrödinger, on the far side of the Moon is named after him. The Erwin Schrödinger International Institute for Mathematical Physics was established in Vienna in 1993.

Color

One of Schrödinger's lesser-known areas of scientific contribution was his work on color, color perception, and colorimetry (*Farbenmetrik*). In 1920, he published three papers in this area:

- "Theorie der Pigmente von größter Leuchtkraft," *Annalen der Physik*, (4), 62, (1920), 603-622
- "Grundlinien einer Theorie der Farbenmetrik im Tagessehen," *Annalen der Physik*, (4), 63, (1920), 397-426; 427-456; 481-520 (Outline of a theory of color measurement for daylight vision)



Erwin Schrödinger's gravesite

- "Farbenmetrik," *Zeitschrift für Physik*, 1, (1920), 459-466 (Color measurement).

The second of these is available in English as "Outline of a Theory of Color Measurement for Daylight Vision" in *Sources of Color Science*, Ed. David L. MacAdam, The MIT Press (1970), 134-182.

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- *My View of the World* Ox Bow Press (1983) ISBN 0-918024-30-7.
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See also

- Entropy and life
- List of Austrian scientists
- List of Austrians
- *Quantum Aspects of Life*
- Subject-object problem

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External links

- Erwin Schrödinger on an Austrian banknote. (<http://www-personal.umich.edu/~jbourj/money1.htm>)
- 1927 Solvay video with opening shot of Schrödinger (<http://uk.youtube.com/watch?v=8GZdZUouzBY>)
- "*biographie*" (<http://www.zbp.univie.ac.at/schrodinger/bio/bio1.htm>)" (in German) or
- "*Biography from the Austrian Central Library for Physics*" (<http://www.zbp.univie.ac.at/schrodinger/ebio/bio1.htm>)" (in English)
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- Critical interdisciplinary review of Schrödinger's "What is life?" (<http://www.disf.org/CosaDevoSapere/Schroedinger.asp>)

John von Neumann

*The native form of this personal name is **Neumann János**. This article uses the Western name order.*

John von Neumann	
 John von Neumann in the 1940s	
Born	December 28, 1903Budapest, Austria-Hungary
Died	February 8, 1957 (aged 53)Washington, D.C., United States
Residence	United States
Nationality	Hungarian and American
Fields	Mathematics and computer science
Institutions	University of Berlin Princeton University Institute for Advanced Study Site Y, Los Alamos
Alma mater	University of Pázmány Péter ETH Zürich
Doctoral advisor	Lipót Fejér
Doctoral students	Donald B. Gillies Israel Halperin John P. Mayberry
Other notable students	Paul Halmos Clifford Hugh Dowker

Known for	von Neumann Equation Abelian von Neumann algebra Game theory von Neumann algebra von Neumann architecture Von Neumann bicommutant theorem Von Neumann cellular automaton Von Neumann universal constructor Von Neumann entropy Von Neumann regular ring Von Neumann–Bernays–Gödel set theory Von Neumann universe Von Neumann conjecture Von Neumann's inequality Stone–von Neumann theorem Von Neumann stability analysis Minimax theorem Von Neumann extractor Von Neumann ergodic theorem Direct integral Ultrastrong topology
Notable awards	Enrico Fermi Award (1956)
Signature 	

John von Neumann (English pronunciation: /vɒn ˈnoɪmən/) (December 28, 1903 – February 8, 1957) was a Hungarian-born American mathematician who made major contributions to a vast range of fields,^[1] including set theory, functional analysis, quantum mechanics, ergodic theory, continuous geometry, economics and game theory, computer science, numerical analysis, hydrodynamics (of explosions), and statistics, as well as many other mathematical fields. He is generally regarded as one of the greatest mathematicians in modern history.^[2] The mathematician Jean Dieudonné called von Neumann "the last of the great mathematicians",^[3] while Peter Lax described him as possessing the most "fearsome technical prowess" and "scintillating intellect" of the century.^[4] Even in Budapest, in the time that produced geniuses like Theodore von Kármán (b. 1881), Leó Szilárd (b. 1898), Eugene Wigner (b. 1902), and Edward Teller (b. 1908), his brilliance stood out.^[5]

Von Neumann was a pioneer of the application of operator theory to quantum mechanics, in the development of functional analysis, a principal member of the Manhattan Project and the Institute for Advanced Study in Princeton (as one of the few originally appointed), and a key figure in the development of game theory^[1] ^[6] and the concepts of cellular automata^[1] and the universal constructor. Along with Teller and Stanisław Ulam, von Neumann worked out key steps in the nuclear physics involved in thermonuclear reactions and the hydrogen bomb.

Biography

The eldest of three brothers, von Neumann was born **Neumann János Lajos** (Hungarian pronunciation: [ˈnoj̯mɒn ˈjaːnof ˈlɔ̌jɒʃ]; in Hungarian the family name comes first) on December 28, 1903 in Budapest, Austro-Hungarian Empire, to somewhat wealthy Jewish parents.^[7] ^[8] ^[9] His father was Neumann Miksa (Max Neumann), a lawyer who worked in a bank. His mother was Kann Margit (Margaret Kann).

János, nicknamed "Jancsi" (Johnny), was a child prodigy who showed an aptitude for languages, memorization, and mathematics. By the age of six, he could exchange jokes in Classical Greek, memorize telephone directories, and display prodigious mental calculation abilities.^[10] He entered the Hungarian-speaking Lutheran high school Fasori Evangélikus Gimnázium in Budapest in 1911. Although he attended school at the grade level appropriate to his age, his father hired private tutors to give him advanced instruction in those areas in which he had displayed an aptitude.

Recognized as a mathematical prodigy, at the age of 15 he began to study under Gábor Szegő. On their first meeting, Szegő was so impressed with the boy's mathematical talent that he was brought to tears.^[11] In 1913, his father was rewarded with ennoblement for his service to the Austro-Hungarian empire. (After becoming semi-autonomous in 1867, Hungary had found itself in need of a vibrant mercantile class.) The Neumann family thus acquiring the title *margittai*, Neumann János became *margittai Neumann János* (John Neumann of Margitta), which he later changed to the German *Johann von Neumann*. He received his Ph.D. in mathematics (with minors in experimental physics and chemistry) from Pázmány Péter University in Budapest at the age of 22.^[1] He simultaneously earned his diploma in chemical engineering from the ETH Zurich in Switzerland^[1] at the behest of his father, who wanted his son to invest his time in a more financially viable endeavour than mathematics. Between 1926 and 1930, he taught as a *Privatdozent* at the University of Berlin, the youngest in its history. By age 25, he had published ten major papers, and by 30, nearly 36.

His father, Max von Neumann died in 1929. In 1930, von Neumann, his mother, and his brothers emigrated to the United States. He anglicized his first name to John, keeping the Austrian-aristocratic surname of von Neumann, whereas his brothers adopted surnames Vonneumann and Neumann (using the *de Neumann* form briefly when first in the U.S.).

Von Neumann was invited to Princeton University, New Jersey in 1930, and, subsequently, was one of the first four people selected for the faculty of the Institute for Advanced Study (two of the others being Albert Einstein and Kurt Gödel), where he remained a mathematics professor from its formation in 1933 until his death.

In 1937, von Neumann became a naturalized citizen of the U.S. In 1938, von Neumann was awarded the Bôcher Memorial Prize for his work in analysis.

Von Neumann married twice. He married Mariette Kövesi in 1930, just prior to emigrating to the United States. They had one daughter (von Neumann's only child), Marina, who is now a distinguished professor of international trade and public policy at the University of Michigan. The couple divorced in 1937. In 1938, von Neumann married Klara Dan, whom he had met during his last trips back to Budapest prior to the outbreak of World War II. The von Neumanns were very active socially within the Princeton academic community, and it is from this aspect of his life that many of the anecdotes which surround von Neumann's legend originate.



Gravestone of John von Neumann

In 1955, von Neumann was diagnosed with what was either bone or pancreatic cancer.^[12] Von Neumann died a year and a half later, in great pain. While at Walter Reed Hospital in Washington, D.C., he invited a Roman Catholic priest, Father Anselm Strittmatter, O.S.B., to visit him for consultation (a move which shocked some of von Neumann's friends).^[13] The priest then administered to him the last Sacraments.^[14] He died under military security lest he reveal military secrets while heavily medicated. John von Neumann was buried at Princeton Cemetery in Princeton, Mercer County, New Jersey.^[15]

Von Neumann wrote 150 published papers in his life; 60 in pure mathematics, 20 in physics, and 60 in applied mathematics. His last work, written while in the hospital and later published in book form as *The Computer and the Brain*, gives an indication of the direction of his interests at the time of his death.

Logic and set theory

The axiomatization of mathematics, on the model of Euclid's *Elements*, had reached new levels of rigor and breadth at the end of the 19th century, particularly in arithmetic (thanks to Richard Dedekind and Giuseppe Peano) and geometry (thanks to David Hilbert). At the beginning of the twentieth century, set theory, the new branch of mathematics discovered by Georg Cantor, and thrown into crisis by Bertrand Russell with the discovery of his famous paradox (on the set of all sets which do not belong to themselves), had not yet been formalized. The problem of an adequate axiomatization of set theory was resolved implicitly about twenty years later (by Ernst Zermelo and Abraham Fraenkel) by way of a series of principles which allowed for the construction of all sets used in the actual practice of mathematics, but which did not explicitly exclude the possibility of the existence of sets which belong to themselves. In his doctoral thesis of 1925, von Neumann demonstrated how it was possible to exclude this possibility in two complementary ways: the *axiom of foundation* and the notion of *class*.

The axiom of foundation established that every set can be constructed from the bottom up in an ordered succession of steps by way of the principles of Zermelo and Fraenkel, in such a manner that if one set belongs to another then the first must necessarily come before the second in the succession (hence excluding the possibility of a set belonging to itself.) To demonstrate that the addition of this new axiom to the others did not produce contradictions, von Neumann introduced a method of demonstration (called the *method of inner models*) which later became an essential instrument in set theory.

The second approach to the problem took as its base the notion of class, and defines a set as a class which belongs to other classes, while a *proper class* is defined as a class which does not belong to other classes. Under the Zermelo/Fraenkel approach, the axioms impede the construction of a set of all sets which do not belong to themselves. In contrast, under the von Neumann approach, the class of all sets which do not belong to themselves can be constructed, but it is a *proper class* and not a set.

With this contribution of von Neumann, the axiomatic system of the theory of sets became fully satisfactory, and the next question was whether or not it was also definitive, and not subject to improvement. A strongly negative answer arrived in September 1930 at the historic mathematical Congress of Königsberg, in which Kurt Gödel announced his first theorem of incompleteness: the usual axiomatic systems are incomplete, in the sense that they cannot prove every truth which is expressible in their language. This result was sufficiently innovative as to confound the majority of mathematicians of the time. But von Neumann, who had participated at the Congress, confirmed his fame as an instantaneous thinker, and in less than a month was able to communicate to Gödel himself an interesting consequence of his theorem: namely that the usual axiomatic systems are unable to demonstrate their own consistency. It is precisely this consequence which has attracted the most attention, even if Gödel originally considered it only a curiosity, and had derived it independently anyway (it is for this reason that the result is called *Gödel's second theorem*, without mention of von Neumann.)

Quantum mechanics

At the International Congress of Mathematicians of 1900, David Hilbert presented his famous list of twenty-three problems considered central for the development of the mathematics of the new century. The sixth of these was *the axiomatization of physical theories*. Among the new physical theories of the century the only one which had yet to receive such a treatment by the end of the 1930s was quantum mechanics. Quantum mechanics found itself in a condition of foundational crisis similar to that of set theory at the beginning of the century, facing problems of both philosophical and technical natures. On the one hand, its apparent non-determinism had not been reduced to an explanation of a deterministic form. On the other, there still existed two independent but equivalent heuristic formulations, the so-called matrix mechanical formulation due to Werner Heisenberg and the wave mechanical formulation due to Erwin Schrödinger, but there was not yet a single, unified satisfactory theoretical formulation.

After having completed the axiomatization of set theory, von Neumann began to confront the axiomatization of quantum mechanics. He immediately realized, in 1926, that a quantum system could be considered as a point in a so-called Hilbert space, analogous to the $6N$ dimension (N is the number of particles, 3 general coordinate and 3 canonical momentum for each) phase space of classical mechanics but with infinitely many dimensions (corresponding to the infinitely many possible states of the system) instead: the traditional physical quantities (e.g., position and momentum) could therefore be represented as particular linear operators operating in these spaces. The *physics* of quantum mechanics was thereby reduced to the *mathematics* of the linear Hermitian operators on Hilbert spaces.

For example, the famous uncertainty principle of Heisenberg, according to which the determination of the position of a particle prevents the determination of its momentum and vice versa, is translated into the *non-commutativity* of the two corresponding operators. This new mathematical formulation included as special cases the formulations of both Heisenberg and Schrödinger, and culminated in the 1932 classic *The Mathematical Foundations of Quantum Mechanics*. However, physicists generally ended up preferring another approach to that of von Neumann (which was considered elegant and satisfactory by mathematicians). This approach was formulated in 1930 by Paul Dirac.

Von Neumann's abstract treatment permitted him also to confront the foundational issue of determinism vs. non-determinism and in the book he demonstrated a theorem according to which quantum mechanics could not possibly be derived by statistical approximation from a deterministic theory of the type used in classical mechanics. This demonstration contained a conceptual error, but it helped to inaugurate a line of research which, through the work of John Stuart Bell in 1964 on Bell's Theorem and the experiments of Alain Aspect in 1982, demonstrated that quantum physics requires a *notion of reality* substantially different from that of classical physics.

Economics and game theory

Von Neumann's first significant contribution to economics was the minimax theorem of 1928. This theorem establishes that in certain zero sum games with perfect information (i.e., in which players know at each time all moves that have taken place so far), there exists a strategy for each player which allows both players to minimize their maximum losses (hence the name minimax). When examining every possible strategy, a player must consider all the possible responses of the player's adversary and the maximum loss. The player then plays out the strategy which will result in the minimization of this maximum loss. Such a strategy, which minimizes the maximum loss, is called optimal for both players just in case their minimaxes are equal (in absolute value) and contrary (in sign). If the common value is zero, the game becomes pointless.

Von Neumann eventually improved and extended the minimax theorem to include games involving imperfect information and games with more than two players. This work culminated in the 1944 classic *Theory of Games and Economic Behavior* (written with Oskar Morgenstern). The public interest in this work was such that The New York Times ran a front page story, something which only Einstein had previously elicited.

Von Neumann's second important contribution in this area was the solution, in 1937, of a problem first described by Léon Walras in 1874, the existence of situations of equilibrium in mathematical models of market development based on supply and demand. He first recognized that such a model should be expressed through disequations and not equations, and then he found a solution to Walras' problem by applying a fixed-point theorem derived from the work of L. E. J. Brouwer. The lasting importance of the work on general equilibria and the methodology of fixed point theorems is underscored by the awarding of Nobel prizes in 1972 to Kenneth Arrow, in 1983 to Gérard Debreu, and in 1994 to John Nash who had improved von Neumann's theory in his Princeton Ph.D thesis.

Von Neumann was also the inventor of the method of proof, used in game theory, known as backward induction (which he first published in 1944 in the book co-authored with Morgenstern, *Theory of Games and Economic Behaviour*).^[16]

Mathematical statistics and econometrics

Von Neumann made some fundamental contributions to mathematical statistics. In 1941, he derived the exact distribution of the ratio of mean square successive difference to the variance for normally distributed variables.^[17] This ratio was applied to the residuals from regression models and is commonly known as the Durbin-Watson statistic^[18] for testing the null hypothesis that the errors are serially independent against the alternative that they follow a stationary first order autoregression. Subsequently, John Denis Sargan and Alok Bhargava^[19] extended the results for testing if the errors on a regression model follow a Gaussian random walk (i.e. possess a unit root) against the alternative that they are a stationary first order autoregression. Von Neumann's contributions to statistics have had a major impact on econometric methodology.

Nuclear weapons

Beginning in the late 1930s, von Neumann began to take more of an interest in applied (as opposed to pure) mathematics. In particular, he developed an expertise in explosions—phenomena which are difficult to model mathematically. This led him to a large number of military consultancies, primarily for the Navy, which in turn led to his involvement in the Manhattan Project. The involvement included frequent trips by train to the project's secret research facilities in Los Alamos, New Mexico.^[1]

Von Neumann's principal contribution to the atomic bomb itself was in the concept and design of the explosive lenses needed to compress the plutonium core of the Trinity test device and the "Fat Man" weapon that was later dropped on Nagasaki. While von Neumann did not originate the "implosion" concept, he was one of its most persistent proponents, encouraging its continued development against the instincts of many of his colleagues, who felt such a design to be unworkable. The lens shape design work was completed by July 1944.



John von Neumann's wartime Los Alamos ID badge photo.

In a visit to Los Alamos in September 1944, von Neumann showed that the pressure increase from explosion shock wave reflection from solid objects was greater than previously believed if the angle of incidence of the shock wave was between 90° and some limiting angle. As a result, it was determined that the effectiveness of an atomic bomb would be enhanced with detonation some kilometers above the target, rather than at ground level.^[20]

Beginning in the spring of 1945, along with four other scientists and various military personnel, von Neumann was included in the target selection committee responsible for choosing the Japanese cities of Hiroshima and Nagasaki as the first targets of the atomic bomb. Von Neumann oversaw computations related to the expected size of the bomb blasts, estimated death tolls, and the distance above the ground at which the bombs should be detonated for optimum shock wave propagation and thus maximum effect.^[21] The cultural capital Kyoto, which had been spared the firebombing inflicted upon militarily significant target cities like Tokyo in World War II, was von Neumann's first choice, a selection seconded by Manhattan Project leader General Leslie Groves. However, this target was dismissed by Secretary of War Henry Stimson.^[22]

On July 16, 1945, with numerous other Los Alamos personnel, von Neumann was an eyewitness to the first atomic bomb blast, conducted as a test of the implosion method device, 35 miles (56 km) southeast of Socorro, New Mexico. Based on his observation alone, von Neumann estimated the test had resulted in a blast equivalent to 5 kilotons of TNT, but Enrico Fermi produced a more accurate estimate of 10 kilotons by dropping scraps of torn-up paper as the shock wave passed his location and watching how far they scattered. The actual power of the explosion had been between 20 and 22 kilotons.^[20]

After the war, Robert Oppenheimer remarked that the physicists involved in the Manhattan project had "known sin". Von Neumann's response was that "sometimes someone confesses a sin in order to take credit for it."

Von Neumann continued unperturbed in his work and became, along with Edward Teller, one of those who sustained the hydrogen bomb project. He then collaborated with Klaus Fuchs on further development of the bomb, and in 1946 the two filed a secret patent on "Improvement in Methods and Means for Utilizing Nuclear Energy", which outlined a scheme for using a fission bomb to compress fusion fuel to initiate a thermonuclear reaction.^[23] The Fuchs-von Neumann patent used radiation implosion, but not in the same way as is used in what became the final hydrogen bomb design, the Teller-Ulam design. Their work was, however, incorporated into the "George" shot of Operation Greenhouse, which was instructive in testing out concepts that went into the final design. The Fuchs-von Neumann work was passed on, by Fuchs, to the USSR as part of his nuclear espionage, but it was not used in the Soviet's own, independent development of the Teller-Ulam design. The historian Jeremy Bernstein has pointed out that ironically, "John von Neumann and Klaus Fuchs, produced a brilliant invention in 1946 that could have changed the whole course of the development of the hydrogen bomb, but was not fully understood until after the bomb had been successfully made."^[24]

Computer science

Von Neumann's hydrogen bomb work was also played out in the realm of computing, where he and Stanisław Ulam developed simulations on von Neumann's digital computers for the hydrodynamic computations. During this time he contributed to the development of the Monte Carlo method, which allowed complicated problems to be approximated using random numbers. Because using lists of "truly" random numbers was extremely slow, von Neumann developed a form of making pseudorandom numbers, using the middle-square method. Though this method has been criticized as crude, von Neumann was aware of this: he justified it as being faster than any other method at his disposal, and also noted that when it went awry it did so obviously, unlike methods which could be subtly incorrect.

While consulting for the Moore School of Electrical Engineering at the University of Pennsylvania on the EDVAC project, von Neumann wrote an incomplete *First Draft of a Report on the EDVAC*. The paper, which was widely distributed, described a computer architecture in which the data and the program are both stored in the computer's memory in the same address space. This architecture is to this day the basis of modern computer design, unlike the earliest computers that were 'programmed' by altering the electronic circuitry. Although the single-memory, stored program architecture is commonly called von Neumann architecture as a result of von Neumann's paper, the architecture's description was based on the work of J. Presper Eckert and John William Mauchly, inventors of the ENIAC at the University of Pennsylvania.^[25]

Von Neumann also created the field of cellular automata without the aid of computers, constructing the first self-replicating automata with pencil and graph paper. The concept of a universal constructor was fleshed out in his posthumous work *Theory of Self Reproducing Automata*.^[26] Von Neumann proved that the most effective way of performing large-scale mining operations such as mining an entire moon or asteroid belt would be by using self-replicating machines, taking advantage of their exponential growth.

He is credited with at least one contribution to the study of algorithms. Donald Knuth cites von Neumann as the inventor, in 1945, of the merge sort algorithm, in which the first and second halves of an array are each sorted recursively and then merged together.^[27] His algorithm for simulating a fair coin with a biased coin^[28] is used in the "software whitening" stage of some hardware random number generators.

He also engaged in exploration of problems in numerical hydrodynamics. With R. D. Richtmyer he developed an algorithm defining *artificial viscosity* that improved the understanding of shock waves. It is possible that we would not understand much of astrophysics, and might not have highly developed jet and rocket engines without that work. The problem was that when computers solve hydrodynamic or aerodynamic problems, they try to put too many computational grid points at regions of sharp discontinuity (shock waves). The *artificial viscosity* was a

mathematical trick to slightly smooth the shock transition without sacrificing basic physics.

Politics and social affairs

Von Neumann obtained at the age of 29 one of the first five professorships at the new Institute for Advanced Study in Princeton, New Jersey (another had gone to Albert Einstein). He was a frequent consultant for the Central Intelligence Agency, the United States Army, the RAND Corporation, Standard Oil, IBM, and others.

Throughout his life von Neumann had a respect and admiration for business and government leaders; something which was often at variance with the inclinations of his scientific colleagues. He enjoyed associating with persons in positions of power, and this led him into government service.^[29]

As President of the Von Neumann Committee for Missiles, and later as a member of the United States Atomic Energy Commission, from 1953 until his death in 1957, he was influential in setting U.S. scientific and military policy. Through his committee, he developed various scenarios of nuclear proliferation, the development of intercontinental and submarine missiles with atomic warheads, and the controversial strategic equilibrium called mutual assured destruction. During a Senate committee hearing he described his political ideology as "violently anti-communist, and much more militaristic than the norm".

Von Neumann's interest in meteorological prediction led him to propose manipulating the environment by spreading colorants on the polar ice caps to enhance absorption of solar radiation (by reducing the albedo), thereby raising global temperatures. He also favored a preemptive nuclear attack on the Soviet Union, believing that doing so could prevent it from obtaining the atomic bomb.^[30]

Personality

Von Neumann invariably wore a conservative grey flannel business suit – he was even known to play tennis wearing his business suit – and he enjoyed throwing large parties at his home in Princeton, occasionally twice a week.^[31] His white clapboard house at 26 Westcott Road was one of the largest in Princeton.^[32] Despite being a notoriously bad driver, he nonetheless enjoyed driving (frequently while reading a book) – occasioning numerous arrests as well as accidents. When Cuthbert Hurd hired him as a consultant to IBM, Hurd often quietly paid the fines for his traffic tickets.^[33]

It was said of him at Princeton that, while he was indeed a demigod, he had made a detailed study of humans and could imitate them perfectly.^[34]

Von Neumann liked to eat and drink heavily; his wife, Klara, said that he could count everything except calories. He enjoyed Yiddish and "off-color" humor (especially limericks).^[14]

Honors

The John von Neumann Theory Prize of the Institute for Operations Research and the Management Sciences (INFORMS, previously TIMS-ORSA) is awarded annually to an individual (or group) who have made fundamental and sustained contributions to theory in operations research and the management sciences.

The IEEE John von Neumann Medal is awarded annually by the IEEE "for outstanding achievements in computer-related science and technology."

The John von Neumann Lecture is given annually at the Society for Industrial and Applied Mathematics (SIAM) by a researcher who has contributed to applied mathematics, and the chosen lecturer is also awarded a monetary prize.

The crater Von Neumann on the Moon is named after him.

The John von Neumann Computing Center in Princeton, New Jersey (40°20'55"N 74°35'32"W) was named in his honour.

The professional society of Hungarian computer scientists, John von Neumann Computer Society, is named after John von Neumann.^[35]

On February 15, 1956, Neumann was presented with the Presidential Medal of Freedom by President Dwight Eisenhower.

On May 4, 2005 the United States Postal Service issued the *American Scientists* commemorative postage stamp series, a set of four 37-cent self-adhesive stamps in several configurations. The scientists depicted were John von Neumann, Barbara McClintock, Josiah Willard Gibbs, and Richard Feynman.

The John von Neumann Award of The Rajk László College for Advanced Studies was named in his honour, and has been given every year since 1995 to professors who have made an outstanding contribution to the exact social sciences and through their work have strongly influenced the professional development and thinking of the members of the college.

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Notes

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- [10] William Poundstone, *Prisoner's dilemma* (Oxford, 1993), introduction
- [11] Impagliazzo, p. 5
- [12] While there is a general agreement that the initially discovered bone tumor was a secondary growth, sources differ as to the location of the primary cancer. While Macrae gives it as pancreatic, the *Life* magazine article says it was prostate.
- [13] The question of whether or not von Neumann had formally converted to Catholicism upon his marriage to Mariette Kövesi (who was Catholic) is addressed by Halmos (ref. 5). He was baptised Roman Catholic but he certainly was not a practicing member of that religion after his divorce.
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- Oral history interview with Alice R. Burks and Arthur W. Burks (<http://www.cbi.umn.edu/oh/display.phtml?id=43>), Charles Babbage Institute, University of Minnesota, Minneapolis. Alice Burks and Arthur Burks describe ENIAC, EDVAC, and IAS computers, and John von Neumann's contribution to the development of computers.
- Oral history interview with Eugene P. Wigner (<http://www.cbi.umn.edu/oh/display.phtml?id=77>), Charles Babbage Institute, University of Minnesota, Minneapolis. Wigner talks about his association with John von Neumann during their school years in Hungary, their graduate studies in Berlin, and their appointments to

Princeton in 1930. Wigner discusses von Neumann's contributions to the theory of quantum mechanics, and von Neumann's interest in the application of theory to the atomic bomb project.

- Oral history interview with Nicholas C. Metropolis (<http://www.cbi.umn.edu/oh/display.phtml?id=81>), Charles Babbage Institute, University of Minnesota. Metropolis, the first director of computing services at Los Alamos National Laboratory, discusses John von Neumann's work in computing. Most of the interview concerns activity at Los Alamos: how von Neumann came to consult at the laboratory; his scientific contacts there, including Metropolis; von Neumann's first hands-on experience with punched card equipment; his contributions to shock-fitting and the implosion problem; interactions between, and comparisons of von Neumann and Enrico Fermi; and the development of Monte Carlo methods. Other topics include: the relationship between Alan Turing and von Neumann; work on numerical methods for non-linear problems; and the ENIAC calculations done for Los Alamos.
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- John von Neumann Postdoctoral Fellowship – Sandia National Laboratories (<http://ascr.sandia.gov/vonNeumannFellowship.htm>)
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- Budapest Tech Polytechnical Institution – John von Neumann Faculty of Informatics (<http://nik.bmf.hu/>)
- John von Neumann speaking at the dedication of the NORD (<http://elm.eeng.dcu.ie/~alife/von-neumann-1954-NORD/>), December 2, 1954 (audio recording)
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- John Von Neumann Memorial (<http://www.findagrave.com/cgi-bin/fg.cgi?page=gr&GRid=7333144>) at Find A Grave

John van Vleck

John Hasbrouck Van Vleck	
Born	March 13, 1899Middletown, Connecticut
Died	October 27, 1980Cambridge, Massachusetts
Nationality	United States
Fields	Physics
Institutions	University of Minnesota University of Wisconsin–Madison Harvard University University of Oxford Balliol College
Alma mater	University of Wisconsin-Madison Harvard University
Doctoral advisor	Edwin C. Kemble
Doctoral students	Robert Serber Edward Mills Purcell Philip Anderson
Notable awards	Elliott Cresson Medal (1971) Nobel Prize in Physics (1977)

John Hasbrouck Van Vleck (March 13, 1899 – October 27, 1980) was an American physicist and mathematician, co-awarded the 1977 Nobel Prize in Physics, for his contributions to the understanding of the behavior of electrons in magnetic solids.

Life and work

Born in Middletown, Connecticut the son of mathematician Edward Burr Van Vleck and grandson of astronomer John Monroe Van Vleck, he grew up in Madison, Wisconsin, received A.B. degree from the University of Wisconsin–Madison in 1920. Then he went to Harvard for graduate studies and earned a Ph.D degree in 1922. He joined the University of Minnesota as an assistant professor in 1923, then moved to the University of Wisconsin–Madison before settling at Harvard. He also earned *Honorary D. Sc.*, or *D. Honoris Causa*, degree from Wesleyan University in 1936.^[1]

J. H. van Vleck established the fundamentals of the quantum mechanical theory of magnetism and the crystal field theory (chemical bonding in metal complexes). He is regarded as the *Father of Modern Magnetism*.^{[2] [3] [4]}

During World War II, J. H. van Vleck worked on radar at the MIT Radiation Lab. He was half time at the Radiation Lab and half time on the staff at Harvard. He showed that at about 1.25-centimeter wavelength water molecules in the atmosphere would lead to troublesome absorption and that at 0.5-centimeter wavelength there would be a similar absorption by oxygen molecules.^{[5] [6] [7] [8]} This was to have important consequences not just for military (and civil) radar systems but later for the new science of radioastronomy.

J. H. van Vleck participated in the Manhattan Project. In June 1942, J. Robert Oppenheimer held a summer study for confirming the concept and feasibility of nuclear weapon at the University of California, Berkeley. Eight theoretical scientists, including J. H. van Vleck, attended it. From July to September, the theoretical study group examined and developed the principles of atomic bomb design.^{[9] [10] [11]} J. H. van Vleck's theoretical work led to establish the Los Alamos Nuclear Weapons Laboratory. He also served on the Los Alamos Review committee in 1943. The committee, established by General Leslie Groves, also consisted of W.K. Lewis of MIT, Chairman; E.L. Rose, of

Jones & Lamson; E.B. Wilson of Harvard; and Richard C. Tolman, Vice Chairman of NDRC. The committee's important contribution (originating with Rose) was a reduction in the size of the firing gun for the Little Boy atomic bomb, a concept which eliminated additional design-weight and sped up production of the bomb for its eventual release over Hiroshima. However it was not employed for the Fat Man bomb at Nagasaki, which relied on implosion of a plutonium shell to reach critical mass.^{[12] [13]}

In the year 1961-62 he was George Eastman Visiting Professor at University of Oxford^[14] and Professorship of Balliol College^[15]. He was awarded the National Medal of Science in 1966^[16] and the Lorentz Medal in 1974.^[17] For his contributions to the understanding of the behavior of electrons in magnetic solids, van Vleck was awarded the Nobel Prize in Physics 1977, along with Philip W. Anderson and Sir Nevill Mott.^[18] Van Vleck transformations and Van Vleck paramagnetism^[19] are also named after him.

Dr. Van Vleck died in Cambridge, Massachusetts, aged 81.^[20]

Writing

- *The Absorption of Radiation by Multiply Periodic Orbits, and its Relation to the Correspondence Principle and the Rayleigh-Jeans Law. Part I. Some Extensions of the Correspondence Principle*^[21], Physical Review, vol. 24, Issue 4, pp. 330–346 (1924)
- *The Absorption of Radiation by Multiply Periodic Orbits, and its Relation to the Correspondence Principle and the Rayleigh-Jeans Law. Part II. Calculation of Absorption by Multiply Periodic Orbits*^[22], Physical Review, vol. 24, Issue 4, pp. 347–365 (1924)
- *Quantum Principles and Line Spectra*, (Bulletin of the National Research Council; v. 10, pt 4, no. 54, 1926)
- *The Theory of Electric and Magnetic Susceptibilities* (Oxford at Clarendon, 1932).
- *Quantum Mechanics, The Key to Understanding Magnetism*^[23], Nobel Lecture, December 8, 1977

Japanese art collector

J. H. van Vleck and his wife Abigail were also important art collectors, particularly in the medium of Japanese woodblock prints (principally *Ukiyo-e*), known as *Van Vleck Collection*. It was inherited from his father Edward Burr Van Vleck. They donated it to the Chazen Museum of Art in Madison, Wisconsin in 1980s.^[24]

Notes

- [1] *Autobiography* (<http://nobelprize.org/physics/laureates/1977/vleck-autobio.html>), John H. van Vleck, The Nobel Prize in Physics 1977
- [2] John H. van Vleck (<http://www.iaqms.org/deceased/van-vleck.php>), International Academy of Quantum Molecular Science
- [3] *On the verge of Umdeutung in Minnesota: Van Vleck and the correspondence principle. Part One.* (<http://www.tc.umn.edu/~janss011/pdf files/dispersion-I.pdf>), Anthony Duncan, Michel Janssen; Elsevier Science, 8 May 2007
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- [5] *Norman F. Ramsey Oral History (1991)* ([http://www.ieeeeghn.org/wiki/index.php/Norman_F._Ramsey_Oral_History_\(1991\)](http://www.ieeeeghn.org/wiki/index.php/Norman_F._Ramsey_Oral_History_(1991))), NORMAN F. RAMSEY: An Interview Conducted by John Bryant, IEEE History Center, 20 June 1991
- [6] *Oral History Transcript* (<http://www.aip.org/history/ohilist/4373.html>), Interview with John H. Van Vleck by Katherine Sopka at Lyman Laboratory of Physics, 28 January 1977
- [7] *A radar history of World War II* (http://books.google.com/books?id=wpFMWeLmp4cC&pg=PA442&vq=Van+Vleck&dq=A+radar+history+of+World+War+II&source=gbs_search_s&cad=0), By Louis Brown, Page 442 and Page 521
- [8] *On the Shape of Collision-Broadened Lines* (http://radiometrics.com/vanvleck_revmodphys45.pdf), J. H. Van Vleck and V. F. Weisskopf; Reviews of Modern Physics, Volume 17, Number 2 and 3, April-July, 1945
- [9] *New Weapons Laboratory Gives Birth to the "Gadget"* (<http://www.lanl.gov/history/atomicbomb/gadget-born.shtml>), 50th Anniversary Article, Los Alamos National Laboratory
- [10] *Berkeley Summer Study Group* (http://www.atomicheritage.org/index.php?option=com_content&task=view&id=296), The Atomic Heritage Foundation
- [11] *Atomic History Timeline 1900- 1942* (http://www.atomicheritage.org/index.php?option=com_content&task=view&id=287), The Atomic Heritage Foundation

- [12] *Oversight Committee Formed as Lab Begins Research* (<http://www.lanl.gov/history/road/oversight.shtml>), 50th Anniversary Article, Los Alamos National Laboratory
- [13] *Now It Can Be Told*: Leslie R. Groves, Lieutenant General, U.S. Army, Retired; Harper, 1962, pp. 162-63.
- [14] Nobel Laureates (http://www.ox.ac.uk/about_the_university/oxford_people/oxonian_award_winners/#aphysics), University of Oxford
- [15] Inspiring minds: the Eastman Professors (http://alumni.balliol.ox.ac.uk/news/fd2006/eastman_professors.asp), Floreat Domus, Balliol College News, Issue 12, June 2006
- [16] The President's National Medal of Science: Recipient Details (http://www.nsf.gov/od/nms/recipient_details.cfm?recipient_id=371), National Science Foundation
- [17] The Lorents medal (<http://www.lorentz.leidenuniv.nl/lorentzmedal/>)
- [18] The Nobel Prize in Physics 1977 (http://nobelprize.org/nobel_prizes/physics/laureates/1977/)
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- [20] *John Van Vleck, Nobel Laureate Known for Work on Magnetism; Earned Three Degree* (<http://select.nytimes.com/gst/abstract.html?res=F00D14FB3D5E12728DDDA10A94D8415B8084F1D3&scp=1&sq=John Van Vleck&st=cse>), The New York Times, October 28, 1980, Tuesday Page A32
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External links

- The Theory Of Electric And Magnetic Susceptibilities (<http://www.archive.org/details/theoryofelectric031070mbp>)
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- Chazen Museum of Art (<http://www.chazen.wisc.edu/home.htm>)
- Oral history interview transcript with John Hasbrouck Van Vleck 14 October 1963, American Institute of Physics, Niels Bohr Library & Archives (http://www.aip.org/history/ohilist/4930_1.html)
- Oral history interview transcript with John Hasbrouck Van Vleck 28 February 1966, American Institute of Physics, Niels Bohr Library & Archives (http://www.aip.org/history/ohilist/4931_1.html)
- Oral history interview transcript with John Hasbrouck Van Vleck 28 January 1977, American Institute of Physics, Niels Bohr Library & Archives (<http://www.aip.org/history/ohilist/4373.html>)

Paul Dirac

Paul Adrien Maurice Dirac	
	
Born	Paul Adrien Maurice Dirac8 August 1902Bristol, England
Died	20 October 1984 (aged 82)Tallahassee, Florida, USA
Nationality	Switzerland (until 1919) United Kingdom (from 1919) ^[1]
Fields	Physics
Institutions	University of Cambridge Florida State University
Alma mater	University of Bristol University of Cambridge
Doctoral advisor	Ralph Fowler
Doctoral students	Homi Bhabha Harish Chandra Mehrotra Dennis Sciama Behram Kurşunoğlu John Polkinghorne
Known for	Dirac equation Dirac comb Dirac delta function Fermi–Dirac statistics Dirac sea Dirac spinor Dirac measure Bra-ket notation Dirac adjoint Dirac large numbers hypothesis Dirac fermion Dirac string Dirac algebra Dirac operator Abraham-Lorentz-Dirac force Dirac bracket Fermi–Dirac integral Negative probability Dirac Picture Dirac-Coulomb-Breit Equation
Notable awards	Nobel Prize in Physics (1933) Copley Medal (1952)

Notes

He is the stepfather of Gabriel Andrew Dirac.

Paul Adrien Maurice Dirac, OM, FRS (pronounced /dɪˈræk/ *di-RAK*; 8 August 1902 – 20 October 1984) was an English theoretical physicist who was one of the founders of quantum mechanics and quantum electrodynamics. Dirac made fundamental contributions to the early development of both quantum mechanics and quantum electrodynamics. He held the Lucasian Chair of Mathematics at the University of Cambridge and spent the last fourteen years of his life at Florida State University.

Among other discoveries, he formulated the Dirac equation, which describes the behaviour of fermions, and predicted the existence of antimatter.

Dirac shared the Nobel Prize in physics for 1933 with Erwin Schrödinger, "for the discovery of new productive forms of atomic theory."^[2]

Early years

Paul Dirac was born in Bristol,^[3] England and grew up in the Bishopston area of the city. His father, Charles Dirac, was an immigrant from Saint-Maurice in the Canton of Valais, Switzerland. His mother was originally from Cornwall and the daughter of a mariner. Paul had an elder brother, Félix, who committed suicide in March 1925, and a younger sister, Béatrice. His early family life appears to have been unhappy due to his father's unusually strict and authoritarian nature. Dirac had a very strained relationship with his father, so much that after his death, he wrote, "I feel much freer now, and I am my own man." He was forced to speak only in French at dinner, and was punished for any grammatical mistakes. Dirac chose silence over punishment. Dirac, later in life mentioned that it was not until he was 23 that he realized that parents were supposed to love their children. His hatred lasted his father's entire life.

Dirac was educated first at Bishop Road Primary School and then at Merchant Venturers' Technical College (later Cotham School), where his father was a French teacher. The school was an institution attached to the University of Bristol, which emphasized scientific subjects and modern languages. This was an unusual arrangement at a time when secondary education in Britain was still dedicated largely to the classics, and something for which Dirac would later express gratitude.

Dirac studied electrical engineering at the University of Bristol, completing his degree in 1921. He then decided that his true calling lay in the mathematical sciences and, after completing a BA in applied mathematics at Bristol in 1923, he received a grant to conduct research at St John's College, Cambridge, where he would remain for most of his career. At Cambridge, Dirac pursued his interests in the theory of general relativity (an interest he gained earlier as a student in Bristol) and in the nascent field of quantum physics, under the supervision of Ralph Fowler.

Career

Dirac noticed an analogy between the Poisson brackets of classical mechanics and the recently proposed quantization rules in Werner Heisenberg's matrix formulation of quantum mechanics. This observation allowed Dirac to obtain the quantization rules in a novel and more illuminating manner. For this work, published in 1926, he received a Ph.D. from Cambridge.

In 1928, building on 2x2 spin matrices which he discovered independently (Abraham Pais quoted Dirac as saying "I believe I got these (matrices) independently of Pauli and possibly Pauli got these independently of me" ^[4]) of Wolfgang Pauli's work on non-relativistic spin systems, he proposed the Dirac equation as a relativistic equation of motion for the wavefunction of the electron.^[5] This work led Dirac to predict the existence of the positron, the electron's antiparticle, which he interpreted in terms of what came to be called the *Dirac sea*.^[6] The positron was observed by Carl Anderson in 1932. Dirac's equation also contributed to explaining the origin of quantum spin as a relativistic phenomenon.

The necessity of fermions i.e. matter being created and destroyed in Enrico Fermi's 1934 theory of beta decay, however, led to a reinterpretation of Dirac's equation as a "classical" field equation for any point particle of spin $\hbar/2$, itself subject to quantization conditions involving anti-commutators. Thus reinterpreted as a (quantum) field equation accurately describing quarks and leptons i.e. all elementary matter particles, this Dirac field equation is as central to theoretical physics as the Maxwell, Yang-Mills and Einstein field equations. Dirac is regarded as the founder of quantum electrodynamics, being the first to use that term. He also introduced the idea of vacuum polarization in the early 1930s. This work was key to the development of quantum mechanics by the next generation of theorists, and in particular Schwinger, Feynman, Sin-Itiro Tomonaga and Dyson in their formulation of quantum electrodynamics.

Dirac's *Principles of Quantum Mechanics*, published in 1930, is a landmark in the history of science. It quickly became one of the standard textbooks on the subject and is still used today. In that book, Dirac incorporated the previous work of Werner Heisenberg on matrix mechanics and of Erwin Schrödinger on wave mechanics into a single mathematical formalism that associates measurable quantities to operators acting on the Hilbert space of vectors that describe the state of a physical system. The book also introduced the delta function. Following his 1939 article,^[7] he also included the bra-ket notation in the third edition of his book,^[8] thereby contributing to its universal use nowadays.

In 1933, following his 1931 paper on magnetic monopoles, Dirac showed that the existence of a single magnetic monopole in the universe would suffice to explain the observed quantization of electrical charge. In 1975,^[9] 1982,^[10] and 2009^{[11] [12] [13]} intriguing results suggested the possible detection of magnetic monopoles, but there is, to date, no direct evidence for their existence.

Dirac was the Lucasian Professor of Mathematics at Cambridge from 1932 to 1969. In 1937, he proposed a speculative cosmological model based on the so-called large numbers hypothesis. During World War II, he conducted important theoretical and experimental research on uranium enrichment by gas centrifuge.

Dirac's quantum electrodynamics made predictions that were - more often than not - infinite and therefore unacceptable. A workaround known as renormalization was developed, but Dirac never accepted this. "I must say that I am very dissatisfied with the situation," he said in 1975, "because this so-called 'good theory' does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small — not neglecting it just because it is infinitely great and you do not want it!"^[14] His refusal to accept renormalization, resulted in his work on the subject moving increasingly out of the mainstream. However, from his once rejected notes he managed to work on putting QED on "logical foundations" based on Hamiltonian formalism that he formulated. He found a rather novel way of deriving the anomalous magnetic moment "Schwinger term" and also the Lamb shift, afresh, using the Heisenberg picture and without using the joining method used by Weisskopf and French, the two pioneers of modern QED, Schwinger and Feynman, in 1963. That was two years before the Tomonaga-Schwinger-Feynman QED was given formal recognition by an award of the Nobel Prize for physics. Weisskopf and French (FW) were the first to obtain the correct result for the Lamb shift and the anomalous magnetic moment of the electron. At first FW results did not agree with the incorrect but independent results of Feynman and Schwinger (Schweber SS 1994 "QED and the men who made it: Dyson, Feynman, Schwinger and Tomonaga", Princeton :PUP). The 1963-1964 lectures Dirac gave on quantum field theory at Yeshiva University were published in 1966 as the Belfer Graduate School of Science, Monograph Series Number, 3. After having relocated to Florida in order to be near his elder daughter, Mary, Dirac spent his last fourteen years (of both life and physics research) at the University of Miami in Coral Gables, Florida and Florida State University in Tallahassee, Florida.

In the 1950s in his search for a better QED, Paul Dirac developed the Hamiltonian theory of constraints (Canad J Math 1950 vol 2, 129; 1951 vol 3, 1) based on lectures that he delivered at the 1949 International Mathematical Congress in Canada. Dirac (1951 "The Hamiltonian Form of Field Dynamics" Canad Jour Math, vol 3 ,1) had also solved the problem of putting the Tomonaga-Schwinger equation into the Schrödinger representation (See Phillips R J N 1987 "Tributes to Dirac" p31 London:Adam Hilger) and given explicit expressions for the scalar meson field

(spin zero pion or pseudoscalar meson), the vector meson field (spin one rho meson), and the electromagnetic field (spin one massless boson, photon).

The Hamiltonian of constrained systems is one of Dirac's many masterpieces. It is a powerful generalization of Hamiltonian theory that remains valid for curved spacetime. The equations for the Hamiltonian involve only six degrees of freedom described by g_{rs} , p^{rs} for each point of the surface on which the state is considered. The g_{m0} ($m = 0,1,2,3$) appear in the theory only through the variables g^{r0} , $(-g^{00})^{-1/2}$ which occur as arbitrary coefficients in the equations of motion. $H = \int d^3x [(-g^{00})^{-1/2} H_L - g^{r0} / g^{00} H_r]$ There are four constraints or weak equations for each point of the surface $x^0 = \text{constant}$. Three of them H_r form the four vector density in the surface. The fourth H_L is a 3-dimensional scalar density in the surface $H_L \approx 0$; $H_r \approx 0$ ($r=1,2,3$) In the late 1950s he applied the Hamiltonian methods he had developed to cast Einstein's general relativity in Hamiltonian form (Proc Roy Soc 1958,A vol 246, 333, Phys Rev 1959, vol 114, 924) and to bring to a technical completion the quantization problem of gravitation and bring it also closer to the rest of physics according to Salam and DeWitt. In 1959 also he gave an invited talk on "Energy of the Gravitational Field" at the New York Meeting of the American Physical Society later published in 1959 Phys Rev Lett 2, 368. In 1964 he published his "Lectures on Quantum Mechanics" (London: Academic) which deals with constrained dynamics of nonlinear dynamical systems including quantization of curved spacetime. He also published a paper entitled "Quantization of the Gravitational Field" in 1967 ICTP/IAEA Trieste Symposium on Contemporary Physics.

If one considers waves moving in the direction x^3 resolved into the corresponding Fourier components ($r,s = 1,2,3$), the variables in the degrees of freedom 13,23,33 are affected by the changes in the coordinate system whereas those in the degrees of freedom 12, (11-22) remain invariant under such changes. The expression for the energy splits up into terms each associated with one of these six degrees of freedom without any cross terms associated with two of them. The degrees of freedom 13, 23, 33 do not appear at all in the expression for energy of gravitational waves in the direction x^3 . The two degrees of freedom 12, (11-22) contribute a positive definite amount of such a form to represent the energy of gravitational waves. These two degrees of freedom correspond in the language of quantum theory, to the gravitational photons (gravitons) with spin +2 or -2 in their direction of motion. The degrees of freedom (11+22) gives rise to the Newtonian potential energy term showing the gravitational force between the two positive mass is attractive and the self energy of every mass is negative.

Amongst his many students was John Polkinghorne, who recalls that Dirac "was once asked what was his fundamental belief. He strode to a blackboard and wrote that the laws of nature should be expressed in beautiful equations."^[15]

Personal life

Dirac married Eugene Wigner's sister, Margit, in 1937. He adopted Margit's two children, Judith and Gabriel. Paul and Margit Dirac had two children together, both daughters, Mary Elizabeth and Florence Monica.

Margit, known as Mancini, visited her brother in 1934 in Princeton from her native Hungary and, while at dinner at the Annex Restaurant (1930s–2006^[16]), met the "lonely-looking man at the next table." This account came from a physicist from Korea who met and was influenced by Dirac, Y.S. Kim, who has also written: "It is quite fortunate for the physics community that Mancini took good care of our respected Paul A.M. Dirac. Dirac published eleven papers during the period 1939-46.... Dirac was able to maintain his normal research productivity only because Mancini was in charge of everything else."^[17]

A reviewer of the 2009 biography writes: "Dirac blamed his [emotional] frailties on his father, a Swiss immigrant who bullied his wife, chivvied his children and insisted Paul spoke only French at home, even though the Diracs lived in Bristol. 'I never knew love or affection when I was a child,' Dirac once said." She also writes that "[t]he problem lay with his genes. Both father and son had autism, to differing degrees. Hence the Nobel winner's reticence, literal-mindedness, rigid patterns of behaviour and self-centredness. [Quoting the biography:] 'Dirac's

traits as a person with autism were crucial to his success as a theoretical physicist: his ability to order information about mathematics and physics in a systematic way, his visual imagination, his self-centredness, his concentration and determination."^[18]

Personality

Dirac was known among his colleagues for his precise and taciturn nature. His colleagues in Cambridge jokingly defined a unit of a dirac which was one word per hour.^[19] When Niels Bohr complained that he did not know how to finish a sentence in a scientific article he was writing, Dirac replied, "I was taught at school never to start a sentence without knowing the end of it."^[20] He criticized the physicist J. Robert Oppenheimer's interest in poetry: "The aim of science is to make difficult things understandable in a simpler way; the aim of poetry is to state simple things in an incomprehensible way. The two are incompatible."^[21]

Dirac himself wrote in his diary during his postgraduate years that he concentrated solely on his research, and only stopped on Sunday, when he took long strolls alone.

An anecdote recounted in a review of the 2009 biography tells of Werner Heisenberg and Dirac sailing on a cruise ship to a conference in Japan in August 1929. "Both still in their twenties, and unmarried, they made an odd couple. Heisenberg was a ladies' man who constantly flirted and danced, while Dirac—'an Edwardian geek', as [biographer] Graham Farmelo puts it—suffered agonies if forced into any kind of socialising or small talk. 'Why do you dance?' Dirac asked his companion. 'When there are nice girls, it is a pleasure,' Heisenberg replied. Dirac pondered this notion, then blurted out: 'But, Heisenberg, how do you know beforehand that the girls are nice?'"^[18]

According to a story told in different versions, a friend or student visited Dirac, not knowing of his marriage. Noticing the visitor's surprise at seeing an attractive woman in the house, Dirac said, "This is... this is Wigner's sister". Margit Dirac told both George Gamow and Anton Capri in the 1960s that her husband had actually said, "Allow me to present Wigner's sister, who is now my wife."^[22] ^[23]

Dirac was also noted for his personal modesty. He called the equation for the time evolution of a quantum-mechanical operator, which he was the first to write down, the "Heisenberg equation of motion". Most physicists speak of Fermi-Dirac statistics for half-integer-spin particles and Bose-Einstein statistics for integer-spin particles. While lecturing later in life, Dirac always insisted on calling the former "Fermi statistics". He referred to the latter as "Einstein statistics" for reasons, he explained, of "symmetry".

Religious views

Heisenberg recalls a friendly conversation among young participants at the 1927 Solvay Conference about Einstein and Planck's views on religion. Wolfgang Pauli, Heisenberg and Dirac took part in it. Dirac's contribution was a criticism of the political purpose of religion, which was much appreciated for its lucidity by Bohr when Heisenberg reported it to him later. Among other things, Dirac said:^[24]

“I cannot understand why we idle discussing religion. If we are honest—and scientists have to be—we must admit that religion is a jumble of false assertions, with no basis in reality. The very idea of God is a product of the human imagination. It is quite understandable why primitive people, who were so much more exposed to the overpowering forces of nature than we are today, should have personified these forces in fear and trembling. But nowadays, when we understand so many natural processes, we have no need for such solutions. I can't for the life of me see how the postulate of an Almighty God helps us in any way. What I do see is that this assumption leads to such unproductive questions as why God allows so much misery and injustice, the exploitation of the poor by the rich and all the other horrors He might have prevented. If religion is still being taught, it is by no means because its ideas still convince us, but simply because some of us want to keep the lower classes quiet. Quiet people are much easier to govern than clamorous and dissatisfied ones. They are also much easier to exploit. Religion is a kind of opium that allows a nation to lull itself into wishful dreams and so forget the injustices that are being perpetrated against the people. Hence the close alliance between those two great political forces, the State and the Church. Both need the illusion that a kindly God rewards—in heaven if not on earth—all those who have not risen up against injustice, who have done their duty quietly and uncomplainingly. That is precisely why the honest assertion that God is a mere product of the human imagination is branded as the worst of all mortal sins.”

Heisenberg's view was tolerant. Pauli, raised as a Catholic had kept silent after some initial remarks, but when finally he was asked for his opinion, jokingly he said: "Well, I'd say that also our friend Dirac has got a religion and the first commandment of this religion is 'There is no God and Paul Dirac is his prophet.'" Everybody burst into laughter, including Dirac.^[24]

Death and after

In 1984, Dirac died in Tallahassee, Florida where he is buried.^[25]

The Dirac-Hellmann Award at FSU was endowed by Dr Bruce P. Hellmann (Dirac's last doctoral student) in 1997 to reward outstanding work in theoretical physics by FSU researchers. The Paul A.M. Dirac Science Library at FSU is named in his honour.

In 1995, a plaque in his honour bearing the Dirac equation was unveiled at Westminster Abbey in London with a speech from Stephen Hawking.

His childhood home in Bristol is commemorated with a blue plaque and the nearby Dirac Road is named in recognition of his links with the city.

A commemorative garden has been established opposite the railway station in Saint-Maurice, Switzerland, the town of origin of his father's family.

Honours

Dirac shared the 1933 Nobel Prize for physics with Erwin Schrödinger "for the discovery of new productive forms of atomic theory."^[2] Dirac was also awarded the Royal Medal in 1939 and both the Copley Medal and the Max Planck medal in 1952. He was elected a Fellow of the Royal Society in 1930, an Honorary Fellow of the American Physical Society in 1948, and an Honorary Fellow of the Institute of Physics, London in 1971. Dirac was awarded the Order of Merit, an outstanding recognition by the land of his birth, in 1973.

In 1975, Dirac gave a series of five lectures at the University of New South Wales which were subsequently published as a book, *Directions of Physics* (Wiley, 1978 – H. Hora and J. Shepanski, eds.). Dirac donated the royalties from this book to the University for the establishment of the Dirac Lecture Series.^[26] The Silver Dirac Medal for the Advancement of Theoretical Physics is awarded by the University of New South Wales on the occasion of the Public Dirac Lecture.

Immediately after his death, two organizations of professional physicists established annual awards in Dirac's memory. The Institute of Physics, the United Kingdom's professional body for physicists, awards the Paul Dirac Medal and Prize for "outstanding contributions to theoretical (including mathematical and computational) physics".^[27] The first three recipients were Stephen Hawking (1987), John Stewart Bell (1988), and Roger Penrose (1989). The Abdus Salam International Centre for Theoretical Physics (ICTP) awards the Dirac Medal of the ICTP each year on Dirac's birthday (8 August). Also, the Dirac Prize is awarded by the International Centre for Theoretical Physics in his memory. Dirac House in Bristol is the headquarters of Institute of Physics Publishing.

The street on which the National High Magnetic Field Laboratory in Tallahassee, Florida, is located was named Paul Dirac Drive. A second location on the Florida State University campus, the Paul A. M. Dirac Science Library, also bears his name. There is also a road named after him in his home town of Bristol, UK. The BBC named its video codec Dirac in his honour.

Legacy

Dirac is widely regarded as one of the world's greatest physicists. He was one of the founders of quantum mechanics and quantum electrodynamics.

His early contributions include the modern operator calculus for quantum mechanics, which he called transformation theory, and an early version of the path integral.^[28] He formulated a many-body formalism for quantum mechanics which allowed each particle to have its own proper time.

His relativistic wave equation for the electron was the first successful attack on the problem of relativistic quantum mechanics. Dirac founded quantum field theory with his reinterpretation of the Dirac equation as a many-body equation, which predicted the existence of antimatter and matter–antimatter annihilation. He was the first to formulate quantum electrodynamics, although he could not calculate arbitrary quantities because the short distance limit requires renormalization.

In an attempt to solve the quantum divergence problem, Dirac gave a classical point particle theory combining advanced and retarded waves to eliminate the classical electron self-energy. Although these classical methods did not immediately solve the problems in quantum electrodynamics, they did lead John Archibald Wheeler and Richard Feynman to formulate an alternative Green's function description for light, which eventually led to Feynman's point particle formulation of quantum field theory.

Dirac discovered the magnetic monopole solutions, the first topological configuration in physics, and used them to give the modern explanation of charge quantization. He developed constrained quantization in the 1960s, identifying the general quantum rules for arbitrary classical systems.

Dirac's quantum-field analysis of the vibrations of a membrane, in the early 1960s, proved extremely useful to modern practitioners of superstring theory and its closely related successor, M-Theory.^[29]

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Werner Heisenberg

Werner Heisenberg	
	
Born	Werner Karl Heisenberg5 December 1901Würzburg, Germany
Died	1 February 1976 (aged 74)Munich, Germany
Nationality	German
Fields	Physics
Institutions	University of Göttingen University of Copenhagen University of Leipzig University of Berlin University of Munich
Alma mater	University of Munich
Doctoral advisor	Arnold Sommerfeld
Other academic advisors	Niels Bohr Max Born
Doctoral students	Felix Bloch Edward Teller Rudolph E. Peierls Reinhard Oehme Friedwardt Winterberg Peter Mittelstaedt Șerban Țițeica Ivan Supek Erich Bagge Hermann Arthur Jahn Raziuddin Siddiqui Heimo Dolch Hans Heinrich Euler Edwin Gora Bernhard Kockel Arnold Siegert Wang Foh-san

Other notable students	William Vermillion Houston Guido Beck Ugo Fano
Known for	Uncertainty Principle Heisenberg's microscope Matrix mechanics Kramers-Heisenberg formula Heisenberg group Isospin
Influenced	Robert Döpel Carl Friedrich von Weizsäcker
Notable awards	Nobel Prize in Physics (1932) Max Planck Medal (1933)
Notes	He was the father of the neurobiologist Martin Heisenberg and the son of August Heisenberg

Werner Heisenberg (5 December 1901 – 1 February 1976) was a German theoretical physicist who made foundational contributions to quantum mechanics and is best known for asserting the uncertainty principle of quantum theory. In addition, he made important contributions to nuclear physics, quantum field theory, and particle physics.

Heisenberg, along with Max Born and Pascual Jordan, set forth the matrix formulation of quantum mechanics in 1925. Heisenberg was awarded the 1932 Nobel Prize in Physics for the creation of quantum mechanics, and its application especially to the discovery of the allotropic forms of hydrogen.^[1]

Following World War II, he was appointed director of the Kaiser Wilhelm Institute for Physics, which was soon thereafter renamed the Max Planck Institute for Physics. He was director of the institute until it was moved to Munich in 1958, when it was expanded and renamed the Max Planck Institute for Physics and Astrophysics.

Heisenberg was also president of the German Research Council, chairman of the Commission for Atomic Physics, chairman of the Nuclear Physics Working Group, and president of the Alexander von Humboldt Foundation.

Biography

Early years

Heisenberg was born in Würzburg, Germany to Kaspar Earnesta August Heisenberg, a secondary school teacher of classical languages who became Germany's only *ordentlicher Professor* (ordinarius professor) of medieval and modern Greek studies in the university system and his wife Annie Wecklein.^[2]

He studied physics and mathematics from 1920 to 1923 at the *Ludwig-Maximilians-Universität München* and the *Georg-August-Universität Göttingen*. At Munich, he studied under Arnold Sommerfeld and Wilhelm Wien. At Göttingen, he studied physics with Max Born and James Franck, and he studied mathematics with David Hilbert. He received his doctorate in 1923, at Munich under Sommerfeld. He completed his Habilitation in 1924, at Göttingen under Born.^{[3] [4]}

Because Sommerfeld had a sincere interest in his students and knew of Heisenberg's interest in Niels Bohr's theories on atomic physics, Sommerfeld took Heisenberg to Göttingen to the *Bohr-Festspiele* (Bohr Festival) in June 1922. At the event, Bohr was a guest lecturer and gave a series of comprehensive lectures on quantum atomic physics. There, Heisenberg met Bohr for the first time, and it had a significant and continuing effect on him.^{[5] [6] [7]}

Heisenberg's doctoral thesis, the topic of which was suggested by Sommerfeld, was on turbulence;^[8] the thesis discussed both the stability of laminar flow and the nature of turbulent flow. The problem of stability was investigated by the use of the Orr–Sommerfeld equation, a fourth order linear differential equation for small

disturbances from laminar flow. He would briefly return to this topic after World War II.^[9]

Heisenberg's paper on the anomalous Zeeman effect^[10] was accepted as his *Habilitationsschrift* under Max Born at Göttingen.^[11]

In his youth he was a member and Scoutleader of the *Neupfadfinder*, a German Scout association and part of the German Youth Movement.^{[12] [13] [14]} In August 1923 Robert Honsell and Heisenberg organized a trip (*Großfahrt*) to Finland with a Scout group of this association from Munich.^[15]

Career

Göttingen, Copenhagen, and Leipzig

From 1924 to 1927, Heisenberg was a Privatdozent at Göttingen. From 17 September 1924 to 1 May 1925, under an International Education Board Rockefeller Foundation fellowship, Heisenberg went to do research with Niels Bohr, director of the Institute of Theoretical Physics at the University of Copenhagen. He returned to Göttingen and with Max Born and Pascual Jordan, over a period of about six months, developed the matrix mechanics formulation of quantum mechanics. On 1 May 1926, Heisenberg began his appointment as a university lecturer and assistant to Bohr in Copenhagen. It was in Copenhagen, in 1927, that Heisenberg developed his uncertainty principle, while working on the mathematical foundations of quantum mechanics. In his paper^[16] on the uncertainty principle, Heisenberg used the word "*Ungenauigkeit*" (imprecision).^{[3] [17] [18]}

In 1927, Heisenberg was appointed *ordentlicher Professor* (ordinarius professor) of theoretical physics and head of the department of physics at the Universität Leipzig; he gave his inaugural lecture on 1 February 1928. In his first paper published from Leipzig,^[19] Heisenberg used the Pauli exclusion principle to solve the mystery of ferromagnetism.^{[3] [4] [17] [20]}

In Heisenberg's tenure at Leipzig, the quality of doctoral students, post-graduate and research associates who studied and worked with Heisenberg there is attested to by the acclaim later earned by these people; at various times, they included: Erich Bagge, Felix Bloch, Ugo Fano, Siegfried Flügge, William Vermillion Houston, Friedrich Hund, Robert S. Mulliken, Rudolf Peierls, George Placzek, Isidor Isaac Rabi, Fritz Sauter, John C. Slater, Edward Teller, John Hasbrouck van Vleck, Victor Frederick Weisskopf, Carl Friedrich von Weizsäcker, Gregor Wentzel and Clarence Zener.^[21]

In early 1929, Heisenberg and Wolfgang Pauli submitted the first of two papers^{[22] [23]} laying the foundation for relativistic quantum field theory. Also in 1929, Heisenberg went on a lecture tour in the United States, Japan, China, and India.^{[17] [21]}

Shortly after the discovery of the neutron by James Chadwick in 1932, Heisenberg submitted the first of three papers^{[24] [25] [26]} on his neutron-proton model of the nucleus. He was awarded the 1932 Nobel Prize in Physics.^{[17] [27]}

In 1928, the British mathematical physicist P. A. M. Dirac had derived the relativistic wave equation of quantum mechanics, which implied the existence of positive electrons, later to be named positrons. In 1932, from a cloud chamber photograph of cosmic rays, the American physicist Carl David Anderson identified a track as having been made by a positron. In mid-1933, Heisenberg presented his theory of the positron. His thinking on Dirac's theory and further development of the theory were set forth in two papers. The first, *Bemerkungen zur Diracschen Theorie des Positrons* (*Remarks on Dirac's theory of the positron*) was published in 1934,^[28] and the second, *Folgerungen aus der Diracschen Theorie des Positrons* (*Consequences of Dirac's Theory of the Positron*), was published in 1936.^{[17] [29] [30]}

In the early 1930s in Germany, the *deutsche Physik* movement was anti-Semitic and anti-theoretical physics, especially including quantum mechanics and the theory of relativity. As applied in the university environment, political factors took priority over the historically applied concept of scholarly ability,^[31] even though its two most prominent supporters were the Nobel Laureates in Physics Philipp Lenard^[32] and Johannes Stark.^[33]

After Adolf Hitler came to power in 1933, Heisenberg was attacked in the press as a "White Jew"^[34] by elements of the *deutsche Physik* (German Physics) movement for his insistence on teaching about the roles of Jewish scientists. As a result, he came under investigation by the SS. This was over an attempt to appoint Heisenberg as successor to Arnold Sommerfeld at the University of Munich. The issue was resolved in 1938 by Heinrich Himmler, head of the SS. While Heisenberg was not chosen as Sommerfeld's successor, he was rehabilitated to the physics community during the Third Reich. Nevertheless, supporters of *deutsche Physik* launched vicious attacks against leading theoretical physicists, including Arnold Sommerfeld and Heisenberg. On 29 June 1936, a National Socialist Party newspaper published a column attacking Heisenberg. On 15 July 1937, he was attacked in a journal of the SS. This was the beginning of what is called the Heisenberg Affair.^[17]

In mid-1936, Heisenberg presented his theory of cosmic-ray showers in two papers.^{[35] [36]} Four more papers^{[37] [38] [39] [40]} appeared in the next two years.^{[17] [41]}

In June 1939, Heisenberg bought a summer home for his family in Urfeld, in southern Germany. He also traveled to the United States in June and July, visiting Samuel Abraham Goudsmit, at the University of Michigan in Ann Arbor. However, Heisenberg refused an invitation to emigrate to the United States. He would not see Goudsmit again until six years later, when Goudsmit was the chief scientific advisor to the American Operation Alsos at the close of World War II. Ironically, Heisenberg would be arrested under Operation Alsos and detained in England under Operation Epsilon.^{[17] [42] [43]}

Matrix Mechanics and the Nobel Prize

Heisenberg's paper establishing quantum mechanics^[44] has puzzled physicists and historians. His methods assume that the reader is familiar with Kramers-Heisenberg transition probability calculations. The main new idea, noncommuting matrices, is justified only by a rejection of unobservable quantities. It introduces the non-commutative multiplication of matrices by physical reasoning, based on the correspondence principle, despite the fact that Heisenberg was not then familiar with the mathematical theory of matrices. The path leading to these results has been reconstructed in MacKinnon, 1977,^[45] and the detailed calculations are worked out in Aitchison et al.^[46]

In Copenhagen, Heisenberg and H. Kramers collaborated on a paper on dispersion, or the scattering from atoms of radiation whose wavelength is larger than the atoms. They showed that the successful formula Kramers had developed earlier could not be based on Bohr orbits, because the transition frequencies are based on level spacings which are not constant. The frequencies which occur in the Fourier transform of sharp classical orbits, by contrast, are equally spaced. But these results could be explained by a semi-classical Virtual State model: the incoming radiation excites the valence, or outer, electron to a virtual state from which it decays. In a subsequent paper Heisenberg showed that this virtual oscillator model could also explain the polarization of fluorescent radiation.

These two successes, and the continuing failure of the Bohr-Sommerfeld model to explain the outstanding problem of the anomalous Zeeman effect, led Heisenberg to use the virtual oscillator model to try to calculate spectral frequencies. The method proved too difficult to immediately apply to realistic problems, so Heisenberg turned to a simpler example, the anharmonic oscillator.

The dipole oscillator consists of a simple harmonic oscillator, which is thought of as a charged particle on a spring, perturbed by an external force, like an external charge. The motion of the oscillating charge can be expressed as a Fourier series in the frequency of the oscillator. Heisenberg solved for the quantum behavior by two different methods. First, he treated the system with the virtual oscillator method, calculating the transitions between the levels that would be produced by the external source.



Scanned at the American
Institute of Physics
Niels Bohr, Werner Heisenberg, and Wolfgang
Pauli, ca. 1935

He then solved the same problem by treating the anharmonic potential term as a perturbation to the harmonic oscillator and using the perturbation methods that he and Born had developed. Both methods led to the same results for the first and the very complicated second order correction terms. This suggested that behind the very complicated calculations lay a consistent scheme.

So Heisenberg set out to formulate these results without any explicit dependence on the virtual oscillator model. To do this, he replaced the Fourier expansions for the spatial coordinates by matrices, matrices which corresponded to the transition coefficients in the virtual oscillator method. He justified this replacement by an appeal to Bohr's correspondence principle and the Pauli doctrine that quantum mechanics must be limited to observables.

On 9 July, Heisenberg gave Born this paper to review and submit for publication. When Born read the paper, he recognized the formulation as one which could be transcribed and extended to the systematic language of matrices,^[47] which he had learned from his study under Jakob Rosanes^[48] at Breslau University. Born, with the help of his assistant and former student Pascual Jordan, began immediately to make the transcription and extension, and they submitted their results for publication; the paper was received for publication just 60 days after Heisenberg's paper.^[49] A follow-on paper was submitted for publication before the end of the year by all three authors.^[50] (A brief review of Born's role in the development of the matrix mechanics formulation of quantum mechanics along with a discussion of the key formula involving the non-commutivity of the probability amplitudes can be found in an article by Jeremy Bernstein, *Max Born and the Quantum Theory*.^[51] A detailed historical and technical account can be found in Mehra and Rechenberg's book *The Historical Development of Quantum Theory. Volume 3. The Formulation of Matrix Mechanics and Its Modifications 1925–1926*.^[52])

Up until this time, matrices were seldom used by physicists; they were considered to belong to the realm of pure mathematics. Gustav Mie had used them in a paper on electrodynamics in 1912 and Born had used them in his work on the lattices theory of crystals in 1921. While matrices were used in these cases, the algebra of matrices with their multiplication did not enter the picture as they did in the matrix formulation of quantum mechanics.^[53]

Born had learned matrix algebra from Rosanes, as already noted, but Born had also learned Hilbert's theory of integral equations and quadratic forms for an infinite number of variables as was apparent from a citation by Born of Hilbert's work *Grundzüge einer allgemeinen Theorie der Linearen Integralgleichungen* published in 1912.^[54] [55] Jordan, too was well equipped for the task. For a number of years, he had been an assistant to Richard Courant at Göttingen in the preparation of Courant and David Hilbert's book *Methoden der mathematischen Physik I*, which was published in 1924.^[56] This book, fortuitously, contained a great many of the mathematical tools necessary for the continued development of quantum mechanics. In 1926, John von Neumann became assistant to David Hilbert, and he would coin the term Hilbert space to describe the algebra and analysis which were used in the development of quantum mechanics.^[57] [58]

In 1928, Albert Einstein nominated Heisenberg, Born, and Jordan for the Nobel Prize in Physics,^[59] but it was not to be. The announcement of the Nobel Prize in Physics for 1932 was delayed until November 1933.^[60] It was at that time that it was announced Heisenberg had won the Prize for 1932 "for the creation of quantum mechanics, the application of which has, inter alia, led to the discovery of the allotropic forms of hydrogen"^[61] [62] and Erwin Schrödinger and Paul Adrien Maurice Dirac shared the 1933 Prize "for the discovery of new productive forms of atomic theory".^[62] One can rightly ask why Born was not awarded the Prize in 1932 along with Heisenberg – Bernstein gives some speculations on this matter. One of them is related to Jordan joining the Nazi Party on 1 May 1933 and becoming a Storm Trooper.^[63] Hence, Jordan's Party affiliations and Jordan's links to Born may have affected Born's chance at the Prize at that time. Bernstein also notes that when Born won the Prize in 1954, Jordan was still alive, and the Prize was awarded for the statistical interpretation of quantum mechanics, attributable alone to Born.^[64]

Heisenberg's reaction to Born for Heisenberg receiving the Prize for 1932 and to Born for Born receiving the Prize in 1954 are also instructive in evaluating whether Born should have shared the Prize with Heisenberg. On 25 November 1933, Born received a letter from Heisenberg in which he said he had been delayed in writing due to a "bad

conscience" that he alone had received the Prize "for work done in Göttingen in collaboration – you, Jordan and I." Heisenberg went on to say that Born and Jordan's contribution to quantum mechanics cannot be changed by "a wrong decision from the outside."^[65] In 1954, Heisenberg wrote an article honoring Max Planck for his insight in 1900. In the article, Heisenberg credited Born and Jordan for the final mathematical formulation of matrix mechanics and Heisenberg went on to stress how great their contributions were to quantum mechanics, which were not "adequately acknowledged in the public eye."^[66]

The *deutsche Physik* movement

On 1 April 1935, the eminent theoretical physicist Arnold Sommerfeld, Heisenberg's doctoral advisor at the University of Munich, achieved emeritus status. However, Sommerfeld stayed in his chair during the selection process for his successor, which took until 1 December 1939. The process was lengthy due to academic and political differences between the Munich Faculty's selection and that of the Reichserziehungsministerium (REM, Reich Education Ministry.) and the supporters of *Deutsche Physik*, which was anti-Semitic and had a bias against theoretical physics, especially quantum mechanics and the theory of relativity. In 1935, the Munich Faculty drew up a list of candidates to replace Sommerfeld as ordinarius professor of theoretical physics and head of the Institute for Theoretical Physics at the University of Munich. There were three names on the list: Werner Heisenberg, who received the Nobel Prize in Physics for 1932, Peter Debye, who would receive the Nobel Prize in Chemistry in 1936, and Richard Becker - all former students of Sommerfeld. The Munich Faculty was firmly behind these candidates, with Heisenberg as their first choice. However, supporters of *Deutsche Physik* and elements in the REM had their own list of candidates and the battle dragged on for over four years. During this time, Heisenberg came under vicious attack by the *Deutsche Physik* supporters. One attack was published in *Das Schwarze Korps*, the newspaper of the Schutzstaffel (SS), headed by Heinrich Himmler. In this, Heisenberg was called a "White Jew" (i.e. an Aryan who acts like a Jew) who should be made to "disappear."^[67] These attacks were taken seriously, as Jews were violently attacked and incarcerated. Heisenberg fought back with an editorial and a letter to Himmler, in an attempt to resolve this matter and regain his honour. At one point, Heisenberg's mother visited Himmler's mother. The two women knew each other as Heisenberg's maternal grandfather and Himmler's father were rectors and members of a Bavarian hiking club. Eventually, Himmler settled the Heisenberg affair by sending two letters, one to SS Gruppenführer Reinhard Heydrich and one to Heisenberg, both on 21 July 1938. In the letter to Heydrich, Himmler said Germany could not afford to lose or silence Heisenberg as he would be useful for teaching a generation of scientists. To Heisenberg, Himmler said the letter came on recommendation of his family and he cautioned Heisenberg to make a distinction between professional physics research results and the personal and political attitudes of the involved scientists. The letter to Heisenberg was signed under the closing "Mit freundlichem Gruss und, Heil Hitler!" (With friendly greetings, Heil Hitler!)"^[68] Overall, the Heisenberg affair was a victory for academic standards and professionalism. However, the appointment of Wilhelm Müller to replace Sommerfeld was a political victory over academic standards. Müller was not a theoretical physicist, had not published in a physics journal, and was not a member of the Deutsche Physikalische Gesellschaft; his appointment was considered a travesty and detrimental to educating theoretical physicists.^{[68] [69] [70] [71] [72]}

During the SS investigation of Heisenberg, the three investigators had training in physics. Heisenberg had participated in the doctoral examination of one of them at the *Universität Leipzig*. The most influential of the three, however, was Johannes Juilfs. During their investigation, they had become supporters of Heisenberg as well as his position against the ideological policies of the *deutsche Physik* movement in theoretical physics and academia.^[73]

World War II

In 1939, shortly after the discovery of nuclear fission, the German nuclear energy project, also known as the *Uranverein* (Uranium Club), was begun. Heisenberg was one of the principal scientists leading research and development in the project.

From 15 to 22 September 1941, Heisenberg traveled to German-occupied Copenhagen to lecture and discuss nuclear research and theoretical physics with Niels Bohr. The meeting, and specifically what it might reveal about Heisenberg's intentions concerning developing nuclear weapons for the Nazi regime, is the subject of the award winning Michael Frayn play titled *Copenhagen*. Documents relating to the Bohr-Heisenberg meeting were released in 2002 by the Niels Bohr Archive and by the Heisenberg family.^{[74] [75]}

On 26 February 1942, Heisenberg presented a lecture to Reich officials on energy acquisition from nuclear fission, after the Army withdrew most of its funding.^[76] The Uranium Club was transferred to the Reich Research Council (RFR) in July 1942. On 4 June 1942, Heisenberg was summoned to report to Albert Speer, Germany's Minister of Armaments, on the prospects for converting the Uranium Club's research toward developing nuclear weapons. During the meeting, Heisenberg told Speer that a bomb could not be built before 1945, and would require significant monetary and manpower resources.^[77] Five days later, on 9 June 1942, Adolf Hitler issued a decree for the reorganization of the RFR as a separate legal entity under the Reich Ministry for Armament and Ammunition; the decree appointed Reich Marshall Göring as the president.^[78]

In September 1942, Heisenberg submitted his first paper of a three-part series on the scattering matrix, or S-matrix, in elementary particle physics. The first two papers were published in 1943^{[79] [80]} and the third in 1944.^[81] The S-matrix described only observables, i.e., the states of incident particles in a collision process, the states of those emerging from the collision, and stable bound states; there would be no reference to the intervening states. This was the same precedent as he followed in 1925 in what turned out to be the foundation of the matrix formulation of quantum mechanics through only the use of observables.^{[17] [41]}

In February 1943, Heisenberg was appointed to the Chair for Theoretical Physics at the *Friedrich-Wilhelms-Universität* (today, the Humboldt-Universität zu Berlin). In April, his election to the *Preußische Akademie der Wissenschaften* (Prussian Academy of Sciences) was approved. That same month, he moved his family to their retreat in Urfeld as Allied bombing increased in Berlin. In the summer, he dispatched the first of his staff at the *Kaiser-Wilhelm Institut für Physik* to Hechingen and its neighboring town of Haigerloch, on the edge of the Black Forest, for the same reasons. From 18–26 October, he traveled to German-occupied Netherlands. In December 1943, Heisenberg visited German occupied Poland.^{[17] [82]}

From 24 January to 4 February 1944, Heisenberg traveled to occupied Copenhagen, after the German Army confiscated Bohr's Institute of Theoretical Physics. He made a short return trip in April. In December, Heisenberg lectured in neutral Switzerland.^[17]

In January 1945, Heisenberg vacated the *Kaiser-Wilhelm Institut für Physik* with about all of his staff for the facilities in the Black Forest.^[17]

Uranium Club

In December 1938, the German chemists Otto Hahn and Fritz Strassmann sent a manuscript to *Naturwissenschaften* reporting they had detected the element barium after bombarding uranium with neutrons,^[83] simultaneously, they communicated these results to Lise Meitner, who had in July of that year fled to the Netherlands and then went to Sweden.^[84] Meitner, and her nephew Otto Robert Frisch, correctly interpreted these results as being nuclear fission.^[85] Frisch confirmed this experimentally on 13 January 1939.^{[86] [87]}

Paul Harteck was director of the physical chemistry department at the University of Hamburg and an advisor to the *Heereswaffenamt* (HWA, Army Ordnance Office). On 24 April 1939, along with his teaching assistant Wilhelm Groth, Harteck made contact with the *Reichskriegsministerium* (RKM, Reich Ministry of War) to alert them to the potential of military applications of nuclear chain reactions. Two days earlier, on 22 April 1939, after hearing a

colloquium paper by Wilhelm Hanle on the use of uranium fission in a *Uranmaschine* (uranium machine, i.e., nuclear reactor), Georg Joos, along with Hanle, notified Wilhelm Dames, at the *Reichserziehungsministerium* (REM, Reich Ministry of Education), of potential military applications of nuclear energy. The communication was given to Abraham Esau, head of the physics section of the *Reichsforschungsrat* (RFR, Reich Research Council) at the REM. On 29 April, a group, organized by Esau, met at the REM to discuss the potential of a sustained nuclear chain reaction. The group included the physicists Walther Bothe, Robert Döpel, Hans Geiger, Wolfgang Gentner (probably sent by Walther Bothe), Wilhelm Hanle, Gerhard Hoffmann and Georg Joos; Peter Debye was invited, but he did not attend. After this, informal work began at the Georg-August University of Göttingen by Joos, Hanle and their colleague Reinhold Mannfopff; the group of physicists was known informally as the first *Uranverein* (Uranium Club) and formally as *Arbeitsgemeinschaft für Kernphysik*. The group's work was discontinued in August 1939, when the three were called to military training.^{[88] [89] [90] [91]}

The second *Uranverein* began after the *Heereswaffenamt* (HWA, Army Ordnance Office) squeezed the *Reichsforschungsrat* (RFR, Reich Research Council) out of the *Reichserziehungsministerium* (REM, Reich Ministry of Education) and started the formal German nuclear energy project under military auspices. The second *Uranverein* was formed on 1 September 1939, the day World War II began, and it had its first meeting on 16 September 1939. The meeting was organized by Kurt Diebner, advisor to the HWA, and held in Berlin. The invitees included Walther Bothe, Siegfried Flügge, Hans Geiger, Otto Hahn, Paul Harteck, Gerhard Hoffmann, Josef Mattauch and Georg Stetter. A second meeting was held soon thereafter and included Klaus Clusius, Robert Döpel, Werner Heisenberg and Carl Friedrich von Weizsäcker. Also at this time, the *Kaiser-Wilhelm Institut für Physik* (KWIP, Kaiser Wilhelm Institute for Physics, after World War II the Max Planck Institute for Physics), in Berlin-Dahlem, was placed under HWA authority, with Diebner as the administrative director, and the military control of the nuclear research commenced.^{[90] [91] [92]}

When it was apparent that the nuclear energy project would not make a decisive contribution to ending the war effort in the near term, control of the KWIP was returned in January 1942 to its umbrella organization, the *Kaiser-Wilhelm Gesellschaft* (KWG, Kaiser Wilhelm Society, after World War II the Max-Planck Gesellschaft), and HWA control of the project was relinquished to the RFR in July 1942. The nuclear energy project thereafter maintained its *kriegswichtig* (important for the war) designation and funding continued from the military. However, the German nuclear power project was then broken down into the following main areas: uranium and heavy water production, uranium isotope separation and the *Uranmaschine* (uranium machine, i.e., nuclear reactor). Also, the project was then essentially split up between a number of institutes, where the directors dominated the research and set their own research agendas.^{[90] [93] [94]} The dominant personnel and facilities were the following:^{[95] [96] [97]}

- *Institut für Physik* (Walther Bothe) of the *Kaiser-Wilhelm Institut für medizinische Forschung* (KWImF, Kaiser Wilhelm Institute for Medical Research),
- Institute for Physical Chemistry (Klaus Clusius) at the Ludwig Maximilian University of Munich,
- HWA *Versuchsstelle* (testing station) in Gottow (Kurt Diebner),
- *Kaiser-Wilhelm-Institut für Chemie* (Otto Hahn),
- Physical Chemistry Department (Paul Harteck) of the University of Hamburg,
- *Kaiser-Wilhelm-Institut für Physik* (Werner Heisenberg),
- Second Experimental Physics Institute (Hans Kopfermann) at the Georg-August University of Göttingen,
- Auergesellschaft (Nikolaus Riehl), and
- *II. Physikalisches Institut* (Georg Stetter) at the University of Vienna.

Heisenberg was appointed director-in-residence of the KWIP on 1 July 1942, as Peter Debye was still officially the director and on leave in the United States; Debye had gone on leave as he was a citizen of The Netherlands and had refused to become a German citizen when the HWA took administrative control of the KWIP. Heisenberg still also had his department of physics at the University of Leipzig where work was done for the *Uranverein* by Robert Döpel and his wife Klara Döpel. During the period Kurt Diebner administered the KWIP under the HWA program,

considerable personal and professional animosity developed between Diebner and the Heisenberg inner circle – Heisenberg, Karl Wirtz, and Carl Friedrich von Weizsäcker.^{[17] [98]}

The point in 1942, when the army relinquished its control of the German nuclear energy project, was the zenith of the project relative to the number of personnel devoting time to the effort. There were only about 70 scientists working on the project, with about 40 devoting more than half their time to nuclear fission research. After this, the number of scientists working on applied nuclear fission diminished dramatically. Many of the scientists not working with the main institutes stopped working on nuclear fission and devoted their efforts to more pressing war related work.^[99]

Over time, the HWA and then the RFR controlled the German nuclear energy project. The most influential people in the project were Kurt Diebner, Abraham Esau, Walther Gerlach and Erich Schumann. Schumann was one of the most powerful and influential physicists in Germany. Schumann was director of the Physics Department II at the Frederick William University (later, University of Berlin), which was commissioned and funded by the *Oberkommando des Heeres* (OKW, Army High Command) to conduct physics research projects. He was also head of the research department of the HWA, assistant secretary of the Science Department of the OKW and *Bevollmächtigter* (plenipotentiary) for high explosives. Diebner, throughout the life of the nuclear energy project, had more control over nuclear fission research than did Walther Bothe, Klaus Clusius, Otto Hahn, Paul Harteck or Werner Heisenberg.^{[100] [101]}

1945: Operation Alsos and Operation Epsilon

Operation Alsos was an Allied effort commanded by the Russian-American Colonel Boris T. Pash. He reported directly to General Leslie Groves, commander of the Manhattan Engineer District, which was developing atomic weapons for the United States. The chief scientific advisor to Operation Alsos was the physicist Samuel Abraham Goudsmit. Goudsmit was selected for this task because of his knowledge of physics, he spoke German, and he personally knew a number of the German scientists working on the German nuclear energy project. He also knew little of the Manhattan Project, so, if he were captured, he would have little intelligence value to the Germans. The objectives of Operation Alsos were to determine if the Germans had an atomic bomb program and to exploit German atomic related facilities, intellectual materials, materiel resources, and scientific personnel for the benefit of the United States. Personnel on this operation generally swept into areas which had just come under control of the Allied military forces, but sometimes they operated in areas still under control by German forces.^{[102] [103] [104]}

Berlin had been a location of many German scientific research facilities. To limit casualties and loss of equipment, many of these facilities were dispersed to other locations in the latter years of the war. The *Kaiser-Wilhelm-Institut für Physik* (KWIP, Kaiser Wilhelm Institute for Physics) had mostly been moved in 1943 and 1944 to Hechingen and its neighboring town of Haigerloch, on the edge of the Black Forest, which eventually became the French occupation zone. This move and a little luck allowed the Americans to take into custody a large number of German scientists associated with nuclear research. The only section of the institute which remained in Berlin was the low-temperature physics section, headed by Ludwig Bewilogua (1906–83), who was in charge of the exponential uranium pile.^{[105] [106]}

Nine of the prominent German scientists who published reports in *Kernphysikalische Forschungsberichte* as members of the *Uranverein*^[107] were picked up by Operation Alsos and incarcerated in England under Operation Epsilon: Erich Bagge, Kurt Diebner, Walther Gerlach, Otto Hahn, Paul Harteck, Werner Heisenberg, Horst Korsching, Carl Friedrich von Weizsäcker and Karl Wirtz. Also, incarcerated was Max von Laue, although he had nothing to do with the nuclear energy project. Goudsmit, the chief scientific advisor to Operation Alsos, thought von Laue might be beneficial to the postwar rebuilding of Germany and would benefit from the high level contacts he would have in England.^[108]

Heisenberg had been captured and arrested by Colonel Pash at Heisenberg's retreat in Urfeld, on 3 May 1945, in what was a true alpine-type operation in territory still under control by German forces. He was taken to Heidelberg,

where, on 5 May, he met Goudsmit for the first time since the Ann Arbor visit in 1939. Germany surrendered just two days later. Heisenberg would not see his family again for eight months. Heisenberg was moved across France and Belgium and flown to England on 3 July 1945.^{[109] [110] [111]}

The 10 German scientists were held at Farm Hall in England. The facility had been a safe house of the British foreign intelligence MI6. During their detention, their conversations were recorded. Conversation thought to be of intelligence value were transcribed and translated into English. The transcripts were released in 1992. Bernstein has published an annotated version of the transcripts in his book *Hitler's Uranium Club: The Secret Recordings at Farm Hall*, along with an introduction to put them in perspective. A complete, unedited publication of the British version of the reports appeared as *Operation Epsilon: The Farm Hall Transcripts*, which was published in 1993 by the Institute of Physics in Bristol and by the University of California Press in the United States.^{[112] [113] [114]}

Post 1945

On 3 January 1946, the 10 Operation Epsilon detainees were transported to Alswede, Germany, which was in the British occupation zone. Heisenberg settled in Göttingen, also in the British zone. In July, he was named director of the *Kaiser-Wilhelm Institut für Physik* (KWIP, Kaiser Wilhelm Institute for Physics), then located in Göttingen. Shortly thereafter, it was renamed the *Max-Planck Institut für Physik*, in honor of Max Planck and to assuage political objections to the continuation of the institute. Heisenberg was its director until 1958. In 1958, the institute was moved to Munich, expanded, and renamed *Max-Planck-Institut für Physik und Astrophysik* (MPIFA). Heisenberg was its director from 1960 to 1970; in the interim, Heisenberg and the astrophysicist Ludwig Biermann were co-directors. Heisenberg resigned his directorship of the MPIFA on 31 December 1970. Upon the move to Munich, Heisenberg also became an *ordentlicher Professor* (ordinarius professor) at the University of Munich.^{[4] [17]}

Just as the Americans did with Operation Alsos, the Russians inserted special search teams into Germany and Austria in the wake of their troops. Their objective, under the Russian Alsos, was also the exploitation of German atomic related facilities, intellectual materials, materiel resources and scientific personnel for the benefit of the Soviet Union. One of the German scientists recruited under this Russian operation was the nuclear physicist Heinz Pose, who was made head of Laboratory V in Obninsk. When he returned to Germany on a recruiting trip for his laboratory, Pose wrote a letter to the Werner Heisenberg inviting him to work in Russia. The letter lauded the working conditions in Russia and the available resources, as well as the favorable attitude of the Russians towards German scientists. A courier hand delivered the recruitment letter, dated 18 July 1946, to Heisenberg; Heisenberg politely declined in a return letter to Pose.^{[115] [116]}

In 1947, Heisenberg presented lectures in Cambridge, Edinburgh and Bristol. Heisenberg also contributed to the understanding of the phenomenon of superconductivity with a paper in 1947^[117] and two papers in 1948,^{[118] [119]} one of them with Max von Laue.^{[17] [120]}

In the period shortly after World War II, Heisenberg briefly returned to the subject of his doctoral thesis, turbulence. Three papers were published in 1948^{[121] [122] [123]} and one in 1950.^{[9] [124]}

In the post-war period, Heisenberg continued his interests in cosmic-ray showers with considerations on multiple production of mesons. He published three papers^{[125] [126] [127]} in 1949, two^{[128] [129]} in 1952, and one^[130] in 1955.^[131]

On 9 March 1949, the *Deutsche Forschungsrat* (German Research Council) was established by the *Max-Planck Gesellschaft* (MPG, Max Planck Society, successor organization to the *Kaiser-Wilhelm Gesellschaft*). Heisenberg was appointed president of the *Deutsche Forschungsrat*. In 1951, the organization was fused with the *Notgemeinschaft der Deutschen Wissenschaft* (NG, Emergency Association of German Science) and that same year renamed the *Deutsche Forschungsgemeinschaft* (DFG, German Research Foundation). With the merger, Heisenberg was appointed to the presidium.^{[17] [132] [133]}

In 1952, Heisenberg served as the chairman of the Commission for Atomic Physics of the DFG. Also that year, he headed the German delegation to the European Council for Nuclear Research.^{[3] [17]}

In 1953, Heisenberg was appointed president of the *Alexander von Humboldt-Stiftung* by Konrad Adenauer. Heisenberg served until 1975. Also, from 1953, Heisenberg's theoretical work concentrated on the unified field theory of elementary particles.^{[3] [4] [17]}

In late 1955 to early 1956, Heisenberg gave the Gifford Lectures at St Andrews University, in Scotland, on the intellectual history of physics. The lectures were later published as *Physics and Philosophy: The Revolution in Modern Science*.^[134]

During 1956 and 1957, Heisenberg was the chairman of the *Arbeitskreis Kernphysik* (Nuclear Physics Working Group) of the *Fachkommission II "Forschung und Nachwuchs"* (Commission II "Research and Growth") of the *Deutschen Atomkommission* (DATK, German Atomic Energy Commission). Other members of the Nuclear Physics Working Group in both 1956 and 1957 were: Walther Bothe, Hans Kopfermann (vice-chairman), Fritz Bopp, Wolfgang Gentner, Otto Haxel, Willibald Jentschke, Heinz Maier-Liebnitz, Josef Mattauch, Wolfgang Riezler, Wilhelm Walcher and Carl Friedrich von Weizsäcker. Wolfgang Paul was also a member of the group during 1957.^[135]

In 1957, Heisenberg was a signatory of the manifesto of the *Göttinger Achtzehn* (Göttingen Eighteen).^[136]

From 1957, Heisenberg was interested in plasma physics and the process of nuclear fusion. He also collaborated with the International Institute of Atomic Physics in Geneva. He was a member of the Institute's Scientific Policy Committee, and for several years was the Committee's chairman.^[3]

In 1973, Heisenberg gave a lecture at Harvard University on the historical development of the concepts of quantum theory.^[137]

On 24 March 1973, Heisenberg gave a speech before the Catholic Academy of Bavaria, accepting the Romano Guardini Prize. An English translation of its title is "Scientific and Religious Truth." And its stated goal was "In what follows, then, we shall first of all deal with the unassailability and value of scientific truth, and then with the much wider field of religion, of which - so far as the Christian religion is concerned - Guardini himself has so persuasively written; finally - and this will be the hardest part to formulate - we shall speak of the relationship of the two truths."^[138] A more detail insight in Planck and Heisenberg on religion has been discussed by Wilfried Schröder in "Natural science and religion" (Bremen 1999, Science edition) and Wilfried Schröder "Naturerkenntnis und Religion" Bremen, science edition 2008).

Personal life

In January 1937 Heisenberg met Elisabeth Schumacher at a private music recital. Elisabeth was the daughter of a well-known Berlin economics professor. They were married on 29 April. The fraternal twins, Maria and Wolfgang, were born to them in January 1938, whereupon, Wolfgang Pauli congratulated Heisenberg on his "pair creation" – a word play on a process from elementary particle physics, pair production. They had five more children over the next 12 years: Barbara, Christine, Jochen, Martin and Verena. Jochen became a physics professor at the University of New Hampshire.^{[139] [140]}

Heisenberg enjoyed classical music and was an accomplished pianist.^[3] He was a Lutheran.^[141]

Heisenberg died of cancer of the kidneys and gall bladder at his home, on 1 February 1976.^[142] The next evening, his colleagues and friends walked in remembrance from the Institute of Physics to his home and each put a candle near the front door.^[143]

Honors and awards

Heisenberg was awarded a number of honors:^[3]

- Honorary doctorates from the University of Bruxelles, the Technological University of Karlsruhe, and the University of Budapest.
- Order of Merit of Bavaria
- Romano Guardini Prize^[138]
- Grand Cross for Federal Service with Star
- Knight of the Order of Merit (Peace Class)
- Fellow of the Royal Society of London
- Member of the Academies of Sciences of Göttingen, Bavaria, Saxony, Prussia, Sweden, Rumania, Norway, Spain, The Netherlands, Rome (Pontifical), the *Deutsche Akademie der Naturforscher Leopoldina* (Halle), the Accademia dei Lincei (Rome), and the American Academy of Sciences.
- 1932–Nobel Prize in Physics "for the creation of quantum mechanics, the application of which has, inter alia, led to the discovery of the allotropic forms of hydrogen".^[61]
- 1933–*Max-Planck-Medaille* of the *Deutsche Physikalische Gesellschaft*

Internal reports

The following reports were published in *Kernphysikalische Forschungsberichte* (*Research Reports in Nuclear Physics*), an internal publication of the German *Uranverein*. The reports were classified Top Secret, they had very limited distribution, and the authors were not allowed to keep copies. The reports were confiscated under the Allied Operation Alsos and sent to the United States Atomic Energy Commission for evaluation. In 1971, the reports were declassified and returned to Germany. The reports are available at the Karlsruhe Nuclear Research Center and the American Institute of Physics.^{[144] [145]}

- Robert Döpel, K. Döpel, and Werner Heisenberg *Bestimmung der Diffusionslänge thermischer Neutronen in Präparat 38*^[146] G-22 (5 December 1940)
- Robert Döpel, K. Döpel, and Werner Heisenberg *Bestimmung der Diffusionslänge thermischer Neutronen in schwerem Wasser* G-23 (7 August 1940)
- Werner Heisenberg *Die Möglichkeit der technischer Energiegewinnung aus der Uranspaltung* G-39 (6 December 1939)
- Werner Heisenberg *Bericht über die Möglichkeit technischer Energiegewinnung aus der Uranspaltung (II)* G-40 (29 February 1940)
- Robert Döpel, K. Döpel, and Werner Heisenberg *Versuche mit Schichtenanordnungen von D_2O und 38* G-75 (28 October 1941)
- Werner Heisenberg *Über die Möglichkeit der Energieerzeugung mit Hilfe des Isotops 238* G-92 (1941)
- Werner Heisenberg *Bericht über Versuche mit Schichtenanordnungen von Präparat 38 und Paraffin am Kaiser Wilhelm Institut für Physik in Berlin-Dahlem* G-93 (May 1941)
- Fritz Bopp, Erich Fischer, Werner Heisenberg, Carl-Friedrich von Weizsäcker, and Karl Wirtz *Untersuchungen mit neuen Schichtenanordnungen aus U-metall und Paraffin* G-127 (March 1942)
- Robert Döpel *Bericht über Unfälle beim Umgang mit Uranmetall* G-135 (9 July 1942)
- Werner Heisenberg *Bemerkungen zu dem geplanten halbtechnischen Versuch mit 1,5 to D_2O und 3 to 38-Metall* G-161 (31 July 1942)
- Werner Heisenberg, Fritz Bopp, Erich Fischer, Carl-Friedrich von Weizsäcker, and Karl Wirtz *Messungen an Schichtenanordnungen aus 38-Metall und Paraffin* G-162 (30 October 1942)
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- Werner Heisenberg *Die Energiegewinnung aus der Atomkernspaltung* G-217 (6 May 1943)

- Fritz Bopp, Walther Bothe, Erich Fischer, Erwin Fünfer, Werner Heisenberg, O. Ritter, and Karl Wirtz *Bericht über einen Versuch mit D_2O und U und 40 cm Kohlerückstreumantel (B7)* G-300 (3 January 1945)
- Robert Döpel, K. Döpel, and Werner Heisenberg *Die Neutronenvermehrung in einem D_2O -38-Metallschichtensystem* G-373 (March 1942)

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- Anna Ludovico, *Effetto Heisenberg. La rivoluzione scientifica che ha cambiato la storia*, Roma: Armando 2001, p. 224 ISBN 88-8358-182-2.
- Barbara Blum, Helmut Heisenberg, Anna Ludovico, *Per Heisenberg*, Roma: Aracne 2006, p. 96 ISBN 88-548-0636-6

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- A. Sommerfeld and W. Heisenberg *Eine Bemerkung über relativistische Röntgendoublets und Linienschärfe*, *Z. Phys.* Volume 10, 393-398 (1922)
- A. Sommerfeld and W. Heisenberg *Die Intensität der Mehrfachlinien und ihrer Zeeman-Komponenten*, *Z. Phys.* Volume 11, 131-154 (1922)
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- W. Heisenberg *Über eine Abänderung der formalen Regeln der Quantentheorie beim Problem der anomalen Zeeman-Effekte*, *Z. Phys.* Volume 26, 291-307 (1924)
- W. Heisenberg, *Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen*, *Zeitschrift für Physik*, **33**, 879-893 (1925). The paper was received on 29 July 1925. [English translation in: B. L. van der Waerden, editor, *Sources of Quantum Mechanics* (Dover Publications, 1968) ISBN 0-486-61881-1 (English title: *Quantum-Theoretical Re-interpretation of Kinematic and Mechanical Relations*).] This is the first paper in the famous trilogy which launched the matrix mechanics formulation of quantum mechanics.
- M. Born and P. Jordan, *Zur Quantenmechanik*, *Zeitschrift für Physik*, **34**, 858-888 (1925). The paper was received on 27 September 1925. [English translation in: B. L. van der Waerden, editor, *Sources of Quantum Mechanics* (Dover Publications, 1968) ISBN 0-486-61881-1 (English title: *On Quantum Mechanics*).] This is the second paper in the famous trilogy which launched the matrix mechanics formulation of quantum mechanics.

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Pop culture references

Television

- *Breaking Bad* protagonist, chemist/chemistry teacher/methamphetamine producer Walter White, adopts "Heisenberg" as his alias. This strategic choice effectively protects Walter's identity when his brother-in-law, DEA agent Hank Schrader, begins tracking down a formidable local meth producer known only as "Heisenberg".
- In a 1990 episode of *The Hogan Family* ("*Snowbound*", aired 01/08/90), Heisenberg was the answer to a historical figure guessing game being played by the family. Clues of Heisenberg's life were given by younger brother Mark. No one guessed the answer correctly.
- In the TV series "*Star Trek*", the writers included a device called "Heisenberg Compensators" in their teleporting devices, which theoretically compensate for the uncertainty in the subatomic measurements, thus making

teleportation feasible.

- In the TV series *Twin Peaks*, Annie Blackburn (played by Heather Graham) quotes Heisenberg to the series' chief protagonist, Dale Cooper (played by Kyle MacLachlan): "What we observe is not nature itself, but nature exposed to our method of questioning." (Season two, episode 20) This quote was originally delivered during a series of lectures at the University of St. Andrews, Scotland in the Winter of 1955-1956.

Video games

- In the 2003 computer game, *Empires: Dawn of the Modern World*, during the "Patton" Campaign, Heisenberg is referenced several times in a fictional representation of Germany countinuing the Vengeance Weapons to the "V10". Being equipped with a nuclear warhead and range capable of hitting the east coast of the United States of America. A presumably fellow, albeit fictional, scientist of Heisenberg's was gunned down by Russian troops ingame with his last words meant for the Americans to notify Heisenberg of the bomb's completion.

See also

- German inventors and discoverers

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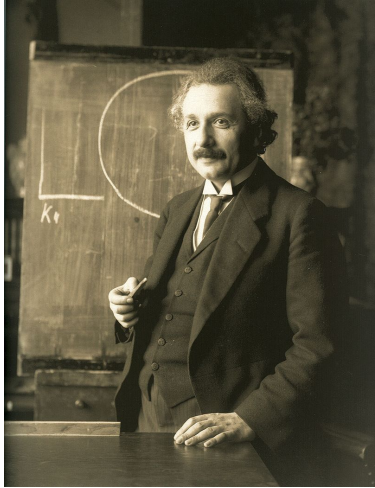
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Albert Einstein

Albert Einstein	
	
Albert Einstein in 1921	
Born	14 March 1879Ulm, Kingdom of Württemberg, German Empire
Died	18 April 1955 (aged 76)Princeton, New Jersey, United States
Resting place	Grounds of the Institute for Advanced Study, Princeton, New Jersey.
Residence	Germany, Italy, Switzerland, United States
Ethnicity	Jewish
Citizenship	- Württemberg/Germany (until 1896) - Stateless (1896–1901) - Switzerland (from 1901) - Austria (1911–12) - Germany (1914–33) - United States (from 1940)^[1]
Alma mater	- ETH Zurich - University of Zurich
Known for	- General relativity and special relativity - Photoelectric effect - Mass-energy equivalence - Quantification of the Brownian motion - Einstein field equations - Bose–Einstein statistics - Unified Field Theory
Spouse	- Mileva Marić (1903–1919) - Elsa Löwenthal, née Einstein, (1919–1936)
Awards	- Nobel Prize in Physics (1921) - Copley Medal (1925) - Max Planck Medal (1929) <i>Time</i> Person of the Century
Signature	



Albert Einstein (pronounced /ˈælbərt ˈaɪnʃtaɪn/; German: [ˈalbɐt ˈaɪnʃtaɪn] (listen); 14 March 1879 – 18 April 1955) was a theoretical physicist, philosopher and author who is widely regarded as one of the most influential and iconic scientists and intellectuals of all time. A German-Swiss Nobel laureate, Einstein is often regarded as the father of modern physics.^[2] He received the 1921 Nobel Prize in Physics "for his services to theoretical physics, and especially for his discovery of the law of the photoelectric effect".^[3]

Near the beginning of his career, Einstein thought that Newtonian mechanics was no longer enough to reconcile the laws of classical mechanics with the laws of the electromagnetic field. This led to the development of his special theory of relativity. He realized, however, that the principle of relativity could also be extended to gravitational fields, and with his subsequent theory of gravitation in 1916, he published a paper on the general theory of relativity. He continued to deal with problems of statistical mechanics and quantum theory, which led to his explanations of particle theory and the motion of molecules. He also investigated the thermal properties of light which laid the foundation of the photon theory of light. In 1917, Einstein applied the general theory of relativity to model the structure of the universe as a whole.^[4]

On the eve of World War II in 1939, he personally alerted President Franklin D. Roosevelt that Germany might be developing an atomic weapon, and recommended that the U.S. begin uranium procurement and nuclear research. As a result, Roosevelt advocated such research, leading to the creation of the top secret Manhattan Project, and the U.S. becoming the first and only country to possess nuclear weapons during the war. Days before his death, however, Einstein signed the Russell–Einstein Manifesto, that highlighted the dangers posed by the military usage of nuclear energy.

Einstein published more than 300 scientific along with over 150 non-scientific works, and received honorary doctorate degrees in science, medicine and philosophy from many European and American universities;^[4] he also wrote about various philosophical and political subjects such as socialism, international relations and the existence of God.^[5] His great intelligence and originality has made the word "Einstein" synonymous with genius.^[6]

Biography

Early life and education

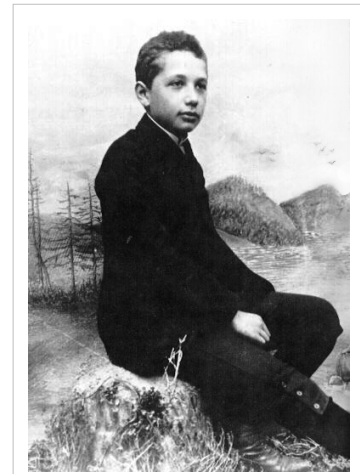


Einstein at the age of 4.

Albert Einstein was born in Ulm, in the Kingdom of Württemberg in the German Empire on 14 March 1879.^[7] His father was Hermann Einstein, a salesman and engineer. His mother was Pauline Einstein (née Koch). In 1880, the family moved to Munich, where his father and his uncle founded *Elektrotechnische Fabrik J. Einstein & Cie*, a company that manufactured electrical equipment based on direct current.^[7]

The Einsteins were non-observant Jews. Their son attended a Catholic elementary school from the age of five until ten.^[8] Although Einstein had early speech difficulties, he was a top student in elementary school.^{[9] [10]}

His father once showed him a pocket compass; Einstein realized that there must be something causing the needle to move, despite the apparent "empty space".^[11] As he grew, Einstein built models and mechanical devices for fun and began to show a talent for mathematics.^[7] In 1889, Max Talmud (later changed to Max Talmey) introduced the ten-year old Einstein to key texts in science, mathematics and philosophy, including Immanuel Kant's *Critique of Pure Reason* and *Euclid's Elements* (which Einstein called the "holy little geometry book").^[12] Talmud was a poor Jewish medical student from Poland. The Jewish community arranged for Talmud to take meals with the Einsteins each week on Thursdays for six years. During this time Talmud wholeheartedly guided Einstein through many secular educational interests.^{[13] [14]}



Albert Einstein in 1893 (age 14).

In 1894, his father's company failed: direct current (DC) lost the War of Currents to alternating current (AC). In search of business, the Einstein family moved to Italy, first to Milan and then, a few months later, to Pavia. When the family moved to Pavia, Einstein stayed in Munich to finish his studies at the Luitpold Gymnasium. His father intended for him to pursue electrical engineering, but Einstein clashed with authorities and resented the school's regimen and teaching method. He later wrote that the spirit of learning and creative thought were lost in strict rote learning. In the spring of 1895, he withdrew to join his family in Pavia, convincing the school to let him go by using a doctor's note.^[7] During this time, Einstein wrote his first scientific work, "The Investigation of the State of Aether in Magnetic Fields".^[15]

Einstein applied directly to the Eidgenössische Polytechnische Schule (ETH) in Zürich, Switzerland. Lacking the requisite Matura certificate, he took an entrance examination, which he failed, although he got exceptional marks in mathematics and physics.^[16] The Einsteins sent Albert to Aarau, in northern Switzerland to finish secondary school.^[7] While lodging with the family of Professor Jost Winteler, he fell in love with the family's daughter, Marie. (His sister Maja later married the Winteler's son Paul.)^[17] In Aarau, Einstein studied Maxwell's electromagnetic theory. At age 17, he graduated, and, with his father's approval, renounced his citizenship in the German Kingdom of Württemberg to avoid military service, and in 1896 he enrolled in the four year mathematics and physics teaching diploma program at the Polytechnic in Zurich. Marie Winteler moved to Olsberg, Switzerland for a teaching post.

Einstein's future wife, Mileva Marić, also enrolled at the Polytechnic that same year, the only woman among the six students in the mathematics and physics section of the teaching diploma course. Over the next few years, Einstein and Marić's friendship developed into romance, and they read books together on extra-curricular physics in which Einstein was taking an increasing interest. In 1900 Einstein was awarded the Zurich Polytechnic teaching diploma, but Marić failed the examination with a poor grade in the mathematics component, theory of functions.^[18] There have been claims that Marić collaborated with Einstein on his celebrated 1905 papers,^{[19] [20]} but historians of physics who have studied the issue find no evidence that she made any substantive contributions.^{[21] [22] [23] [24]}

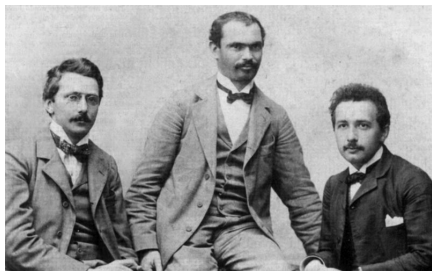
Marriages and children

In early 1902, Einstein and Mileva Marić had a daughter they named Lieserl in their correspondence, who was born in Novi Sad where Marić's parents lived.^[25] Her full name is not known, and her fate is uncertain after 1903.^[26]

Einstein and Marić married in January 1903. In May 1904, the couple's first son, Hans Albert Einstein, was born in Bern, Switzerland. Their second son, Eduard, was born in Zurich in July 1910. In 1914, Einstein moved to Berlin, while his wife remained in Zurich with their sons. Marić and Einstein divorced on 14 February 1919, having lived apart for five years.

Einstein married Elsa Löwenthal (née Einstein) on 2 June 1919, after having had a relationship with her since 1912. She was his first cousin maternally and his second cousin paternally. In 1933, they emigrated permanently to the United States. In 1935, Elsa Einstein was diagnosed with heart and kidney problems and died in December 1936.^[27]

Patent office



Left to right: Conrad Habicht, Maurice Solovine and Einstein, who founded the Olympia Academy

After graduating, Einstein spent almost two frustrating years searching for a teaching post, but a former classmate's father helped him secure a job in Bern, at the Federal Office for Intellectual Property, the patent office, as an assistant examiner.^[28] He evaluated patent applications for electromagnetic devices. In 1903, Einstein's position at the Swiss Patent Office became permanent, although he was passed over for promotion until he "fully mastered machine technology".^[29]

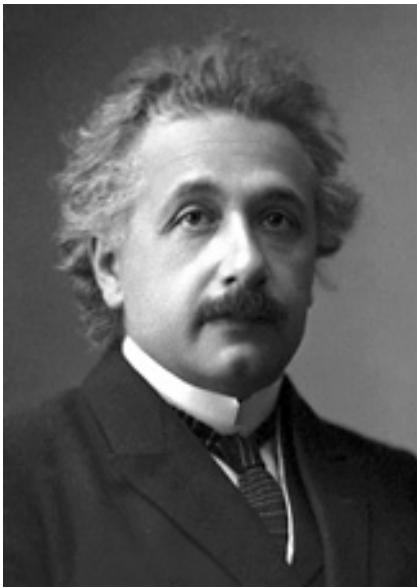
Much of his work at the patent office related to questions about transmission of electric signals and electrical-mechanical synchronization of time, two technical problems that show up conspicuously in the thought experiments that eventually led Einstein to his radical conclusions about the nature of light and the fundamental connection between space and time.^[30]

With a few friends he met in Bern, Einstein started a small discussion group, self-mockingly named "The Olympia Academy", which met regularly to discuss science and philosophy. Their readings included the works of Henri Poincaré, Ernst Mach, and David Hume, which influenced his scientific and philosophical outlook.



Einstein's home in Bern

Academic career



Einstein's official 1921 portrait after receiving the Nobel Prize in Physics.

In 1901, Einstein had a paper on the capillary forces of a straw published in the prestigious *Annalen der Physik*.^[31] On 30 April 1905, he completed his thesis, with Alfred Kleiner, Professor of Experimental Physics, serving as pro-forma advisor. Einstein was awarded a PhD by the University of Zurich. His dissertation was entitled "A New Determination of Molecular Dimensions".^[32] That same year, which has been called Einstein's *annus mirabilis* or "miracle year", he published four groundbreaking papers, on the photoelectric effect, Brownian motion, special relativity, and the equivalence of matter and energy, which were to bring him to the notice of the academic world.

By 1908, he was recognized as a leading scientist, and he was appointed lecturer at the University of Berne. The following year, he quit the patent office and the lectureship to take the position of physics docent^[33] at the University of Zurich. He became a full professor at Karl-Ferdinand University in Prague in 1911. In 1914, he returned to Germany after being appointed director of the Kaiser Wilhelm Institute for Physics (1914–1932)^[34] and a professor at the Humboldt University of Berlin, although with a special clause in his contract that freed him

from most teaching obligations. He became a member of the Prussian Academy of Sciences. In 1916, Einstein was appointed president of the German Physical Society (1916–1918).^{[35] [36]}

In 1911, he had calculated that, based on his new theory of general relativity, light from another star would be bent by the Sun's gravity. That prediction was claimed confirmed by observations made by a British expedition led by Sir Arthur Eddington during the solar eclipse of May 29, 1919. International media reports of this made Einstein world famous. On 7 November 1919, the leading British newspaper *The Times* printed a banner headline that read: "Revolution in Science – New Theory of the Universe – Newtonian Ideas Overthrown".^[37] (Much later, questions were raised whether the measurements were accurate enough to support Einstein's theory.)

In 1921, Einstein was awarded the Nobel Prize in Physics. Because relativity was still considered somewhat controversial, it was officially bestowed for his explanation of the photoelectric effect. He also received the Copley Medal from the Royal Society in 1925.

Travels abroad

Einstein visited New York City for the first time on 2 April 1921. When asked where he got his scientific ideas, Einstein explained that he believed scientific work best proceeds from an examination of physical reality and a search for underlying axioms, with consistent explanations that apply in all instances and avoid contradicting each other. He also recommended theories with visualizable results.(Einstein 1954)^[38]

In 1922, he traveled throughout Asia and later to Palestine, as part of a six-month excursion and speaking tour. His travels included Singapore, Ceylon, and Japan, where he gave a series of lectures to thousands of Japanese. His first lecture in Tokyo lasted four hours, after which he met the emperor and empress at the Imperial Palace where thousands came to watch. Einstein later gave his impressions of the Japanese in a letter to his sons:^[39] :307 "Of all the people I have met, I like the Japanese most, as they are modest, intelligent, considerate, and have a feel for art."^[39] :308

On his return voyage, he also visited Palestine for twelve days in what would become his only visit to that region. "He was greeted with great British pomp, as if he were a head of state rather than a theoretical physicist", writes Isaacson. This included a cannon salute upon his arrival at the residence of the British high commissioner, Sir

Herbert Samuel. During one reception given to him, the building was "stormed by throngs who wanted to hear him". In Einstein's talk to the audience, he expressed his happiness over the event:

I consider this the greatest day of my life. Before, I have always found something to regret in the Jewish soul, and that is the forgetfulness of its own people. Today, I have been made happy by the sight of the Jewish people learning to recognize themselves and to make themselves recognized as a force in the world.^[40] :308

Emigration from Germany

In 1933, Einstein was compelled to immigrate to the United States due to the rise to power of the Nazis under Germany's new chancellor, Adolf Hitler.^[41] While visiting American universities in April, 1933, he learned that the new German government had passed a law barring Jews from holding any official positions, including teaching at universities. A month later, the Nazi book burnings occurred, with Einstein's works being among those burnt, and Nazi propaganda minister Joseph Goebbels proclaimed, "Jewish intellectualism is dead."^[40] Einstein also learned that his name was on a list of assassination targets, with a "\$5,000 bounty on his head". One German magazine included him in a list of enemies of the German regime with the phrase, "not yet hanged".^[40] ^[42]

Among other German scientists forced to flee were fourteen Nobel laureates and twenty-six of the sixty professors of theoretical physics in the country. Among the other scientists who left Germany, or the other countries it came to dominate, were Edward Teller, Niels Bohr, Enrico Fermi, Otto Stern, Victor Weisskopf, Hans Bethe, and Lise Meitner, many of whom made certain that the Allies would develop nuclear weapons first, before the Nazis.^[40] With so many other Jewish scientists now forced by circumstances to live in America, often working side by side, Einstein wrote to a friend, "For me the most beautiful thing is to be in contact with a few fine Jews—a few millennia of a civilized past do mean something after all." In another letter he writes, "In my whole life I have never felt so Jewish as now."^[40]

He took up a position at the Institute for Advanced Study at Princeton, New Jersey,^[43] an affiliation that lasted until his death in 1955. There, he tried unsuccessfully to develop a unified field theory and to refute the accepted interpretation of quantum physics. He and Kurt Gödel, another Institute member, became close friends. They would take long walks together discussing their work. His last assistant was Bruria Kaufman, who later became a renowned physicist.

World War II and the Manhattan Project

In the summer of 1939, a few months before the beginning of World War II, Einstein was persuaded to write a letter to President Franklin D. Roosevelt and warn him that Nazi Germany might be developing an atomic bomb. The letter, written with the help of Hungarian emigre physicist Leo Szilard, gave the letter more prestige, with Einstein also recommending that the U.S. begin uranium enrichment and nuclear research. According to F.G. Gosling of the U.S. Department of Energy, Einstein, Szilard, and other refugees including Edward Teller and Eugene Wigner, "regarded it as their responsibility to alert Americans to the possibility that German scientists might win the race to build an atomic bomb, and to warn that Hitler would be more than willing to resort to such a weapon."^[44]

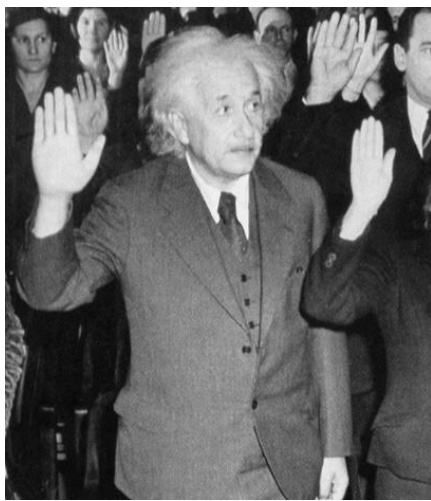
British columnist Ambrose Evans-Pritchard notes, however, that Washington at first "brushed off with disbelief" the fears they expressed. He then describes how quickly Roosevelt changed his mind:



Next to Oliver Locker-Lampson, being protected in Norfolk, England, after escaping Nazi Germany in 1933

Albert Einstein interceded through the Belgian queen mother, eventually getting a personal envoy into the Oval Office. Roosevelt initially fobbed him off. He listened more closely at a second meeting over breakfast the next day, then made up his mind within minutes. "This needs action," he told his military aide. It was the birth of the Manhattan Project.^[45]

Gosling adds that "the President was a man of considerable action once he had chosen a direction," and believed that the U.S. "could not take the risk of allowing Hitler" to possess nuclear bombs.^[44] Other weapons historians agree that the letter was "arguably the key stimulus for the U.S. adoption of serious investigations into nuclear weapons on the eve of the U.S. entry into World War II". As a result of Einstein's letter, and his meetings with Roosevelt, the U.S. entered the "race" to develop the bomb first, drawing on its "immense material, financial, and scientific resources". It became the only country to develop an atomic bomb during World War II as a result of its Manhattan Project.^[46] Einstein said to his old friend, Linus Pauling, in 1954, the last year of his life: "I made one great mistake in my life — when I signed the letter to President Roosevelt recommending that atom bombs be made; but there was some justification — the danger that the Germans would make them..."^[47]



Taking oath of allegiance for U.S. citizenship,
(1940)

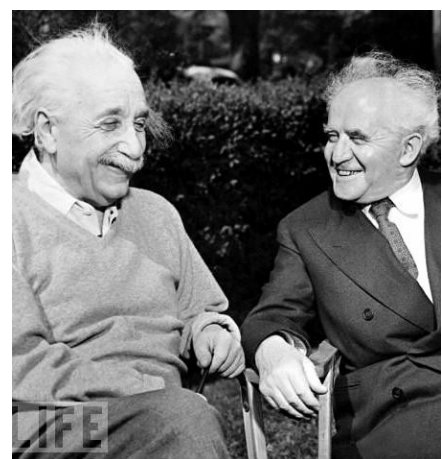
U.S. citizenship

Einstein became an American citizen in 1940. Not long after settling into his career at Princeton, he expressed his appreciation of the "meritocracy" in American culture when compared to Europe. According to Isaacson, he recognized the "right of individuals to say and think what they pleased", without social barriers, and as result, the individual was "encouraged" to be more creative, a trait he valued from his own early education. Einstein writes:

What makes the new arrival devoted to this country is the democratic trait among the people. No one humbles himself before another person or class. . . American youth has the good fortune not to have its outlook troubled by outworn traditions.^[40]
:432

As a member of the NAACP at Princeton who campaigned for the civil rights of African Americans, Einstein corresponded with civil rights activist W. E. B. Du Bois, and in 1946 Einstein called racism America's "worst disease".^[48] He later stated, "Race prejudice has unfortunately become an American tradition which is uncritically handed down from one generation to the next. The only remedies are enlightenment and education".^[49]

After the death of Israel's first president, Chaim Weizmann, in November 1952, Prime Minister David Ben-Gurion offered Einstein the position of President of Israel, a mostly ceremonial post.^[50] The offer was presented by Israel's ambassador in Washington, Abba Eban, who explained that the offer "embodies the deepest respect which the Jewish people can repose in any of its sons".^[39] :522 However, Einstein declined, and wrote in his response that he was "deeply moved", and "at once saddened and ashamed" that he could not accept it:

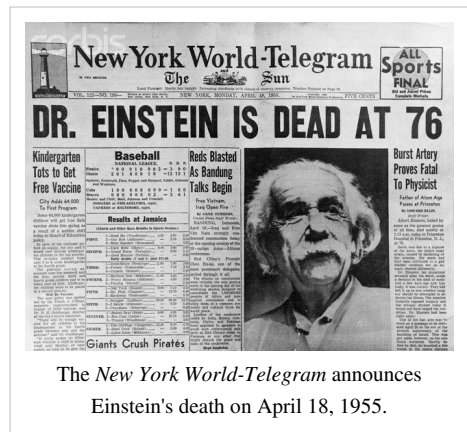


Einstein with David Ben Gurion, 1951

All my life I have dealt with objective matters, hence I lack both the natural aptitude and the experience to deal properly with people and to exercise official function. I am the more more distressed over these circumstances because my relationship with the Jewish people became my strongest human tie once I achieved complete clarity about our precarious position among the nations of the world.^[39] :522 [50] [51]

Death

On April 17, 1955, Albert Einstein experienced internal bleeding caused by the rupture of an abdominal aortic aneurysm, which had previously been reinforced surgically by Dr. Rudolph Nissen in 1948.^[52] He took the draft of a speech he was preparing for a television appearance commemorating the State of Israel's seventh anniversary with him to the hospital, but he did not live long enough to complete it.^[53] Einstein refused surgery, saying: "I want to go when I want. It is tasteless to prolong life artificially. I have done my share, it is time to go. I will do it elegantly."^[54] He died in Princeton Hospital early the next morning at the age of 76, having continued to work until near the end.



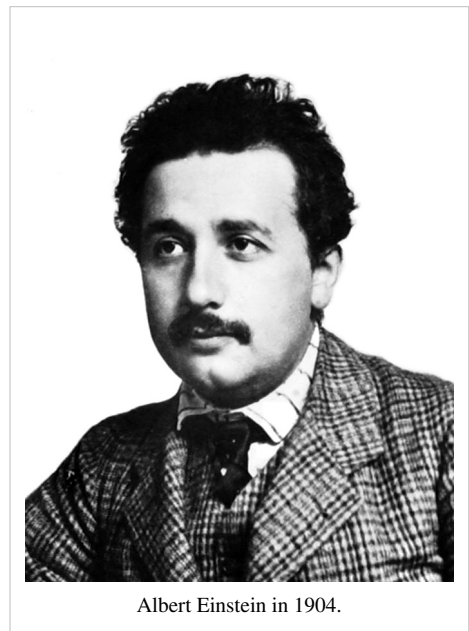
Einstein's remains were cremated and his ashes were scattered around the grounds of the Institute for Advanced Study.^[55] ^[56] During the autopsy, the pathologist of Princeton Hospital, Thomas Stoltz Harvey, removed Einstein's brain for preservation, without the permission of his family, in hope that the neuroscience of the future would be able to discover what made Einstein so intelligent.^[57]

Scientific career

Throughout his life, Einstein published hundreds of books and articles. Most were about physics, but a few expressed leftist political opinions about pacifism, socialism, and zionism.^[5] ^[7] In addition to the work he did by himself he also collaborated with other scientists on additional projects including the Bose–Einstein statistics, the Einstein refrigerator and others.^[58]

Physics in 1900

Einstein's early papers all come from attempts to demonstrate that atoms exist and have a finite nonzero size. At the time of his first paper in 1902, it was not yet completely accepted by physicists that atoms were real, even though chemists had good evidence ever since Antoine Lavoisier's work a century earlier. The reason physicists were skeptical was because no 19th century theory could fully explain the properties of matter from the properties of atoms.



Ludwig Boltzmann was a leading 19th century atomist physicist, who had struggled for years to gain acceptance for atoms. Boltzmann had given an interpretation of the laws of thermodynamics, suggesting that the law of entropy increase is statistical. In Boltzmann's way of thinking, the entropy is the logarithm of the number of ways a system could be configured inside. The reason the entropy goes up is only because it is more likely for a system to go from a special state with only a few possible internal configurations to a more generic state with many. While Boltzmann's statistical interpretation of entropy is

universally accepted today, and Einstein believed it, at the turn of the 20th century it was a minority position.

The statistical idea was most successful in explaining the properties of gases. James Clerk Maxwell, another leading atomist, had found the distribution of velocities of atoms in a gas, and derived the surprising result that the viscosity of a gas should be independent of density. Intuitively, the friction in a gas would seem to go to zero as the density goes to zero, but this is not so, because the mean free path of atoms becomes large at low densities. A subsequent experiment by Maxwell and his wife confirmed this surprising prediction. Other experiments on gases and vacuum, using a rotating slitted drum, showed that atoms in a gas had velocities distributed according to Maxwell's distribution law.

In addition to these successes, there were also inconsistencies. Maxwell noted that at cold temperatures, atomic theory predicted specific heats that are too large. In classical statistical mechanics, every spring-like motion has thermal energy $k_B T$ on average at temperature T , so that the specific heat of every spring is Boltzmann's constant k_B . A monatomic solid with N atoms can be thought of as N little balls representing N atoms attached to each other in a box grid with $3N$ springs, so the specific heat of every solid is $3Nk_B$, a result which became known as the Dulong–Petit law. This law is true at room temperature, but not for colder temperatures. At temperatures near zero, the specific heat goes to zero.

Similarly, a gas made up of a molecule with two atoms can be thought of as two balls on a spring. This spring has energy $k_B T$ at high temperatures, and should contribute an extra k_B to the specific heat. It does at temperatures of about 1000 degrees, but at lower temperature, this contribution disappears. At zero temperature, all other contributions to the specific heat from rotations and vibrations also disappear. This behavior was inconsistent with classical physics.

The most glaring inconsistency was in the theory of light waves. Continuous waves in a box can be thought of as infinitely many spring-like motions, one for each possible standing wave. Each standing wave has a specific heat of k_B , so the total specific heat of a continuous wave like light should be infinite in classical mechanics. This is obviously wrong, because it would mean that all energy in the universe would be instantly sucked up into light waves, and everything would slow down and stop.

These inconsistencies led some people to say that atoms were not physical, but mathematical. Notable among the skeptics was Ernst Mach, whose positivist philosophy led him to demand that if atoms are real, it should be possible to see them directly.^[59] Mach believed that atoms were a useful fiction, that in reality they could be assumed to be infinitesimally small, that Avogadro's number was infinite, or so large that it might as well be infinite, and k_B was infinitesimally small. Certain experiments could then be explained by atomic theory, but other experiments could not, and this is the way it will always be.

Einstein opposed this position. Throughout his career, he was a realist. He believed that a single consistent theory should explain all observations, and that this theory would be a description of what was really going on, underneath it all. So he set out to show that the atomic point of view was correct. This led him first to thermodynamics, then to statistical physics, and to the theory of specific heats of solids.

In 1905, while he was working in the patent office, the leading German language physics journal *Annalen der Physik* published four of Einstein's papers. The four papers eventually were recognized as revolutionary, and 1905 became known as Einstein's "Miracle Year", and the papers as the *Annus Mirabilis Papers*.

Thermodynamic fluctuations and statistical physics

Einstein's earliest papers were concerned with thermodynamics. He wrote a paper establishing a thermodynamic identity in 1902, and a few other papers which attempted to interpret phenomena from a statistical atomic point of view.

His research in 1903 and 1904 was mainly concerned with the effect of finite atomic size on diffusion phenomena. As in Maxwell's work, the finite nonzero size of atoms leads to effects which can be observed. This research, and the thermodynamic identity, were well within the mainstream of physics in his time. They would eventually form the content of his PhD thesis.^[60]

His first major result in this field was the theory of thermodynamic fluctuations. When in equilibrium, a system has a maximum entropy and, according to the statistical interpretation, it can fluctuate a little bit. Einstein pointed out that the statistical fluctuations of a macroscopic object, like a mirror suspended on spring, would be completely determined by the second derivative of the entropy with respect to the position of the mirror.

Searching for ways to test this relation, his great breakthrough came in 1905. The theory of fluctuations, he realized, would have a visible effect for an object which could move around freely. Such an object would have a velocity which is random, and would move around randomly, just like an individual atom. The average kinetic energy of the object would be $k_B T$, and the time decay of the fluctuations would be entirely determined by the law of friction.

The law of friction for a small ball in a viscous fluid like water was discovered by George Stokes. He showed that for small velocities, the friction force would be proportional to the velocity, and to the radius of the particle (see Stokes' law). This relation could be used to calculate how far a small ball in water would travel due to its random thermal motion, and Einstein noted that such a ball, of size about a micrometre, would travel about a few micrometres per second. This motion could be easily detected with a microscope and indeed, as Brownian motion, had actually been observed by the botanist Robert Brown. Einstein was able to identify this motion with that predicted by his theory. Since the fluctuations which give rise to Brownian motion are just the same as the fluctuations of the velocities of atoms, measuring the precise amount of Brownian motion using Einstein's theory would show that Boltzmann's constant is non-zero and would measure Avogadro's number.

These experiments were carried out a few years later by Jean Baptiste Perrin, and gave a rough estimate of Avogadro's number consistent with the more accurate estimates due to Max Planck's theory of blackbody light and Robert Millikan's measurement of the charge of the electron.^[61] Unlike the other methods, Einstein's required very few theoretical assumptions or new physics, since it was directly measuring atomic motion on visible grains.

Einstein's theory of Brownian motion was the first paper in the field of statistical physics. It established that thermodynamic fluctuations were related to dissipation. This was shown by Einstein to be true for time-independent fluctuations, but in the Brownian motion paper he showed that dynamical relaxation rates calculated from classical mechanics could be used as statistical relaxation rates to derive dynamical diffusion laws. These relations are known as Einstein relations.

The theory of Brownian motion was the least revolutionary of Einstein's *Annus mirabilis* papers, but it is the most frequently cited, and had an important role in securing the acceptance of the atomic theory by physicists.

Thought experiments and a-priori physical principles

Einstein's thinking underwent a transformation in 1905. He had come to understand that quantum properties of light mean that Maxwell's equations were only an approximation. He knew that new laws would have to replace these, but he did not know how to go about finding those laws. He felt that guessing formal relations would not go anywhere.

So he decided to focus on a-priori principles instead, which are statements about physical laws which can be understood to hold in a very broad sense even in domains where they have not yet been shown to apply. A well accepted example of an a-priori principle is rotational invariance. If a new force is discovered in physics, it is assumed to be rotationally invariant almost automatically, without thought. Einstein sought new principles of this sort, to guide the production of physical ideas. Once enough principles are found, then the new physics will be the simplest theory consistent with the principles and with previously known laws.

The first general a-priori principle he found was the principle of relativity, that uniform motion is indistinguishable from rest. This was understood by Hermann Minkowski to be a generalization of rotational invariance from space to space-time. Other principles postulated by Einstein and later vindicated are the principle of equivalence and the principle of adiabatic invariance of the quantum number. Another of Einstein's general principles, Mach's principle, is fiercely debated, and whether it holds in our world or not is still not definitively established.

The use of a-priori principles is a distinctive unique signature of Einstein's early work, and has become a standard tool in modern theoretical physics.

Special relativity

His 1905 paper on the electrodynamics of moving bodies introduced his theory of special relativity, which showed that the observed independence of the speed of light on the observer's state of motion required fundamental changes to the notion of simultaneity. Consequences of this include the time-space frame of a moving body slowing down and contracting (in the direction of motion) relative to the frame of the observer. This paper also argued that the idea of a luminiferous aether – one of the leading theoretical entities in physics at the time – was superfluous.^[62] In his paper on *mass–energy equivalence*, which had previously been considered to be distinct concepts, Einstein deduced from his equations of special relativity what has been called the 20th century's best-known equation: $E = mc^2$.^[63] ^[64] This equation suggests that tiny amounts of mass could be converted into huge amounts of energy and presaged the development of nuclear power.^[65] Einstein's 1905 work on relativity remained controversial for many years, but was accepted by leading physicists, starting with Max Planck.^[66] ^[67]

Photons

In a 1905 paper,^[68] Einstein postulated that light itself consists of localized particles (*quanta*). Einstein's light quanta were nearly universally rejected by all physicists, including Max Planck and Niels Bohr. This idea only became universally accepted in 1919, with Robert Millikan's detailed experiments on the photoelectric effect, and with the measurement of Compton scattering.

Einstein's paper on the light particles was almost entirely motivated by thermodynamic considerations. He was not at all motivated by the detailed experiments on the photoelectric effect, which did not confirm his theory until fifteen years later. Einstein considers the entropy of light at temperature T , and decomposes it into a low-frequency part and a high-frequency part. The high-frequency part, where the light is described by Wien's law, has an entropy which looks exactly the same as the entropy of a gas of classical particles.

Since the entropy is the logarithm of the number of possible states, Einstein concludes that the number of states of short wavelength light waves in a box with volume V is equal to the number of states of a group of localizable particles in the same box. Since (unlike others) he was comfortable with the statistical interpretation, he confidently postulates that the light itself is made up of localized particles, as this is the only reasonable interpretation of the entropy.

This leads him to conclude that each wave of frequency f is associated with a collection of photons with energy hf each, where h is Planck's constant. He does not say much more, because he is not sure how the particles are related to the wave. But he does suggest that this idea would explain certain experimental results, notably the photoelectric effect.^[69]

Quantized atomic vibrations

Einstein continued his work on quantum mechanics in 1906, by explaining the specific heat anomaly in solids. This was the first application of quantum theory to a mechanical system. Since Planck's distribution for light oscillators had no problem with infinite specific heats, the same idea could be applied to solids to fix the specific heat problem there. Einstein showed in a simple model that the hypothesis that solid motion is quantized explains why the specific heat of a solid goes to zero at zero temperature.

Einstein's model treats each atom as connected to a single spring. Instead of connecting all the atoms to each other, which leads to standing waves with all sorts of different frequencies, Einstein imagined that each atom was attached to a fixed point in space by a spring. This is not physically correct, but it still predicts that the specific heat is $3Nk_B$, since the number of independent oscillations stays the same.

Einstein then assumes that the motion in this model is quantized, according to the Planck law, so that each independent spring motion has energy which is an integer multiple of hf , where f is the frequency of oscillation. With this assumption, he applied Boltzmann's statistical method to calculate the average energy of the spring. The result was the same as the one that Planck had derived for light: for temperatures where $k_B T$ is much smaller than hf , the motion is frozen, and the specific heat goes to zero.

So Einstein concluded that quantum mechanics would solve the main problem of classical physics, the specific heat anomaly. The particles of sound implied by this formulation are now called phonons. Because all of Einstein's springs have the same stiffness, they all freeze out at the same temperature, and this leads to a prediction that the specific heat should go to zero exponentially fast when the temperature is low. The solution to this problem is to solve for the independent normal modes individually, and to quantize those. Then each normal mode has a different frequency, and long wavelength vibration modes freeze out at colder temperatures than short wavelength ones. This was done by Peter Debye, and after this modification Einstein's quantization method reproduced quantitatively the behavior of the specific heats of solids at low temperatures.

This work was the foundation of condensed matter physics.

Adiabatic principle and action-angle variables

Throughout the 1910s, quantum mechanics expanded in scope to cover many different systems. After Ernest Rutherford discovered the nucleus and proposed that electrons orbit like planets, Niels Bohr was able to show that the same quantum mechanical postulates introduced by Planck and developed by Einstein would explain the discrete motion of electrons in atoms, and the periodic table of the elements.

Einstein contributed to these developments by linking them with the 1898 arguments Wilhelm Wien had made. Wien had shown that the hypothesis of adiabatic invariance of a thermal equilibrium state allows all the blackbody curves at different temperature to be derived from one another by a simple shifting process. Einstein noted in 1911 that the same adiabatic principle shows that the quantity which is quantized in any mechanical motion must be an adiabatic invariant. Arnold Sommerfeld identified this adiabatic invariant as the action variable of classical mechanics. The law that the action variable is quantized was the basic principle of the quantum theory as it was known between 1900 and 1925.

Wave-particle duality

Although the patent office promoted Einstein to Technical Examiner Second Class in 1906, he had not given up on academia. In 1908, he became a *privatdozent* at the University of Bern.^[70] In "über die Entwicklung unserer Anschauungen über das Wesen und die Konstitution der Strahlung" ("The Development of Our Views on the Composition and Essence of Radiation"), on the quantization of light, and in an earlier 1909 paper, Einstein showed that Max Planck's energy quanta must have well-defined momenta and act in some respects as independent, point-like particles. This paper introduced the *photon* concept (although the name *photon* was introduced later by Gilbert N. Lewis in 1926) and inspired the notion of wave-particle duality in quantum mechanics.

Theory of critical opalescence

Einstein returned to the problem of thermodynamic fluctuations, giving a treatment of the density variations in a fluid at its critical point. Ordinarily the density fluctuations are controlled by the second derivative of the free energy with respect to the density. At the critical point, this derivative is zero, leading to large fluctuations. The effect of density fluctuations is that light of all wavelengths is scattered, making the fluid look milky white. Einstein relates this to Raleigh scattering, which is what happens when the fluctuation size is much smaller than the wavelength, and which explains why the sky is blue.^[71]



Einstein at the Solvay Conference in 1911.

Zero-point energy

Einstein's physical intuition led him to note that Planck's oscillator energies had an incorrect zero point. He modified Planck's hypothesis by stating that the lowest energy state of an oscillator is equal to $\frac{1}{2}hf$, to half the energy spacing between levels. This argument, which was made in 1913 in collaboration with Otto Stern, was based on the thermodynamics of a diatomic molecule which can split apart into two free atoms.

Principle of equivalence

In 1907, while still working at the patent office, Einstein had what he would call his "happiest thought". He realized that the principle of relativity could be extended to gravitational fields. He thought about the case of a uniformly accelerated box not in a gravitational field, and noted that it would be indistinguishable from a box sitting still in an unchanging gravitational field.^[72] He used special relativity to see that the rate of clocks at the top of a box accelerating upward would be faster than the rate of clocks at the bottom. He concludes that the rates of clocks depend on their position in a gravitational field, and that the difference in rate is proportional to the gravitational potential to first approximation.

Although this approximation is crude, it allowed him to calculate the deflection of light by gravity, and show that it is nonzero. This gave him confidence that the scalar theory of gravity proposed by Gunnar Nordström was incorrect. But the actual value for the deflection that he calculated was too small by a factor of two, because the approximation he used doesn't work well for things moving at near the speed of light. When Einstein finished the full theory of general relativity, he would rectify this error and predict the correct amount of light deflection by the sun.

From Prague, Einstein published a paper about the effects of gravity on light, specifically the gravitational redshift and the gravitational deflection of light. The paper challenged astronomers to detect the deflection during a solar

eclipse.^[73] German astronomer Erwin Finlay-Freundlich publicized Einstein's challenge to scientists around the world.^[74]

Einstein thought about the nature of the gravitational field in the years 1909–1912, studying its properties by means of simple thought experiments. A notable one is the rotating disk. Einstein imagined an observer making experiments on a rotating turntable. He noted that such an observer would find a different value for the mathematical constant π than the one predicted by Euclidean geometry. The reason is that the radius of a circle would be measured with an uncontracted ruler, but, according to special relativity, the circumference would seem to be longer because the ruler would be contracted.

Since Einstein believed that the laws of physics were local, described by local fields, he concluded from this that spacetime could be locally curved. This led him to study Riemannian geometry, and to formulate general relativity in this language.

Hole argument and Entwurf theory

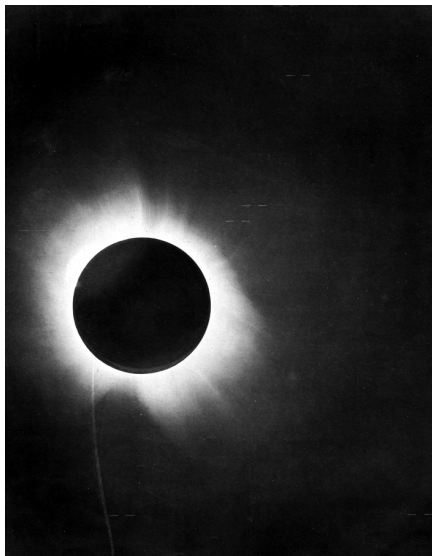
While developing general relativity, Einstein became confused about the gauge invariance in the theory. He formulated an argument that led him to conclude that a general relativistic field theory is impossible. He gave up looking for fully generally covariant tensor equations, and searched for equations that would be invariant under general linear transformations only.

In June, 1913 the Entwurf ("draft") theory was the result of these investigations. As its name suggests, it was a sketch of a theory, with the equations of motion supplemented by additional gauge fixing conditions. Simultaneously less elegant and more difficult than general relativity, after more than two years of intensive work Einstein abandoned the theory in November, 1915 after realizing that the hole argument was mistaken.^[75]

General relativity

In 1912, Einstein returned to Switzerland to accept a professorship at his *alma mater*, the ETH. Once back in Zurich, he immediately visited his old ETH classmate Marcel Grossmann, now a professor of mathematics, who introduced him to Riemannian geometry and, more generally, to differential geometry. On the recommendation of Italian mathematician Tullio Levi-Civita, Einstein began exploring the usefulness of general covariance (essentially the use of tensors) for his gravitational theory. For a while Einstein thought that there were problems with the approach, but he later returned to it and, by late 1915, had published his general theory of relativity in the form in which it is used today.^[76] This theory explains gravitation as distortion of the structure of spacetime by matter, affecting the inertial motion of other matter. During World War I, the work of Central Powers scientists was available only to Central Powers academics, for national security reasons. Some of Einstein's work did reach the United Kingdom and the United States through the efforts of the Austrian Paul Ehrenfest and physicists in the Netherlands, especially 1902 Nobel Prize-winner Hendrik Lorentz and Willem de Sitter of Leiden University. After the war ended, Einstein maintained his relationship with Leiden University, accepting a contract as an *Extraordinary Professor*; for ten years, from 1920 to 1930, he travelled to Holland regularly to lecture.^[77]

In 1917, several astronomers accepted Einstein's 1911 challenge from Prague. The Mount Wilson Observatory in California, U.S., published a solar spectroscopic analysis that showed no gravitational redshift.^[78] In 1918, the Lick Observatory, also in California, announced that it too had disproved Einstein's prediction, although its findings were not published.^[79]



Eddington's photograph of a solar eclipse, which confirmed Einstein's theory that light "bends".

However, in May 1919, a team led by the British astronomer Arthur Stanley Eddington claimed to have confirmed Einstein's prediction of gravitational deflection of starlight by the Sun while photographing a solar eclipse with dual expeditions in Sobral, northern Brazil, and Príncipe, a west African island.^[74] Nobel laureate Max Born praised general relativity as the "greatest feat of human thinking about nature";^[80] fellow laureate Paul Dirac was quoted saying it was "probably the greatest scientific discovery ever made".^[81] The international media guaranteed Einstein's global renown.

There have been claims that scrutiny of the specific photographs taken on the Eddington expedition showed the experimental uncertainty to be comparable to the same magnitude as the effect Eddington claimed to have demonstrated, and that a 1962 British expedition concluded that the method was inherently unreliable.^[37] The deflection of light during a solar eclipse was confirmed by later, more accurate observations.^[82] Some resented the newcomer's fame, notably among some German physicists, who later started the *Deutsche Physik* (German Physics)

movement.^{[83] [84]}

Cosmology

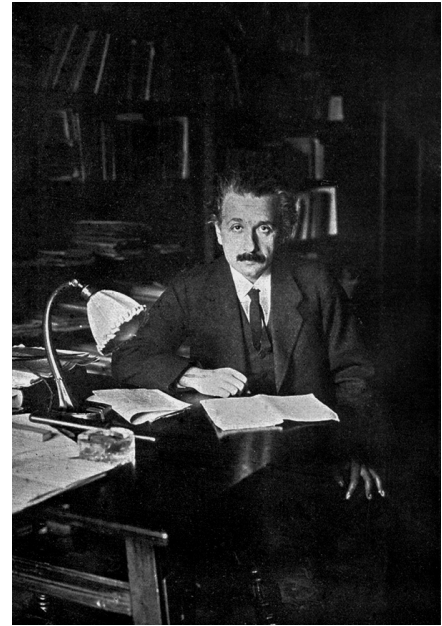
In 1917, Einstein applied the General theory of relativity to model the structure of the universe as a whole. He wanted the universe to be eternal and unchanging, but this type of universe is not consistent with relativity. To fix this, Einstein modified the general theory by introducing a new notion, the cosmological constant. With a positive cosmological constant, the universe could be an eternal static sphere.^[85]

Einstein believed a spherical static universe is philosophically preferred, because it would obey Mach's principle. He had shown that general relativity incorporates Mach's principle to a certain extent in frame dragging by gravitomagnetic fields, but he knew that Mach's idea would not work if space goes on forever. In a closed universe, he believed that Mach's principle would hold.

Mach's principle has generated much controversy over the years.

Modern quantum theory

In 1917, at the height of his work on relativity, Einstein published an article in *Physikalische Zeitschrift* that proposed the possibility of stimulated emission, the physical process that makes possible the maser and the laser.^[86] This article showed that the statistics of absorption and emission of light would only be consistent with Planck's distribution law if the emission of light into a mode with n photons would be enhanced statistically compared to the emission of light into an empty mode. This paper was enormously influential in the later development of quantum mechanics, because it was the first paper to show that the statistics of atomic transitions had simple laws. Einstein discovered Louis de Broglie's work, and supported his ideas, which were received skeptically at first. In another major paper from this era, Einstein gave a wave equation for de Broglie waves, which Einstein suggested was the Hamilton–Jacobi equation of mechanics. This paper would inspire Schrödinger's work of 1926.



Einstein in his office at the University of Berlin.

Bose–Einstein statistics

In 1924, Einstein received a description of a statistical model from Indian physicist Satyendra Nath Bose, based on a counting method that assumed that light could be understood as a gas of indistinguishable particles. Einstein noted that Bose's statistics applied to some atoms as well as to the proposed light particles, and submitted his translation of Bose's paper to the *Zeitschrift für Physik*. Einstein also published his own articles describing the model and its implications, among them the Bose–Einstein condensate phenomenon that some particulates should appear at very low temperatures.^[87] It was not until 1995 that the first such condensate was produced experimentally by Eric Allin Cornell and Carl Wieman using ultra-cooling equipment built at the NIST–JILA laboratory at the University of Colorado at Boulder.^[88] Bose–Einstein statistics are now used to describe the behaviors of any assembly of bosons. Einstein's sketches for this project may be seen in the Einstein Archive in the library of the Leiden University.^[1]

Energy momentum pseudotensor

General relativity includes a dynamical spacetime, so it is difficult to see how to identify the conserved energy and momentum. Noether's theorem allows these quantities to be determined from a Lagrangian with translation invariance, but general covariance makes translation invariance into something of a gauge symmetry. The energy and momentum derived within general relativity by Noether's prescriptions do not make a real tensor for this reason.

Einstein argued that this is true for fundamental reasons, because the gravitational field could be made to vanish by a choice of coordinates. He maintained that the non-covariant energy momentum pseudotensor was in fact the best description of the energy momentum distribution in a gravitational field. This approach has been echoed by Lev Landau and Evgeny Lifshitz, and others, and has become standard.

The use of non-covariant objects like pseudotensors was heavily criticized in 1917 by Erwin Schrödinger and others.

Unified field theory

Following his research on general relativity, Einstein entered into a series of attempts to generalize his geometric theory of gravitation, which would allow the explanation of electromagnetism. In 1950, he described his "unified field theory" in a *Scientific American* article entitled "On the Generalized Theory of Gravitation".^[89] Although he continued to be lauded for his work, Einstein became increasingly isolated in his research, and his efforts were ultimately unsuccessful. In his pursuit of a unification of the fundamental forces, Einstein ignored some mainstream developments in physics, most notably the strong and weak nuclear forces, which were not well understood until many years after his death. Mainstream physics, in turn, largely ignored Einstein's approaches to unification. Einstein's dream of unifying other laws of physics with gravity motivates modern quests for a theory of everything and in particular string theory, where geometrical fields emerge in a unified quantum-mechanical setting.

Wormholes

Einstein collaborated with others to produce a model of a wormhole. His motivation was to model elementary particles with charge as a solution of gravitational field equations, in line with the program outlined in the paper "Do Gravitational Fields play an Important Role in the Constitution of the Elementary Particles?". These solutions cut and pasted Schwarzschild black holes to make a bridge between two patches.

If one end of a wormhole was positively charged, the other end would be negatively charged. These properties led Einstein to believe that pairs of particles and antiparticles could be described in this way.

Einstein–Cartan theory

In order to incorporate spinning point particles into general relativity, the affine connection needed to be generalized to include an antisymmetric part, called the torsion. This modification was made by Einstein and Cartan in the 1920s.

Equations of motion

The theory of general relativity has a fundamental law – the Einstein equations which describe how space curves, the geodesic equation which describes how particles move may be derived from the Einstein equations.

Since the equations of general relativity are non-linear, a lump of energy made out of pure gravitational fields, like a black hole, would move on a trajectory which is determined by the Einstein equations themselves, not by a new law. So Einstein proposed that the path of a singular solution, like a black hole, would be determined to be a geodesic from general relativity itself.

This was established by Einstein, Infeld and Hoffmann for pointlike objects without angular momentum, and by Roy Kerr for spinning objects.

Einstein's controversial beliefs in physics

In addition to his well-accepted results, some of Einstein's views are regarded as controversial:

- In the special relativity paper (in 1905), Einstein noted that, given a specific definition of the word "force" (a definition which he later agreed was not advantageous), and if we choose to maintain (by convention) the equation mass $\times$ acceleration = force, then one arrives at $m/(1-v^2/c^2)$ as the expression for the transverse mass of a fast moving particle. This differs from the accepted expression today, because, as noted in the footnotes to Einstein's paper added in the 1913 reprint, "it is more to the point to define force in such a way that the laws of energy and momentum assume the simplest form", as was done, for example, by Max Planck in 1906, who gave the now familiar expression $m/\sqrt{1-v^2/c^2}$ for the transverse mass. As Miller points out, this is equivalent to the transverse mass predictions of both Einstein and Lorentz. Einstein had commented already in the 1905 paper that "With a different definition of force and acceleration, we should naturally obtain other expressions for the masses. This shows that in comparing different theories... we must proceed very cautiously."^[90]

- Einstein published (in 1922) a qualitative theory of superconductivity based on the vague idea of electrons shared in orbits. This paper predated modern quantum mechanics, and today is regarded as being incorrect. The current theory of low temperature superconductivity was only worked out in 1957, thirty years after the establishing of modern quantum mechanics. However, even today, superconductivity is not well understood, and alternative theories continue to be put forward, especially to account for high-temperature superconductors.
- After introducing the concept of gravitational waves in 1917, Einstein subsequently entertained doubts about whether they could be physically realized. In 1937 he published a paper saying that the focusing properties of geodesics in general relativity would lead to an instability which causes plane gravitational waves to collapse in on themselves. While this is true to a certain extent in some limits, because gravitational instabilities can lead to a concentration of energy density into black holes, for plane waves of the type Einstein and Rosen considered in their paper, the instabilities are under control. Einstein retracted this position a short time later.
- Einstein denied several times that black holes could form. In 1939 he published a paper that argues that a star collapsing would spin faster and faster, spinning at the speed of light with infinite energy well before the point where it is about to collapse into a black hole. This paper received no citations, and the conclusions are well understood to be wrong. Einstein's argument itself is inconclusive, since he only shows that stable spinning objects have to spin faster and faster to stay stable before the point where they collapse. But it is well understood today (and was understood well by some even then) that collapse cannot happen through stationary states the way Einstein imagined. Nevertheless, the extent to which the models of black holes in classical general relativity correspond to physical reality remains unclear, and in particular the implications of the central singularity implicit in these models are still not understood. Efforts to conclusively prove the existence of event horizons have still not been successful.
- Closely related to his rejection of black holes, Einstein believed that the exclusion of singularities might restrict the class of solutions of the field equations so as to force solutions compatible with quantum mechanics, but no such theory has ever been found.
- In the early days of quantum mechanics, Einstein tried to show that the uncertainty principle was not valid, but by 1927 he had become convinced that it was valid.
- In the EPR paper, Einstein argued that quantum mechanics cannot be a complete realistic and local representation of phenomena, given specific definitions of "realism", "locality", and "completeness". The modern consensus is that Einstein's concept of realism is too restrictive.
- Einstein himself considered the introduction of the cosmological term in his 1917 paper founding cosmology as a "blunder".^[91] The theory of general relativity predicted an expanding or contracting universe, but Einstein wanted a universe which is an unchanging three dimensional sphere, like the surface of a three dimensional ball in four dimensions. He wanted this for philosophical reasons, so as to incorporate Mach's principle in a reasonable way. He stabilized his solution by introducing a cosmological constant, and when the universe was shown to be expanding, he retracted the constant as a blunder. This is not really much of a blunder – the cosmological constant is necessary within general relativity as it is currently understood, and it is widely believed to have a nonzero value today.
- Einstein did not immediately appreciate the value of Minkowski's four-dimensional formulation of special relativity, although within a few years he had adopted it as the basis for his theory of gravitation.
- Finding it too formal, Einstein believed that Heisenberg's matrix mechanics was incorrect. He changed his mind when Schrödinger and others demonstrated that the formulation in terms of the Schrödinger equation, based on Einstein's wave-particle duality was equivalent to Heisenberg's matrices.

Collaboration with other scientists

In addition to long time collaborators Leopold Infeld, Nathan Rosen, Peter Bergmann and others, Einstein also had some one-shot collaborations with various scientists.

Einstein-de Haas experiment

Einstein and De Haas demonstrated that magnetization is due to the motion of electrons, nowadays known to be the spin. In order to show this, they reversed the magnetization in an iron bar suspended on a torsion pendulum. They confirmed that this leads the bar to rotate, because the electron's angular momentum changes as the magnetization changes. This experiment needed to be sensitive, because the angular momentum associated with electrons is small, but it definitively established that electron motion of some kind is responsible for magnetization.

Schrödinger gas model

Einstein suggested to Erwin Schrödinger that he might be able to reproduce the statistics of a Bose–Einstein gas by considering a box. Then to each possible quantum motion of a particle in a box associate an independent harmonic oscillator. Quantizing these oscillators, each level will have an integer occupation number, which will be the number of particles in it.

This formulation is a form of second quantization, but it predates modern quantum mechanics. Erwin Schrödinger applied this to derive the thermodynamic properties of a semiclassical ideal gas. Schrödinger urged Einstein to add his name as co-author, although Einstein declined the invitation.^[92]

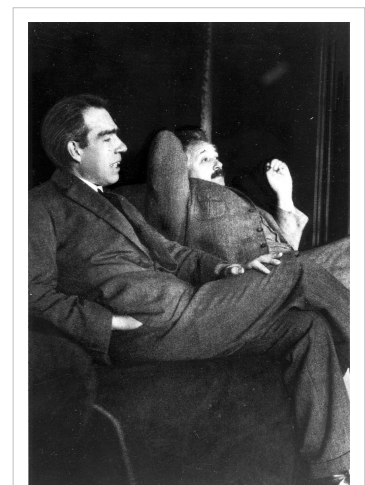
Einstein refrigerator

In 1926, Einstein and his former student Leó Szilárd co-invented (and in 1930, patented) the Einstein refrigerator. This absorption refrigerator was then revolutionary for having no moving parts and using only heat as an input.^[93] On 11 November 1930, U.S. Patent 1781541^[94] was awarded to Albert Einstein and Leó Szilárd for the refrigerator. Their invention was not immediately put into commercial production, as the most promising of their patents were quickly bought up by the Swedish company Electrolux to protect its refrigeration technology from competition.^[94]

Bohr versus Einstein

In the 1920s, quantum mechanics developed into a more complete theory. Einstein was unhappy with the Copenhagen interpretation of quantum theory developed by Niels Bohr and Werner Heisenberg, both in its outcomes and its instrumentalist methodology, Einstein being a scientific realist. In this interpretation, quantum phenomena are inherently probabilistic, with definite states resulting only upon interaction with classical systems. A public debate between Einstein and Bohr followed, lasting on and off for many years (including during the Solvay Conferences). Einstein formulated thought experiments against the Copenhagen interpretation, which were all rebutted by Bohr. In a 1926 letter to Max Born, Einstein wrote: "I, at any rate, am convinced that He [God] does not throw dice."^[95]

Einstein was never satisfied by what he perceived to be quantum theory's intrinsically incomplete description of nature, and in 1935 he further explored the issue in collaboration with Boris Podolsky and Nathan Rosen, noting that the theory seems to require non-local interactions; this is known as the EPR paradox.^[96] The EPR experiment has since been performed, with results confirming quantum theory's predictions.^[97] Repercussions of the Einstein–Bohr debate have found their way into philosophical discourse.



Einstein and Niels Bohr, 1925

Einstein–Podolsky–Rosen paradox

In 1935, Einstein returned to the question of quantum mechanics. He considered how a measurement on one of two entangled particles would affect the other. He noted, along with his collaborators, that by performing different measurements on the distant particle, either of position or momentum, different properties of the entangled partner could be discovered without disturbing it in any way.

He then used a hypothesis of local realism to conclude that the other particle had these properties already determined. The principle he proposed is that if it is possible to determine what the answer to a position or momentum measurement would be, without in any way disturbing the particle, then the particle actually has values of position or momentum.

This principle distilled the essence of Einstein's objection to quantum mechanics. As a physical principle, it has since been shown to be incompatible with experiments.

Political views

Einstein flouted the ascendant Nazi movement and later tried to be a voice of moderation in the tumultuous formation of the State of Israel.^[98] Fred Jerome in his *Einstein on Israel and Zionism: His Provocative Ideas About the Middle East* argues that Einstein was a Cultural Zionist who supported the idea of a Jewish homeland but opposed the establishment of a Jewish state in Palestine "with borders, an army, and a measure of temporal power." Instead, he preferred a bi-national state with "continuously functioning, mixed, administrative, economic, and social organizations."^[99] ^[100] However Ami Isseroff in his article *Was Einstein a Zionist*, argues that Einstein supported the recognition of the State of Israel and declared it "the fulfillment of our dream" when President Harry Truman recognize Israel in May 1948 and in presidential election 1948 Einstein supported Henry A. Wallace's Progressive Party which advocate pro-Soviet and pro-Israel foreign policy.^[101] ^[102]



Albert Einstein, seen here with his wife Elsa Einstein and Zionist leaders, including future President of Israel Chaim Weizmann, his wife Dr. Vera Weizmann, Menahem Ussishkin, and Ben-Zion Mossinson on arrival in New York City in 1921.

Throughout the November Revolution in Germany Einstein signed an appeal for the foundation of a nationwide liberal and democratic party,^[103] ^[104] which was published in the Berliner Tageblatt on 16 November 1918,^[105] and became a member of the German Democratic Party.^[106]

In his article *Why Socialism?*,^[107] published in 1949 in the *Monthly Review*, Einstein described a chaotic capitalist society, a source of evil to be overcome, as the "predatory phase of human development". He came to the following conclusion:

I am convinced there is only one way to eliminate these grave evils [capitalism], namely through the establishment of a socialist economy, accompanied by an educational system which would be oriented toward social goals. In such an economy, the means of production are owned by society itself and are utilized in a planned fashion. A planned economy, which adjusts production to the needs of the community, would distribute the work to be done among all those able to work and would guarantee a livelihood to every man, woman, and child. The education of the individual, in addition to promoting his own innate abilities, would attempt to develop in him a sense of responsibility for his fellow men in place of the glorification of power and success in our present society.^[107]

He braved anti-communist politics and resistance to the civil rights movement in the United States. On the floor of the US Congress, Einstein was accused by John E. Rankin of Mississippi of being a "foreign-born agitator" who

sought "to further the spread of Communism throughout the world".^[108] He also participated in the 1927 congress of the League against Imperialism in Brussels.^[109]

After World War II, as enmity between the former allies became a serious issue, Einstein wrote, "I do not know how the third World War will be fought, but I can tell you what they will use in the Fourth – rocks!"^[110] (Einstein 1949) With Albert Schweitzer and Bertrand Russell, Einstein lobbied to stop nuclear testing and future bombs. Days before his death, Einstein signed the Russell–Einstein Manifesto, which led to the Pugwash Conferences on Science and World Affairs.^[111]

Einstein was a member of several civil rights groups, including the Princeton chapter of the NAACP. When the aged W. E. B. Du Bois was accused of being a Communist spy, Einstein volunteered as a character witness, and the case was dismissed shortly afterward. Einstein's friendship with activist Paul Robeson, with whom he served as co-chair of the American Crusade to End Lynching, lasted twenty years.^[112]

Einstein said "Politics is for the moment, equation for the eternity."^[113] He declined the presidency of Israel in 1952.^[114]

Religious views

The question of scientific determinism gave rise to questions about Einstein's position on theological determinism, and whether or not he believed in God, or in a god. He once said:

You may call me an agnostic... I do not share the crusading spirit of the professional atheist whose fervor is mostly due to a painful act of liberation from the fetters of religious indoctrination received in youth. I prefer an attitude of humility corresponding to the weakness of our intellectual understanding of nature and of our own being.^[115]

Non-scientific legacy

While travelling, Einstein wrote daily to his wife Elsa and adopted stepdaughters Margot and Ilse. The letters were included in the papers bequeathed to The Hebrew University. Margot Einstein permitted the personal letters to be made available to the public, but requested that it not be done until twenty years after her death (she died in 1986^[116]). Barbara Wolff, of The Hebrew University's Albert Einstein Archives, told the BBC that there are about 3,500 pages of private correspondence written between 1912 and 1955.^[117]

Einstein bequeathed the royalties from use of his image to The Hebrew University of Jerusalem. Corbis, successor to The Roger Richman Agency, licenses the use of his name and associated imagery, as agent for the university.^[118]
[119]

In popular culture

In the period before World War II, Einstein was so well-known in America that he would be stopped on the street by people wanting him to explain "that theory". He finally figured out a way to handle the incessant inquiries. He told his inquirers "Pardon me, sorry! Always I am mistaken for Professor Einstein."^[120]

Einstein has been the subject of or inspiration for many novels, films, plays, and works of music.^[121] He is a favorite model for depictions of mad scientists and absent-minded professors; his expressive face and distinctive hairstyle have been widely copied and exaggerated. *Time* magazine's Frederic Golden wrote that Einstein was "a cartoonist's dream come true".^[122]

Awards and honors

In 1922, Einstein was awarded the 1921 Nobel Prize in Physics,^[123] "for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect". This refers to his 1905 paper on the photoelectric effect, "On a Heuristic Viewpoint Concerning the Production and Transformation of Light", which was well supported by the experimental evidence by that time. The presentation speech began by mentioning "his theory of relativity [which had] been the subject of lively debate in philosophical circles [and] also has astrophysical implications which are being rigorously examined at the present time". (Einstein 1923)

It was long reported that Einstein gave the Nobel prize money to his first wife, Mileva Marić, in compliance with their 1919 divorce settlement. However, personal correspondence made public in 2006^[124] shows that he invested much of it in the United States, and saw much of it wiped out in the Great Depression.

In 1929, Max Planck presented Einstein with the Max Planck medal of the German Physical Society in Berlin, for extraordinary achievements in theoretical physics.^[125]

In 1936, Einstein was awarded the Franklin Institute's Franklin Medal for his extensive work on relativity and the photo-electric effect.^[125]

The International Union of Pure and Applied Physics named 2005 the "World Year of Physics" in commemoration of the 100th anniversary of the publication of the annus mirabilis papers.^[126]

The *Albert Einstein Science Park* is located on the hill Telegrafenberg in Potsdam, Germany. The best known building in the park is the Einstein Tower which has a bronze bust of Einstein at the entrance. The Tower is an astrophysical observatory that was built to perform checks of Einstein's theory of General Relativity.^[127]

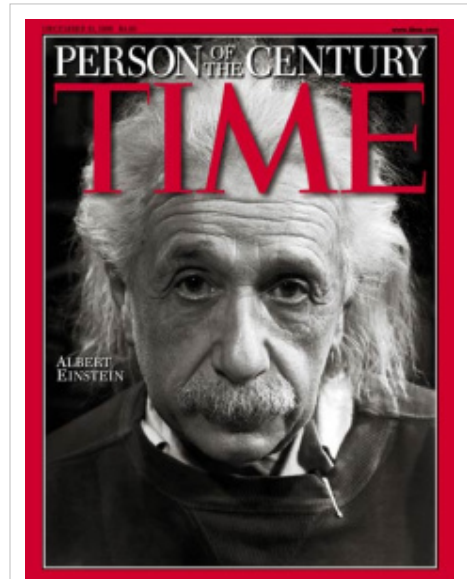
The *Albert Einstein Memorial* in central Washington, D.C. is a monumental bronze statue depicting Einstein seated with manuscript papers in hand. The statue, commissioned in 1979, is located in a grove of trees at the southwest corner of the grounds of the National Academy of Sciences on Constitution Avenue.

The chemical element 99, einsteinium, was named for him in August 1955, four months after Einstein's death.^[128]
^[129] 2001 Einstein is an inner main belt asteroid discovered on 5 March 1973.^[130]

In 1999 *Time* magazine named him the Person of the Century,^[122] ^[131] ahead of Mahatma Gandhi and Franklin Roosevelt, among others. In the words of a biographer, "to the scientifically literate and the public at large, Einstein is synonymous with genius".^[132] Also in 1999, an opinion poll of 100 leading physicists ranked Einstein the "greatest physicist ever".^[133] A Gallup poll recorded him as the fourth most admired person of the 20th century in the U.S.^[134]

In 1990, his name was added to the Walhalla temple for "laudable and distinguished Germans",^[135] which is located east of Regensburg, in Bavaria, Germany.^[136]

The United States Postal Service honored Einstein with a Prominent Americans series (1965–1978) 8¢ postage stamp.



In 1999 Albert Einstein was named the Person of the Century.

Awards named after him

The Albert Einstein Award (sometimes called the Albert Einstein Medal because it is accompanied with a gold medal) is an award in theoretical physics, established to recognize high achievement in the natural sciences. It was endowed by the Lewis and Rosa Strauss Memorial Fund in honor of Albert Einstein's 70th birthday. It was first awarded in 1951 and included a prize money of \$ 15,000,^{[137] [138]} which was later reduced to \$ 5,000.^{[139] [140]} The winner is selected by a committee (the first of which consisted of Einstein, Oppenheimer, von Neumann and Weyl^[141]) of the Institute for Advanced Study, which administers the award.^[138]

The Albert Einstein Medal is an award presented by the Albert Einstein Society in Bern, Switzerland. First given in 1979, the award is presented to people who have "rendered outstanding services" in connection with Einstein.^[142]

The Albert Einstein Peace Prize is given yearly by the Chicago, Illinois-based Albert Einstein Peace Prize Foundation. Winners of the prize receive \$50,000.^[143]

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- [13] www.chem.harvard.edu/herschbach/Einstein_Student.pdf Albert's intellectual growth was strongly fostered at home. His mother, a talented pianist, ensured the children's musical education. His father regularly read Schiller and Heine aloud to the family. Uncle Jakob challenged Albert with mathematical problems, which he solved with "a deep feeling of happiness". Most remarkable was Max Talmud, a poor Jewish medical student from Poland, "for whom the Jewish community had obtained free meals with the Einstein family". Talmud came on Thursday nights for about six years, and "invested his whole person in examining everything that engaged [Albert's] interest". Talmud had Albert read and discuss many books with him. These included a series of twenty popular science books that convinced Albert "a lot in the Bible stories could not be true", and a textbook of plane geometry that launched Albert on avid self-study of mathematics, years ahead of the school curriculum. Talmud even had Albert read Kant; as a result Einstein began preaching to his schoolmates about Kant, with "forcefulness"
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External links

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Max Born

Max Born	
 Max Born (1882-1970)	
Born	11 December 1882Breslau, Germany
Died	5 January 1970 (aged 87)Göttingen, Germany
Residence	Göttingen, Germany
Citizenship	Germany/United Kingdom
Nationality	German
Fields	Physicist
Institutions	University of Frankfurt am Main University of Göttingen University of Edinburgh
Alma mater	University of Göttingen
Doctoral advisor	Carl Runge
Other academic advisors	Joseph Larmor J. J. Thomson
Doctoral students	Victor Frederick Weisskopf J. Robert Oppenheimer Lothar Wolfgang Nordheim Max Delbrück Walter Elsasser Friedrich Hund Pascual Jordan Maria Goeppert-Mayer Herbert S. Green Cheng Kaijia Siegfried Flügge Edgar Krahn Maurice Pryce Antonio Rodríguez Bertha Swirles Paul Weiss Peng Huanwu
Other notable students	Emil Wolf

Known for	Born-Haber cycle Born rigidity Born approximation Born-Infeld theory Born-Oppenheimer approximation Born's Rule Born-Landé equation
Notable awards	Nobel Prize in Physics (1954)

Max Born (11 December 1882 – 5 January 1970) was a German born physicist and mathematician who was instrumental in the development of quantum mechanics. He also made contributions to solid-state physics and optics and supervised the work of a number of notable physicists in the 1920s and 30s. Born won the 1954 Nobel Prize in Physics (shared with Walther Bothe).

Early life and education

Max was born on December 11, 1882 in Breslau (now Wrocław, Poland), which at Born's birth was in the Prussian Province of Silesia in the German Empire. He was one of two children born to Gustav Born, (b. 22 April 1850, Kempen, d. 6 July 1900, Breslau), an anatomist and embryologist, and Margarethe ('Gretchen') Kauffmann (b. 22 January 1856, Tannhausen, d. 29 August 1886, Breslau), from a Silesian family of industrialists.

Gustav and Gretchen married on 7 May 1881. She died when Max was just four years old, on 29 August 1886.

Max had a sister Käthe (b. 5 March 1884), and a half-brother Wolfgang (b. 21 October 1892) from his father's second marriage (m. 13 September 1891) with Bertha Lipstein.

Initially educated at the König-Wilhelm-Gymnasium, Born went on to study at the University of Breslau followed by Heidelberg University and the University of Zurich. During study for his Ph.D.^[1] and Habilitation ^[2] at the University of Göttingen, he came into contact with many prominent scientists and mathematicians including Klein, Hilbert, Minkowski, Runge, Schwarzschild, and Voigt. In 1908-1909 he studied at Gonville and Caius College, Cambridge.

When Born arrived in Göttingen in 1904, Klein, Hilbert, and Minkowski^[3] were the high priests of mathematics and were known as the “mandarins.” Very quickly after his arrival, Born formed close ties to the latter two men. From the first class he took with Hilbert, Hilbert identified Born as having exceptional abilities and selected him as the lecture scribe, whose function was to write up the class notes^[4] for the students' mathematics reading room at the University of Göttingen. Being class scribe put Born into regular, invaluable contact with Hilbert, during which time Hilbert's intellectual largesse benefited Born's fertile mind. Hilbert became Born's mentor and Hilbert eventually selected him to be the first to hold the unpaid, semi-official position of Hilbert's assistant. Born's introduction to Minkowski came through Born's stepmother, Bertha, as she knew Minkowski from dancing classes in Königsberg. The introduction netted Born invitations to the Minkowski household for Sunday dinners. In addition, while performing his duties as scribe and assistant, Born often saw Minkowski at Hilbert's house. Born's outstanding work on elasticity - a subject near and dear to Klein - became the core of his *magna cum laude* Ph.D. thesis, in spite of some of Born's irrationalities in dealing with Klein.^[5]

Born married Hedwig, née Ehrenberg, who was, like Born, of Jewish descent (although a practising Christian), on 2 August 1913, and converted to the Lutheran faith soon thereafter; the marriage produced three children including G. V. R. Born. His daughter Irene was the mother of British-born Australian singer and actress Olivia Newton-John. Via marriage, he is the great uncle of British alternative comedian Ben Elton.

Career

After Max's Habilitation in 1909, he settled in as a young academic at Göttingen as a Privatdozent (Associate Professor).^[6] In Göttingen, Born stayed at a boarding house run by Sister Annie at Dahlmannstraße 17, known as El BoKaReBo. The name was derived from the first letters of the last names of its boarders: "El" for Ella Philipson (a medical student), "Bo" for Born and Hans Bolza (a physics student), "Ka" for Theodore von Kármán (a Privatdozent), and "Re" for Albrecht Renner (a medical student). A frequent visitor to the boarding house was Paul Peter Ewald, a doctoral student of Arnold Sommerfeld on loan to David Hilbert at Göttingen as a special assistant for physics.^[7] Richard Courant, a mathematician and Privatdozent, called these people the "in group."^[8]

From 1915 to 1919, except for a period in the German army, Born was extraordinarius professor of theoretical physics at the University of Berlin, where he formed a life-long friendship with Albert Einstein. In 1919, he became ordinarius professor on the science faculty at the University of Frankfurt am Main. While there, the University of Göttingen was looking for a replacement for Peter Debye, and the Philosophy Faculty had Born at the top of their list. In negotiating for the position with the education ministry, Born arranged for another chair at Göttingen and for his long-time friend and colleague James Franck to fill it.^[9] In 1921, Born became ordinarius professor of theoretical physics and Director of the new Institute of Theoretical Physics at Göttingen.^[10] While there, he formulated^[11] the now-standard interpretation of the *probability density function* for $\psi^*\psi$ in the Schrödinger equation of quantum mechanics, published in July 1926^[12] and for which he was awarded the Nobel Prize in Physics in 1954, some three decades later.

For the 12 years Born and Franck were at Göttingen (1921–1933), Born had a collaborator with shared views on basic scientific concepts — a distinct advantage for teaching and his research on the developing quantum theory. The approach of close collaboration between theoretical physicists and experimental physicists was also shared by Born at Göttingen and Arnold Sommerfeld at the University of Munich, who was ordinarius professor of theoretical physics and Director of the Institute of Theoretical Physics — also a prime mover in the development of quantum theory. Born and Sommerfeld not only shared their approach in using experimental physics to test and advance their theories, Sommerfeld, in 1922 when he was in the United States lecturing at the University of Wisconsin–Madison, sent his student Werner Heisenberg to be Born's assistant. Heisenberg again returned to Göttingen in 1923 and completed his Habilitation under Born in 1924 and became a Privatdozent at Göttingen — the year before Heisenberg and Born published their first papers on matrix mechanics.^[13] ^[14]

In 1925, Born and Werner Heisenberg formulated the matrix mechanics representation of quantum mechanics. On 9 July, Heisenberg gave Born a paper to review and submit for publication.^[15] In the paper, Heisenberg formulated quantum theory avoiding the concrete but unobservable representations of electron orbits by using parameters such as transition probabilities for quantum jumps, which necessitated using two indexes corresponding to the initial and final states.^[16] When Born read the paper, he recognized the formulation as one which could be transcribed and extended to the systematic language of matrices,^[17] which he had learned from his study under Jakob Rosanes^[18] at Breslau University. Born, with the help of his assistant and former student Pascual Jordan, began immediately to make the transcription and extension, and they submitted their results for publication; the paper was received for publication just 60 days after Heisenberg's paper.^[19] A follow-on paper was submitted for publication before the end of the year by all three authors.^[20] (A brief review of Born's role in the development of the matrix mechanics formulation of quantum mechanics along with a discussion of the key formula involving the non-commutativity of the probability amplitudes can be found in an article by Jeremy Bernstein.^[21] A detailed historical and technical account can be found in Mehra and Rechenberg's book *The Historical Development of Quantum Theory. Volume 3. The Formulation of Matrix Mechanics and Its Modifications 1925–1926*.^[22])

Up until this time, matrices were seldom used by physicists; they were considered to belong to the realm of pure mathematics. Gustav Mie had used them in a paper on electrodynamics in 1912 and Born had used them in his work on the lattices theory of crystals in 1921. While matrices were used in these cases, the algebra of matrices with their multiplication did not enter the picture as they did in the matrix formulation of quantum mechanics.^[23]

Born, however, had learned matrix algebra from Rosanes, as already noted, but Born had also learned Hilbert's theory of integral equations and quadratic forms for an infinite number of variables as was apparent from a citation by Born of Hilbert's work *Grundzüge einer allgemeinen Theorie der Linearen Integralgleichungen* published in 1912.^{[24] [25]} Jordan, too was well equipped for the task. For a number of years, he had been an assistant to Richard Courant at Göttingen in the preparation of Courant and David Hilbert's book *Methoden der mathematischen Physik I*, which was published in 1924.^[26] This book, fortuitously, contained a great many of the mathematical tools necessary for the continued development of quantum mechanics. In 1926, John von Neumann became assistant to David Hilbert, and he would coin the term Hilbert space to describe the algebra and analysis which were used in the development of quantum mechanics.^{[27] [28]}

In 1928, Albert Einstein nominated Heisenberg, Born, and Jordan for the Nobel Prize in Physics,^[29] but it was not to be. The announcement of the Nobel Prize in Physics for 1932 was delayed until November 1933.^[30] It was at that time that it was announced Heisenberg had won the Prize for 1932 "for the creation of quantum mechanics, the application of which has led to the discovery of the allotropic forms of hydrogen"^[31] and Erwin Schrödinger and Paul Adrien Maurice Dirac shared the 1933 Prize "for the discovery of new productive forms of atomic theory".^[31] One can rightly ask why Born was not awarded the Prize in 1932 along with Heisenberg – Bernstein gives some speculations on this matter. One of them is related to Jordan joining the Nazi Party on 1 May 1933 and becoming a Storm Trooper.^[32] Hence, Jordan's Party affiliations and Jordan's links to Born may have affected Born's chance at the Prize at that time. Bernstein also notes that when Born won the Prize in 1954, Jordan was still alive, and the Prize was awarded for the statistical interpretation of quantum mechanics, attributable alone to Born.^[33]

Heisenberg's reaction to Born for Heisenberg himself receiving the Prize for 1932 and Born receiving the Prize in 1954 is also instructive in evaluating whether Born should have shared the Prize with Heisenberg. On 25 November 1933 Born received a letter from Heisenberg in which he said he had been delayed in writing due to a "bad conscience" that he alone had received the Prize "for work done in Göttingen in collaboration — you, Jordan and I." Heisenberg went on to say that Born and Jordan's contribution to quantum mechanics cannot be changed by "a wrong decision from the outside."^[34] In 1954, Heisenberg wrote an article honoring Max Planck for his insight in 1900. In the article, Heisenberg credited Born and Jordan for the final mathematical formulation of matrix mechanics and Heisenberg went on to stress how great their contributions were to quantum mechanics, which were not "adequately acknowledged in the public eye."^[35]

Those who received their Ph.D. degrees under Born at Göttingen included Max Delbrück, Walter Elsasser, Friedrich Hund, Pascual Jordan, Maria Goeppert-Mayer, Lothar Wolfgang Nordheim, J. Robert Oppenheimer, and Victor Weisskopf.^[36] Born's assistants at the University of Göttingen's Institute for Theoretical Physics included Enrico Fermi, Werner Heisenberg, Gerhard Herzberg, Friedrich Hund, Pascual Jordan, Wolfgang Pauli, Léon Rosenfeld, Edward Teller, and Eugene Wigner.^{[36] [37] [38]} Walter Heitler became an assistant to Born in 1928 and under Born completed his Habilitation in 1929.^[39] Born not only recognized talent to work with him, but he let his "superstars stretch past him."^[40] His Ph.D. student Delbrück, and six of his assistants (Fermi, Heisenberg, Goeppert-Mayer, Herzberg, Pauli, Wigner) went on to win Nobel Prizes.

In a letter to Born in 1926, Einstein made his famous remark regarding quantum mechanics, often paraphrased as "The Old One does not play dice."^[41]

In 1933 Born emigrated from Germany. He had strong and public pacifist opinions; moreover, though Born was a Lutheran, he was classified as a "Jew" by the Nazi racial laws due to his ancestry, and was thus stripped of his professorship. He took up a position as Stokes Lecturer at the University of Cambridge. From 1936 to 1953 he was Tait Professor of Natural Philosophy at the University of Edinburgh. He became a British subject and a Fellow of the Royal Society of London in 1939.^[42]

Born had a dislike for nuclear weapons research, but he still acknowledged "it might be the only way out."^[43] Much of the theoretical power behind the development of the first atomic bomb was due to many of those surrounding him at Göttingen and working on atomic physics and quantum mechanics: three of his Ph.D. students (Maria

Goeppert-Mayer, Oppenheimer and Weisskopf), three of his assistants (Fermi, Teller, and Wigner), the Director of the Second Institute for Experimental Physics (James Franck), and David Hilbert's assistant (John von Neumann).^[44]

Max and Hedwig Born retired to Bad Pyrmont (10 km south of Hamelin) in West Germany, in 1954.^[45]

Born was one of the 11 signatories to the Russell-Einstein Manifesto.

Born is also the great-grandfather of the famous TV editor and percussionist Kip Thompson-Born.

Born died in Göttingen, Germany. He is buried there in the same cemetery as Walther Nernst, Wilhelm Weber, Max von Laue, Max Planck, and David Hilbert.

Max Born Prize

In memory of his important contributions, the Max Born prize was created by the German Physical Society and the British Institute of Physics. It is awarded annually.

Published works

During his life, Born wrote several semi-popular and technical books. His volumes on topics like atomic physics and optics were very well-received and are considered classics in their fields which are still in print. The following is a listing of his major works:

- *Über das Thomson'sche Atommodell* Habilitations-Vortrag (FAM, 1909) - The Habilitation was done at the University of Göttingen, on 23 October 1909.^[46]
- *Dynamik der Kristallgitter* (Teubner, 1915)^[47] - After its publication, the physicist Arnold Sommerfeld asked Born to write an article based on it for the 5th volume of the *Mathematical Encyclopedia*. World War I delayed the start of work on this article, but it was taken up in 1919 and finished in 1922. It was published as a revised edition under the title *Atomic Theory of Solid States*.^[48]
 - *Dynamical Theory of Crystal Lattices*, with Kun Huang. (Oxford, Clarendon Press, 1954)^[49]
- *Die Relativitätstheorie Einsteins und ihre physikalischen Grundlagen* (Springer, 1920) - Based on Born's lectures at the University of Frankfurt am Main.^[50]
 - Available in English under the title *Einstein's Theory of Relativity*.^[51]
- *Vorlesungen über Atommechanik* (Springer, 1925)^[47]
 - *Mechanics of the Atom* (George Bell & Sons, 1927) - Translated by J. W. Fisher and revised by D. R. Hartree.^[52]
- *Problems of Atomic Dynamics* (MIT Press, 1926) – A first account of matrix mechanics being developed in Germany, based on two series of lectures given at MIT, over three months, in late 1925 and early 1926.^[53] ^[54]
- *Elementare Quantenmechanik (Zweiter Band der Vorlesungen über Atommechanik)*, with Pascual Jordan. (Springer, 1930) - This was the first volume of what was intended as a two-volume work. This volume was limited to the work Born did with Jordan on matrix mechanics. The second volume was to deal with Erwin Schrödinger's wave mechanics. However, the second volume was not even started by Born, as he believed his friend and colleague Hermann Weyl had written it before he could do so.^[55] ^[56]
- *Optik: Ein Lehrbuch der elektromagnetische Lichttheorie* (Springer, 1933) - The book was released just as the Borns were emigrating to England.
 - *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*,^[57] with Emil Wolf. (Pergamon, 1959) - This book is not an English translation of *Optik*, but rather a substantially new book. Shortly after World War II, a number of scientists suggested that Born update and translate his work into English. Since there had been many advances in optics in the intervening years, updating was warranted. In 1951, Emil Wolf began as Born's private assistant on the book; it was eventually published in 1959 by Robert Maxwell's Pergamon Press^[58] - the delay being due to the lengthy time needed "to resolve all the financial and

publishing tricks created by Maxwell.”^[59]

- *Moderne Physik* (1933) -- Based on seven lectures given at the Technischen Hochschule Berlin.^[60]
 - *Atomic Physics* (Blackie, London, 1935) - Authorized translation of *Moderne Physik* by John Dougall, with updates.^[61]
- *The Restless Universe*^[62] (Blackie and Son Limited, 1935) - A popularized rendition of the workshop of nature. Born's nephew, Otto Königsberger, whose successful career as an architect in Berlin was brought to an end when the Nazis took over, was temporarily brought to England to illustrate the book.^[60]
- *Experiment and Theory in Physics* (Cambridge University Press, 1943) – The address given King's College, Newcastle-on-Tyne, at the request of the Durham Philosophical Society and the Pure Science Society. An expanded version of the lecture appeared in a 1956 Dover Publications edition.^[63]
- *Natural Philosophy of Cause and Chance* (Oxford University Press, 1949) – Based on Born's 1948 Waynflete lectures, given at the College of St. Mary Magdalen, Oxford University. A later edition (Dover, 1964) included two appendices: “Symbol and Reality” and Born's lecture given at the Nobel laureates 1964 meeting in Landau, Germany.^[64]
- *A General Kinetic Theory of Liquids* with H. S. Green (Cambridge University Press, 1949) -- The six papers in this book were reproduced with permission from the *Proceedings of the Royal Society*.
- *Physics in My Generation: A Selection of Papers* (Pergamon, 1956)^[65]
- *Physik im Wandel meiner Zeit* (Vieweg, 1957)
- *Physik und Politik* (VandenHoeck und Ruprecht, 1960)
- *Zur Begründung der Matrizenmechanik*, with Werner Heisenberg and Pascual Jordan (Battenberg, 1962) - Published in honor of Max Born's 80th birthday. This edition reprinted the authors' articles on matrix mechanics published in *Zeitschrift für Physik*, Volumes **26** and **33-35**, 1924-1926.^[66]
- *My Life and My Views: A Nobel Prize Winner in Physics Writes Provocatively on a Wide Range of Subjects* (Scribner, 1968) - Part II (pp. 63–206) is a translation of *Verantwortung des Naturwissenschaftlers*.^[67]
- *Briefwechsel 1916-1955, kommentiert von Max Born* with Hedwig Born and Albert Einstein (Nymphenburger, 1969)
 - *The Born-Einstein Letters: Correspondence between Albert Einstein and Max and Hedwig Born from 1916–1955, with commentaries by Max Born* (Macmillan, 1971).^[68]
- *Mein Leben: Die Erinnerungen des Nobelpreisträgers* (Munich: Nymphenburger, 1975). Born's published memoirs.
 - *My Life: Recollections of a Nobel Laureate* (Scribner, 1978).^[69] Translation of *Mein Leben*.
- Born Nobel Prize Speech^[70] - 1954
- Born Nobel Prize Lecture^[71] - 1954
- Published papers^[72] (as listed on the Smithsonian/NASA Astrophysics Data System (ADS))
- Published papers^[73] (as listed on HistCite)
- Published Books^[74] (based on the Library of Congress citations)
- Published Works - Berlin-Brandenburgische Akademie der Wissenschaften Akademiebibliothek^[75]

Selected journal literature

While links have been provided in this article to journal publications by Born, a few of his papers are worth highlighting here along with citations to translations in English.

Matrix Mechanics A trilogy of papers launched the matrix mechanics formulation of quantum mechanics:

- W. Heisenberg, *Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen*, *Zeitschrift für Physik*, **33**, 879-893, 1925 (received 29 July 1925). [English translation in: B. L. van der Waerden, editor, *Sources of Quantum Mechanics* (Dover Publications, 1968) ISBN 0-486-61881-1 (English title: *Quantum-Theoretical Re-interpretation of Kinematic and Mechanical Relations*).]
- M. Born and P. Jordan, *Zur Quantenmechanik*, *Zeitschrift für Physik*, **34**, 858-888, 1925 (received 27 September 1925). [English translation in: B. L. van der Waerden, editor, *Sources of Quantum Mechanics* (Dover Publications, 1968) ISBN 0-486-61881-1 (English title: *On Quantum Mechanics*).]
- M. Born, W. Heisenberg, and P. Jordan, *Zur Quantenmechanik II*, *Zeitschrift für Physik*, **35**, 557-615, 1926 (received 16 November 1925). [English translation in: B. L. van der Waerden, editor, *Sources of Quantum Mechanics* (Dover Publications, 1968) ISBN 0-486-61881-1]

Probability Density The now-standard interpretation of the probability density function for $\psi^*\psi$ in the Schrödinger equation of quantum mechanics was published by Born in the first of these two papers, and it is this for which he was awarded the Nobel Prize in Physics in 1954. The second paper is a continuation and extension of the analysis provided in the first paper.

- Max Born (1926). "Zur Quantenmechanik der Stoßvorgänge" ^[76]. *Zeitschrift für Physik* **37**: 863–867. doi:10.1007/BF01397477. Retrieved 2009-12-16.
- Max Born (1926). "Quantenmechanik der Stoßvorgänge" ^[77]. *Zeitschrift für Physik* **38**: 803–827. doi:10.1007/BF01397184. Retrieved 2009-12-18. "English translation in: Gunther Ludwig, editor, *Wave Mechanics* (Pergamon, 1968) ISBN 08-103204-8. Under the title: *Quantum Mechanics of Collision Processes*".

Awards and honors

- 1934 - Stokes Medal of Cambridge ^[78]
- 1939 - Fellow of the Royal Society ^[78]
- 1945 - MacDougall-Brisbane Medal of the Royal Society of Edinburgh ^[79]
- 1945 - Gunning-Victoria Jubilee Prize of the Royal Society of Edinburgh ^[78]
- 1948 - Max Planck Medaille der Deutschen Physikalischen Gesellschaft ^[78]
- 1950 - Hughes Medal of the Royal Society of London ^[78]
- 1953 - Honorary citizen of the town of Göttingen ^[78]
- 1954 - Nobel Prize in Physics The award was for Born's fundamental research in quantum mechanics, especially for his statistical interpretation of the wavefunction. ^[78]
 - 1954 - Nobel Prize Speech ^[70]
 - 1954 - Born Nobel Prize Lecture ^[71]
- 1956 - Hugo Grotius Medal for International Law, Munich ^[78]
- 1959 - Grand Cross of Merit with Star of the Order of Merit of the German Federal Republic ^[80]
- 1982 - Ceremony at the University of Göttingen in the 100th Birth Year of Max Born and James Franck, Institute Directors 1921 - 1933. ^[81]
- Max-Born Institut für Nichtlineare Optik und Kurzzeitspektroskopie im Forschungsverbund Berlin e.V. ^[82] - Institute named in his honor.

Bibliography

- Jeremy Bernstein *Max Born and the Quantum Theory*, *Am. J. Phys.* 73 (11) 999-1008 (2005). Department of Physics, Stevens Institute of Technology, Hoboken, New Jersey 07030. Received 14 April 2005; accepted 29 July 2005.
- Max Born *The statistical interpretation of quantum mechanics*. Nobel Lecture ^[71] – 11 December 1954.
- Nancy Thorndike Greenspan, "The End of the Certain World: The Life and Science of Max Born" (Basic Books, 2005) ISBN 0-7382-0693-8. Also published in Germany: *Max Born - Baumeister der Quantenwelt. Eine Biographie* (Spektrum Akademischer Verlag, 2005), ISBN 3-8274-1640-X.
- Max Jammer *The Conceptual Development of Quantum Mechanics* (McGraw-Hill, 1966)
- Christa Jungnickel and Russell McCormmach. *Intellectual Mastery of Nature. Theoretical Physics from Ohm to Einstein, Volume 2: The Now Mighty Theoretical Physics, 1870 to 1925*. University of Chicago Press, Paper cover, 1990. ISBN 0-226-41585-6
- Jagdish Mehra and Helmut Rechenberg *The Historical Development of Quantum Theory. Volume 3. The Formulation of Matrix Mechanics and Its Modifications 1925–1926*. (Springer, 2001) ISBN 0-387-95177-6
- B. L. van der Waerden, editor, *Sources of Quantum Mechanics* (Dover Publications, 1968) ISBN 0-486-61881-1
- Kurt Gottfried, *Born to Greatness?*, *Nature*, Vol. 435, 739, 9 June 2005. [83]

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- [1] His Ph.D. thesis in mathematics was defended at the University of Göttingen on 13 June 1906: Untersuchungen über die Stabilität der elastischen Linie in Ebene und Raum, unter verschiedenen Grenzbedingungen. It was awarded magna cum laude. See Greenspan, 2005, pp. 35-36 and Max Born's Life (<http://do.nw.schule.de/mbr/schule/maxbornengl.htm>).
- [2] The Habilitation was done at the University of Göttingen, on 23 October 1909: *Über das Thomson'sche Atommodell* Habilitations-Vortrag (FAM, 1909). See Greenspan, 2005, pp. 49, 51, and 353.
- [3] Hilbert and Minkowski had been colleagues at the University of Königsberg. Klein brought Hilbert to Göttingen. Then, Hilbert brought Minkowski.
- [4] Since the lectures were the creation of the lecturer and not take from a textbook, the scribe performed a very important function.
- [5] Greenspan, 2005, 26-34.
- [6] Greenspan, 2005, pp. 49, 53, and 353.
- [7] It was at the boarding house that Paul Peter Ewald met Ella Philipson, who was to become his wife. They had a daughter, Rose. Hans Bethe, who got his doctorate from Sommerfeld in 1928, met Rose at Duke University in 1937, and they were married in 1938. See Greenspan, 2005, p. 53. Also see Hans Bethe (<http://www.nytimes.com/2005/03/07/science/08cnd-bethe.html?pagewanted=3&ei=5090&en=537d823fc7c24201&ex=1267938000>) – New York Times.
- [8] Greenspan, 2005, p. 53.
- [9] James Franck took the dual position of ordinarius professor and Director of the Second Institute for Experimental Physics at Göttingen. See Nobel Prize Biography (http://nobelprize.org/nobel_prizes/physics/laureates/1925/franck-bio.html).
- [10] Greenspan, 2005, pp. 96-97.
- [11] Max Born (http://nobelprize.org/nobel_prizes/physics/laureates/1954/born-lecture.pdf) – Nobel Lecture, 11 December 1954: *The statistical interpretation of quantum mechanics*.
- [12] Max Born (1926). "Zur Quantenmechanik der Stoßvorgänge" (<http://www.springerlink.com/content/h06w8465t710u328/>). *Zeitschrift für Physik* **37**: 863–867. doi:10.1007/BF01397477. . Retrieved 2008-12-16.
- [13] Greenspan, 2005, pp. 113, 120, and 123.
- [14] Jungnickel, Christa and Russell McCormmach. *Intellectual Mastery of Nature. Theoretical Physics from Ohm to Einstein, Volume 2: The Now Mighty Theoretical Physics, 1870 to 1925*. University of Chicago Press, Paper cover, 1990. ISBN 0-226-41585-6. pp. 274, and 281-285 and 350-354.
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- [16] Emilio Segrè, *From X-Rays to Quarks: Modern Physicists and their Discoveries* (W. H. Freeman and Company, 1980) ISBN 0-7167-1147-8, pp. 153–157.
- [17] Abraham Pais, *Niels Bohr's Times in Physics, Philosophy, and Polity* (Clarendon Press, 1991) ISBN 0-19-852049-2, pp. 275–279.
- [18] Max Born (http://nobelprize.org/nobel_prizes/physics/laureates/1954/born-lecture.pdf) — Nobel Lecture (1954)
- [19] M. Born and P. Jordan, *Zur Quantenmechanik*, *Zeitschrift für Physik*, **34**, 858–888, 1925 (received 27 September 1925). [English translation in: B. L. van der Waerden, editor, *Sources of Quantum Mechanics* (Dover Publications, 1968) ISBN 0-486-61881-1]

- [20] M. Born, W. Heisenberg, and P. Jordan, *Zur Quantenmechanik II*, *Zeitschrift für Physik*, **35**, 557-615, 1925 (received 16 November 1925).
[English translation in: B. L. van der Waerden, editor, *Sources of Quantum Mechanics* (Dover Publications, 1968) ISBN 0-486-61881-1]
- [21] Jeremy Bernstein *Max Born and the Quantum Theory*, *Am. J. Phys.* **73** (11) 999–1008 (2005)
- [22] Mehra, Volume 3 (Springer, 2001)
- [23] Jammer, 1966, pp. 206-207.
- [24] van der Waerden, 1968, p. 51.
- [25] The citation by Born was in Born and Jordan's paper, the second paper in the trilogy which launched the matrix mechanics formulation. See van der Waerden, 1968, p. 351.
- [26] Constance Ried *Courant* (Springer, 1996) p. 93.
- [27] John von Neumann *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, *Mathematische Annalen* **102** 49–131 (1929)
- [28] When von Neumann left Göttingen in 1932, his book on the mathematical foundations of quantum mechanics, based on Hilbert's mathematics, was published under the title *Mathematische Grundlagen der Quantenmechanik*. See: Norman Macrae, *John von Neumann: The Scientific Genius Who Pioneered the Modern Computer, Game Theory, Nuclear Deterrence, and Much More* (Reprinted by the American Mathematical Society, 1999) and Constance Reid, *Hilbert* (Springer-Verlag, 1996) ISBN 0-387-94674-8.
- [29] Bernstein, 2004, p. 1004.
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- [31] Nobel Prize in Physics and 1933 (http://nobelprize.org/nobel_prizes/physics/laureates/1933/press.html) – Nobel Prize Presentation Speech.
- [32] Bernstein, 2005, p. 1004.
- [33] Bernstein, 2005, p. 1006.
- [34] Greenspan, 2005, p. 191.
- [35] Greenspan, 2005, pp. 285-286.
- [36] *Sources for History of Quantum Physics* (<http://www.amphilsoc.org/library/guides/ahqp/index.htm>)
- [37] Greenspan, 2005, pp. 178 and 262.
- [38] Biography on Gerhard Herzberg (http://www.nserc.gc.ca/award_e.asp?nav=herzberg&lbi=scientist) - National Sciences and Engineering Research Council of Canada
- [39] Author Catalog: Heitler (<http://www.amphilsoc.org/library/guides/ahqp/>) – American Philosophical Society
- [40] Greenspan, 2005, p. 143.
- [41] Letter from A. Einstein to M. Born dated 12 December 1926 [Max Born, *Physics in my generation*, Springer-Verlag, New York (1969), p. 113]. Also: "...Even the great initial success of the quantum theory does not make me believe in the fundamental dice-game..." — Albert Einstein, 7 September 1944 (p. 149). *The Correspondence between Albert Einstein and Max and Hedwig Born 1916-1955* with commentaries by Max Born, ISBN 0-8027-0326-7
- [42] Born was elected to the Royal Society in March and received his Certificate of Naturalization on 31 August 1939, one day before World War II broke out in Europe. See Greenspan, 2005, p. 226.
- [43] Greenspan, 2005, p. 239.
- [44] Students, assistants, and colleagues of Born at Göttingen who worked on the Manhattan Project:
- Maria Goeppert-Mayer – Worked on the Manhattan Project with Harold Urey at Columbia University on isotope separation.
 - Robert Oppenheimer – Director of Los Alamos Scientific Laboratory (LASL) - One of the four major sites of the Manhattan Engineering District.
 - Victor Weisskopf – Head of T-3 Group, Experiments, Efficiency Calculations, and Radiation Hydrodynamics, LASL
 - Enrico Fermi – Director of Research, Met Lab of the University of Chicago - One of the four major sites of the Manhattan Engineering District.
 - Edward Teller – Head of T-1 Group, Hydrodynamics of Implosion and Super, LASL
 - Eugene Wigner – Director of Theoretical Studies, Met Lab
 - James Franck – Director of the Chemistry Division, Met Lab, and Chairman of the Committee on Political and Social Problems
 - John von Neumann – LASL consultant on implosion mechanism for the plutonium bomb. (Neumann was assistant to David Hilbert at Göttingen and was greatly influenced by both David Hilbert's and Max Born's work. Neumann applied the mathematics of Hilbert space to Born's quantum mechanics, and, in 1932, his foundational book on the mathematical underpinnings of quantum mechanics, *Mathematische Grundlagen der Quantenmechanik*, was published.)
- [45] Biographical Sketch from the German Historical Museum (<http://www.dhm.de/lemo/html/biografien/BornMax/index.html>)
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- [55] Greenspan, 2005, pp. 159-160.
- [56] Jungnickel, Volume 2, 1990, p. 378.
- [57] *Principles of Optics* is now in its 7th revised printing, ISBN 0-521-64222-1. The first 5 revised editions were done by Pergamon Press (1959 - 1975). The last 2 were done by Cambridge University Press in 1980 and 1999.
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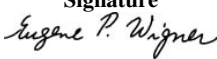
External links

- American Institute of Physics History Search: Max Born (<http://www.aip.org/servlet/plainHistory?collection=HISTORY&queryText=Max+Born>)
 - Encyclopaedia Britannica, Max Born - full article (<http://www.britannica.com/eb/article-9080764/Max-Born>)
 - Annotated bibliography for Max Born from the Alsos Digital Library for Nuclear Issues (<http://alsos.wlu.edu/qsearch.aspx?browse=people/Born,+Max>)
 - Freeview video of Gustav Born (son of Max) with conversation and film on Gustav's memories of his father by the Vega Science Trust (<http://www.vega.org.uk/video/programme/92>)
 - O'Connor, John J.; Robertson, Edmund F., "Max Born" (<http://www-history.mcs.st-andrews.ac.uk/Biographies/Born.html>), *MacTutor History of Mathematics archive*, University of St Andrews.
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 - Max Born (<http://genealogy.math.ndsu.nodak.edu/id.php?id=18245>) at the Mathematics Genealogy Project
 - Max Born information from Nobel Winners site (http://www.nobel-winners.com/Physics/max_born.html)
 - Nobel Laureate biography (<http://www.nobel.se/physics/laureates/1954/born-bio.html>)
 - Papers of Professor Max Born (1882-1970) (<http://www.archiveshub.ac.uk/news/0412born.html>) Held at the Edinburgh University Library, Special Collections Division
 - Recollections of Max Born ([http://www.springerlink.com/\(csrmskyqgy1ghombklkrigzj\)/app/home/contribution.asp?referrer=parent&backto=issue,25,26;journal,152,584;linkingpublicationresults,1:100241,1](http://www.springerlink.com/(csrmskyqgy1ghombklkrigzj)/app/home/contribution.asp?referrer=parent&backto=issue,25,26;journal,152,584;linkingpublicationresults,1:100241,1)), by Emil Wolf, in *Astrophysics and Space Science*, Volume 227, Numbers 1-2. (Biographical tribute)
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 - Oral History interview transcript with Max Born June 1960, 17 & 18 October 1962, American Institute of Physics, Niels Bohr Library and Archives (http://www.aip.org/history/ohilist/4522_1.html)
-

Eugene Wigner

*The native form of this personal name is **Wigner Jenő**. This article uses the Western name order.*

Eugene P. Wigner	
 Eugene Paul Wigner (1902–1995)	
Born	November 17, 1902Budapest, Austria-Hungary
Died	January 1, 1995 (aged 92)Princeton, New Jersey, United States
Residence	USA
Citizenship	American (post-1937) Hungarian (pre-1937)
Fields	Theoretical Physics Atomic Physics Nuclear Physics Solid State Physics
Institutions	University of Göttingen University of Wisconsin–Madison Princeton University Manhattan project
Alma mater	Technische Hochschule Berlin
Doctoral advisor	Michael Polanyi
Other academic advisors	László Rátz Richard Becker
Doctoral students	John Bardeen Victor Frederick Weisskopf Marcos Moshinsky Abner Shimony Edwin Thompson Jaynes Frederick Seitz Conyers Herring Jack H. Irving Frederick Tappert

Known for	Law of conservation of parity Wigner D-matrix Wigner–Eckart theorem Wigner's friend Wigner semicircle distribution Wigner's classification Wigner quasi-probability distribution Wigner crystal Wigner effect Wigner–Seitz cell Relativistic Breit–Wigner distribution Modified Wigner distribution function Wigner-d'Espagnat inequality Gabor–Wigner transform Wigner's theorem Wigner distribution Jordan–Wigner transformation Newton–Wigner localization Wigner–Seitz radius 6-j symbol 9-j symbol
Influenced	Eugene Feenberg George Cowan Robert Serber Igal Talmi
Notable awards	Enrico Fermi Award (1958) Max Planck Medal (1961) Nobel Prize in Physics (1963) National Medal of Science (1969)
Signature 	
Notes	He was Paul Dirac's brother-in-law and the uncle of Gabriel Andrew Dirac.

Eugene Paul "E. P." Wigner (Hungarian **Wigner Jenő Pál**; November 17, 1902 – January 1, 1995) was a Hungarian American physicist and mathematician.

He received a share of the Nobel Prize in Physics in 1963 "for his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles"; the other half of the award was shared between Maria Goeppert-Mayer and J. Hans D. Jensen. Some contemporaries referred to Wigner as *the Silent Genius* and some even considered him the intellectual equal to Albert Einstein, though without his prominence. Wigner is important for having laid the foundation for the theory of symmetries in quantum mechanics as well as for his research into the structure of the atomic nucleus, and for his several mathematical theorems. It was Eugene Wigner who first identified Xe-135 "poisoning" in nuclear reactors, and for this reason it is sometimes referred to as *Wigner poisoning*.^[1]

Early life



Werner Heisenberg + Eugene Wigner (1928)

Wigner was born in Budapest, Austria-Hungary, into a middle class Jewish family. At the age of 11, Wigner contracted what his parents believed to be tuberculosis. They sent him to live for six weeks in a sanitarium in the Austrian mountains. During this period, Wigner developed an interest in mathematical problems. From 1915 through 1919, together with John von Neumann, Wigner studied at the Fasori Evangélikus Gimnázium, where they both benefited from the instruction of the noted mathematics teacher László Ráth. In 1919, to escape the Béla Kun communist regime, the Wigner family briefly moved to Austria, returning to Hungary after Kun's downfall. Partly as a reaction to the prominence of Jews in the Kun regime, the family

converted to Lutheranism.^[2]

In 1921, Wigner studied chemical engineering at the Technische Hochschule in Berlin (today the Technische Universität Berlin). He also attended the Wednesday afternoon colloquia of the German Physical Society. These colloquia featured such luminaries as Max Planck, Max von Laue, Rudolf Ladenburg, Werner Heisenberg, Walther Nernst, Wolfgang Pauli, and Albert Einstein. Wigner also met the physicist Leo Szilard, who at once became Wigner's closest friend. A third experience in Berlin was formative. Wigner worked at the Kaiser Wilhelm Institute for Physical Chemistry and Elektrochemie (now the Fritz Haber Institute), and there he met Michael Polanyi, who became, after László Ráth, Wigner's greatest teacher.

Middle years

In the late 1920s, Wigner deeply explored the field of quantum mechanics. A period at Göttingen as an assistant to the great mathematician David Hilbert proved a disappointment, as Hilbert was no longer active in his works. Wigner nonetheless studied independently. He laid the foundation for the theory of symmetries in quantum mechanics and in 1927 introduced what is now known as the Wigner D-matrix.^[3] It is safe to state that he and Hermann Weyl carry the whole responsibility for the introduction of group theory into quantum mechanics (they spread the "Gruppenpest"). See Wigner's 1931 monograph for a survey of his work on group theory. In the late 1930s, he extended his research into atomic nuclei. He developed an important general theory of nuclear reactions (see for instance the Wigner–Eckart theorem). By 1929, his papers were drawing notice in the world of physics. In 1930, Princeton University recruited Wigner, which was very timely, since the Nazis soon rose to power in Germany. At Princeton in 1934, Wigner introduced his sister Manci to the physicist Paul Dirac, whom she married.

In 1936, Princeton University did not rehire Wigner, hence he searched for new employment. He found this at the University of Wisconsin. There he met his first wife, Amelia Frank, who was a physics student there. However she died unexpectedly in 1937, naturally leaving Wigner distraught.

On January 8, 1937, Wigner became a naturalized citizen of the United States. Princeton University soon invited Wigner back into its employment, and he rejoined its faculty in Fall 1938. Although he was a professed political amateur, in 1939 and 1940 he played a major role in prompting the U.S. Government to establish the Manhattan Project, which developed the first atomic bomb by 1945. However, by his personal beliefs, Wigner was at heart a pacifist. Wigner was present at a converted squash courts at the University of Chicago's abandoned Stagg Field on December 2, 1942, when the world's first atomic reactor, Chicago Pile One (CP-1) achieved a nuclear chain reaction (a critical reaction).^[4] He later contributed to civil defense in the U.S. In 1946, Wigner accepted a position as the Director of Research and Development at the Clinton Laboratory (now the Oak Ridge National Laboratory) in Oak Ridge, Tennessee. When his duties there did not work out especially well, Wigner returned to Princeton University.

In 1941, Wigner married his second wife, Mary Annette Wheeler, a professor of physics at Vassar College, who had completed her Ph.D. at Yale University in 1932. They remained married until her death in 1977, and they were the parents of two children.

Last years



Patricia Eileen (left) and Eugene Paul Wigner at their home in Princeton.

In 1960, Wigner published a now classic article on the philosophy of mathematics and of physics, which has become his best-known work outside of technical mathematics and physics, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences". He argued that biology and cognition could be the origin of physical concepts, as we humans perceive them, and that the happy coincidence that mathematics and physics were so well matched, seemed to be "unreasonable" and hard to explain. His reasoning was resisted by the Harvard mathematician Andrew M. Gleason.

In 1963, Wigner was awarded the Nobel Prize in Physics. Wigner professed to never have considered the possibility that this might occur, and he added: "I never expected to get my name in the newspapers without doing something wicked." Wigner later won the Enrico Fermi award, and the National Medal of Science. In 1992, at the age of 90, Wigner published a memoir, *The Recollections of Eugene P. Wigner* with Andrew Szanton. Wigner died three years later in Princeton, New Jersey. One of his significant students was Abner Shimony. Wigner's third wife was Eileen Clare-Patton Hamilton Wigner ("Pat"), the widow of another physicist, Donald Ross Hamilton, who had died in 1972. (He had been the Dean of the Graduate School at Princeton University.)

Near the end of his life, Wigner's thoughts turned more philosophical. In his memoirs, Wigner said: "The full meaning of life, the collective meaning of all human desires, is fundamentally a mystery beyond our grasp. As a young man, I chafed at this state of affairs. But by now I have made peace with it. I even feel a certain honor to be associated with such a mystery." He became interested in the Vedanta philosophy of Hinduism, particularly its ideas of the universe as an all pervading consciousness. [6] In his collection of essays *Symmetries and Reflections – Scientific Essays*, he commented "It was not possible to formulate the laws (of quantum theory) in a fully consistent way without reference to consciousness."

Wigner also conceived the *Wigner's friend* thought experiment in physics, which is an extension of the *Schrödinger's cat* thought experiment. The *Wigner's friend* experiment asks the question: "At what stage does a 'measurement' take place?" Wigner designed the experiment to highlight how he believed that consciousness is necessary to the quantum-mechanical measurement processes.



Feza Gursey (right) with Eugene Wigner, photo by Y. S. Kim [5] (1988) (permission of Prof. Kim to release it to public domain).

Honors

- Franklin Medal, 1950
- Atoms for Peace Award, 1959
- Eugene P. Wigner Reactor Physicist Award ^[7] at the American Nuclear Society.
- Enrico Fermi Award ^[8].
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- (with Creutz, E. C. & R. R. Wilson) "Absorption of Thermal Neutrons in Uranium, ^[12]" Princeton University, United States Department of Energy (through predecessor agency the Atomic Energy Commission), (Sept. 26, 1941).
- "Radioactivity of the Cooling Water, ^[13]" Metallurgical Laboratory of the University of Chicago, United States Department of Energy (through predecessor agency the Atomic Energy Commission), (March 1, 1943).
- "Solutions of Boltzmann's Equation for Mono-energetic Neutrons in an Infinite Homogeneous Medium, ^[14]" Metallurgical Laboratory of the University of Chicago, United States Department of Energy (through predecessor agency the Atomic Energy Commission), (Nov. 30, 1943).
- (with Weinberg, A. M. & J. Stephenson) "Recalculation of the Critical Size and Multiplication Constant of a Homogeneous $\text{UO}_2 - \text{D}_2\text{O}$ Mixtures, ^[15]" Metallurgical Laboratory of the University of Chicago, (Feb. 11, 1944).
- (with F.L. Friedman) "On the Boundary Condition Between Two Multiplying Media, ^[16]" Metallurgical Laboratory of the University of Chicago, (April 19, 1944).
- (with J. E. Wilkins, Jr.) "Effect of the Temperature of the Moderator on the Velocity Distribution of Neutrons with Numerical Calculations for H as Moderator, ^[17]" Oak Ridge National Laboratory (ORNL), United States Department of Energy (through predecessor agency the Atomic Energy Commission), (Sept. 14, 1944).
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- [3] Wigner, E., 1927, *Zeitschrift f. Physik* 43: 624–52.
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- [6] http://www.vedanta-newyork.org/thoughtson_rk.htm
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- [15] http://www.osti.gov/cgi-bin/rd_accomplishments/display_biblio.cgi?id=ACC0145&numPages=8&fp=N
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- [17] http://www.osti.gov/cgi-bin/rd_accomplishments/display_biblio.cgi?id=ACC0147&numPages=13&fp=N
- [18] http://www.osti.gov/cgi-bin/rd_accomplishments/display_biblio.cgi?id=ACC0148&numPages=16&fp=N
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- [20] <http://www.dartmouth.edu/~matc/MathDrama/reading/Wigner.html>

External links

- Annotated bibliography for Eugene Wigner from the Alsos Digital Library for Nuclear Issues (<http://alsos.wlu.edu/qsearch.aspx?browse=people/Wigner,+Eugene>)
- Biography and Bibliographic Resources (<http://www.osti.gov/accomplishments/wigner.html>), from the Office of Scientific and Technical Information, United States Department of Energy
- Oral history interview with Eugene P. Wigner (<http://www.cbi.umn.edu/oh/display.phtml?id=77>) Charles Babbage Institute, University of Minnesota, Minneapolis – Wigner talks about his association with John von Neumann during their school years in Hungary, their graduate studies in Berlin, and their appointments to Princeton in 1930. Wigner discusses von Neumann's contributions to the theory of quantum mechanics, Wigner's own work in this area, and von Neumann's interest in the application of theory to the atomic bomb project.
- Eugene Wigner Biography (http://www.nobel-winners.com/Physics/eugene_paul_wigner.html)
- Nobel Prize Biography (http://nobelprize.org/nobel_prizes/physics/laureates/1963/wigner-bio.html)
- National Academy of Sciences biography (<http://www.nap.edu/readingroom/books/biomems/ewigner.html>)
- O'Connor, John J.; Robertson, Edmund F., "Eugene Wigner" (<http://www-history.mcs.st-andrews.ac.uk/Biographies/Wigner.html>), *MacTutor History of Mathematics archive*, University of St Andrews.
- his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles (<http://generatorp.bravehost.com/dmx/wigner-bio.html>)
- An interview with Wigner about his experience at Princeton (http://www.princeton.edu/~mudd/finding_aids/mathoral/pmc44.htm)
- Oral history interview transcript with Eugene Wigner 21 November 1963, American Institute of Physics, Niels Bohr Library & Archives (http://www.aip.org/history/ohilist/4963_1.html)
- Oral history interview transcript with Eugene Wigner 24 January 1981, American Institute of Physics, Niels Bohr Library & Archives (<http://www.aip.org/history/ohilist/4965.html>)

Alexandru Proca

Alexandru Proca	
Born	October 16, 1897Bucharest, Romania
Died	December 13, 1955 (aged 58)Paris, France
Citizenship	France
Nationality	Romania
Fields	Physicist (theoretical)
Alma mater	Paris-Sorbonne University in France
Doctoral advisor	Louis de Broglie
Known for	Proca's equations
Notable awards	Honorary Member of the Romanian Academy of Arts and Sciences, elected post mortem in 1990.

Alexandru Proca (October 16, 1897, Bucharest – December 13, 1955, Paris) was a Romanian physicist. He developed the meson theory of nuclear forces and the mathematical physics equations that bear his name (Proca's equations). He became a French citizen in 1931.

Education

High-school and college

In Romania, he was one of the eminent students of the school "Gheorghe Lazar" and the Polytechnic School in Bucharest. With a very strong interest in theoretical physics, he went to Paris where he graduated in Science from the Paris-Sorbonne University, receiving from the hand of Marie Curie his diploma of the Bachelor of Science degree. Then, he was employed as a researcher/physicist at the Radium Institute in Paris in 1925.

Ph.D. studies

He carried out Ph.D. studies in theoretical physics under the supervision of Nobel laureate Louis de Broglie. He defended successfully his Ph.D. thesis entitled "*On the relativistic theory of Dirac's electron*" in front of an examination committee chaired by the Nobel laureate Jean Perrin.

Scientific achievements

He also studied and worked with Nobel laureates Niels Bohr and Marie Curie,^[1] . Alexandru Proca became to be known as one of the most influential Romanian theoretical physicists of the last century.^[2] having developed the meson theory of nuclear forces ahead of the first reports of Nobel laureate Hideki Yukawa. Proca's equations for the vectorial mesonic field were employed by Yukawa who subsequently received the Nobel Prize for an explanation of the nuclear forces by using this field.

Publications at the Library of Congress

- Library of Congress^[3]

Notes

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External links

- Brief History of IFIN-HH: PRECURSORS Hon. Acad. Alexandru Proca (1897 - 1955) (<http://www.nipne.ro/about/history/>) and Acad. Prof. Dr. Horia Hulubei (1896-1972).

Hideki Yukawa

Hideki Yukawa 湯川 秀樹	
	
Born	23 January 1907Tokyo, Japan
Died	8 September 1981 (aged 74)Kyoto, Japan
Nationality	Japan
Fields	Theoretical Physics
Institutions	Osaka Imperial University Kyoto Imperial University Imperial University of Tokyo Institute for Advanced Study Columbia University
Alma mater	Kyoto Imperial University
Influences	Enrico Fermi
Notable awards	Nobel Prize in Physics (1949)

Hideki Yukawa FRSE (湯川 秀樹 *Yukawa Hideki*, 23 January 1907 – 8 September 1981) né **Ogawa** (小川), was a Japanese theoretical physicist and the first Japanese Nobel laureate.

Biography

Yukawa was born in Tokyo, Japan. In 1929, after receiving his degree from Kyoto Imperial University, he stayed on as a lecturer for four years. After graduation, he was interested in theoretical physics, particularly in the theory of elementary particles. In 1932, he married Sumi (スミ); they had two sons, Harumi and Takaaki. In 1933 he became an assistant professor at Osaka University, at age 26.

In 1935 he published his theory of mesons, which explained the interaction between protons and neutrons, and was a major influence on research into elementary particles. In 1940 he became a professor in Kyoto University. In 1940 he won the Imperial Prize of the Japan Academy, in 1943 the Decoration of Cultural Merit from the Japanese government. In 1949 he became a professor at Columbia University, the same year he received the Nobel Prize in Physics, after the discovery by Cecil Frank Powell, Giuseppe Occhialini and César Lattes of Yukawa's predicted pion in 1947. Yukawa also worked on the theory of K-capture, in which a low energy electron is absorbed by the nucleus, after its initial prediction by G. C. Wick.^[1]

Yukawa became the first chairman of Yukawa Institute for Theoretical Physics in 1953. He received a Doctorate, *honoris causa*, from the University of Paris and honorary memberships in the Royal Society of Edinburgh, the Indian Academy of Sciences, the International Academy of Philosophy and Sciences, and the Pontificia Academia Scientiarum.

He was an editor of *Progress of Theoretical Physics*,^[2] and published the papers *Introduction to Quantum Mechanics* (1946) and *Introduction to the Theory of Elementary Particles* (1948).

In 1955, he joined ten other leading scientists and intellectuals in signing the Russell-Einstein Manifesto, calling for nuclear disarmament.

Solo violinist Diana Yukawa (ダイアナ湯川) is a relative of Hideki Yukawa.

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External links

- Hideki Yukawa - Biography (<http://www.nobel.se/physics/laureates/1949/yukawa-bio.html>)
- About Hideki Yukawa (http://www.nobel-winners.com/Physics/hideki_yukawa.html)
- his prediction of the existence of mesons on the basis of theoretical work on nuclear forces. (<http://holiker.narod.ru/four/yukawa-press.html>)

Neville Mott

Nevill Francis Mott	
	
Born	30 September 1905Leeds, England
Died	8 August 1996 (aged 90)Milton Keynes, Buckinghamshire, England
Nationality	United Kingdom
Fields	Physics
Institutions	University of Manchester Gonville and Caius College, Cambridge University of Bristol
Alma mater	St John's College, Cambridge
Notable awards	Nobel Prize in Physics (1977)

Sir Nevill Francis Mott, CH, FRS (30 September 1905 – 8 August 1996) was an English physicist. He won the Nobel Prize for Physics in 1977 for his work on the electronic structure of magnetic and disordered systems, especially amorphous semiconductors. The award was shared with Philip W. Anderson and J. H. Van Vleck, who had pursued independent research.

Biography

Early years

Mott was born in Leeds to Lilian Mary Reynolds and Charles Francis Mott. and grew up first in the village of Giggleswick, in the West Riding of Yorkshire, where his father was Senior Science Master at the local school. It was a generally secular childhood. The family moved (due to his father's jobs) first to Staffordshire, then to Chester and finally Liverpool, where his father had been appointed Director of Education. Mott was at first educated at home by his mother, who was a Cambridge Mathematics Tripos graduate. His parents had actually met in the Cavendish Laboratory, when both engaged in Physics research. At ten years of age he began formal education at Clifton College in Bristol, then at St. John's College, Cambridge, where he read the Mathematics Tripos.

Career

Mott was appointed to a lectureship at Manchester University in 1929. He returned to Cambridge in 1930 as a Fellow and lecturer of Gonville and Caius College and in 1933 moved to Bristol University as Melville Wills Professor in Theoretical Physics.

In 1948 he became Henry Overton Wills Professor of Physics and Director of the Henry Herbert Wills Physical Laboratory at Bristol. In 1954 he was appointed Cavendish Professor of Physics at Cambridge, a post he held until 1971. Additionally he served as Master of Gonville and Caius College, 1959-1966.

Mott's accomplishments include explaining theoretically the effect of light on a photographic emulsion (see latent image) and outlining the transition of substances from metallic to nonmetallic states (Mott transition). The term Mott insulator is also named for him.

Mott was elected a Fellow of the Royal Society in 1936. Mott served as president of the Physical Society in 1957. In the early 1960s he was chairman of the British Pugwash group. He was knighted in 1962.^[1] He continued to work until he was about ninety. He was made a Companion of Honour in 1995.

Personal life

Mott was married to Ruth Eleanor Horder, and had two daughters, Elizabeth and Alice. He died in Milton Keynes in Buckinghamshire.

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External links

- Delightful BBC video of 1985 interview with Nevill Mott^[3] (accessed Oct 8, 2010)
- Nobel lecture^[4] (PDF)
- Sir Nevill Francis Mott^[5]
- Mott's memories^[6] University of Bristol (accessed Jan 2006)
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- [5] http://www.nobel-winners.com/Physics/nevill_francis_mott.html
- [6] <http://www.phy.bris.ac.uk/history/11.%20Mott's%20Memories.pdf>
- [7] <http://www.bath.ac.uk/ncuacs/rsip-nfm.htm>

Paul W. Anderson

Paul W. S. Anderson	
	
Anderson at WonderCon 2010	
Born	4 March 1965Newcastle-upon-Tyne, England
Occupation	Film director, producer and screenwriter
Spouse	Milla Jovovich (2009–present)
Children	Ever Gabo Anderson (3 November 2007)

Paul William Scott Anderson (born 4 March 1965), also known as **Paul W.S. Anderson** or **Paul Anderson**, is an English film director who regularly works in science fiction movies and video game adaptations.

Life and career

Anderson was born in Newcastle upon Tyne, England. Educated at Newcastle's Royal Grammar School, Anderson went on to graduate from the University of Warwick as the youngest student to achieve a BA in Film & Literature. He made his debut as the writer-director of *Shopping*, which starred Sean Pertwee, Jude Law and Sadie Frost as thieves who smashed cars into storefronts. When released in the United Kingdom it was banned in some cinemas, and only gained a release in the United States as an edited, direct to video release.

After this, he directed the successful 1995 video game adaptation *Mortal Kombat*. While prior video game movies, like *Street Fighter* and *Super Mario Bros.*, had been all-out disasters, *Mortal Kombat* was well received by fans, and some critics. He declined to direct the sequel, *Mortal Kombat: Annihilation*, which was not well received by critics or fans. Anderson was asked to direct a third movie, *Mortal Kombat: Devastation*, but declined again.

The success of *Mortal Kombat* gave Anderson free rein to choose his next project, *Soldier*, written by *Blade Runner* screenwriter David Webb Peoples. Intended as a "sidequel" to *Blade Runner*, the movie was set in the same universe (but not the same planet), and contained numerous references to the earlier film. Kurt Russell was attached to star, but was unavailable at the time, which delayed the production. In the meantime, Anderson made the science fiction horror film *Event Horizon*. It was poorly received at the box office, and Anderson blamed the failure on studio-enforced cuts. While not a box-office success, the film gained a small cult following. *Soldier* was eventually completed and released in 1998, but was a disaster both commercially and critically.

After the poor performance of both *Event Horizon* and *Soldier*, Anderson was forced to think smaller. His planned remake of the cult film *Death Race 2000* was put on hold, and he set about writing and directed a TV movie, *The Sight*, in 2000. It was a minor success, and Anderson returned to cinema screens in 2002 when he wrote and directed an adaptation of the survival horror series *Resident Evil*. At that point he began to credit himself as "Paul W. S. Anderson", to avoid confusion with the American director Paul Thomas Anderson.

Produced on a moderate budget in comparison to his earlier movies, *Resident Evil* was a commercial success in cinemas and on DVD, prompting Anderson to write (but not direct) the sequels, *Resident Evil: Apocalypse* and *Resident Evil: Extinction* both were commercially successful but failed critically.

Anderson's next project was the much-anticipated *Alien vs. Predator*, a concept hinted at in *Predator 2* and later popularized by a series of Dark Horse Comics. A movie version had been stuck in development hell for several years despite the franchise crossing into every other form of media, from books to comics to video games. The film was finally released in August 2004, and proved to be Anderson's most successful film to date, grossing \$172,544,654^[1] internationally on a budget of \$60 million, despite his success it received mostly negative reviews. After completing *Alien vs. Predator* Anderson rebooted his *Death Race 2000* remake and finally got it released as *Death Race* in 2008. Anderson will next direct an adaptation of *The Three Musketeers*, who will be played by Logan Lerman, Matthew Macfadyen, Ray Stevenson and Luke Evans.^[2]

Personal life

In April 2007, *People Magazine* announced that he and actress Milla Jovovich were expecting a baby girl in November 2007. The two met when Anderson directed her in the first *Resident Evil*. They were engaged in March 2003, and were married on August 22, 2009.^[3] ^[4] Jovovich gave birth to their first child, a daughter, Ever Gabo Anderson, on 3 November 2007 at Cedars-Sinai Medical Center in Los Angeles, California, one day before her due date of November 4.^[5]

Criticism

Screenwriter Peter Briggs, who had penned the very first *Alien vs. Predator* screenplay, disputed some of Anderson's other comments in an online interview, saying Anderson's claim that Briggs' original screenplay was "locked down" was incorrect, and that many elements of Anderson's screenplay were suspiciously similar.^[6]

Filmography

Year	Film	Credited as		
		Director	Writer	Producer
1994	<i>Shopping</i>	Yes	Yes	
1995	<i>Mortal Kombat</i>	Yes		
1997	<i>Event Horizon</i>	Yes		
1998	<i>Soldier</i>	Yes		
2000	<i>The Sight</i>	Yes	Yes	Yes
2002	<i>Resident Evil</i>	Yes	Yes	Yes
2004	<i>Alien vs. Predator</i>	Yes	Yes	
	<i>Resident Evil: Apocalypse</i>		Yes	Yes
2005	<i>The Dark</i>			Yes
2007	<i>DOA: Dead or Alive</i>			Yes
	<i>Resident Evil: Extinction</i>		Yes	Yes
2008	<i>Death Race</i>	Yes	Yes	Yes
2009	<i>Pandorum</i>			Yes
2010	<i>Resident Evil: Afterlife</i>	Yes	Yes	Yes
2011	<i>The Three Musketeers</i>	Yes		Yes

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External links

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 - Paul W. S. Anderson (http://www.rottentomatoes.com/p/paul_anderson/) at Rotten Tomatoes
 - Paul Anderson (<http://www.the-numbers.com/people/directors/PANDE.php>) at The Numbers
 - Paul W. S. Anderson (<http://www.boxofficemojo.com/people/chart/?view=Director&id=paulwsanderson.htm>) at Box Office Mojo
 - Paul W. S. Anderson interview at JoBlo.com (<http://www.joblo.com/arrow/index.php?id=615>)
 - Paul W. S. Anderson Unofficial Fansite (<http://www.paulwsanderson.co.uk>)
 - Paul WS Anderson Interview on TheCinemaSource for Resident Evil: Afterlife (<http://www.thecinemasource.com/blog/interviews/paul-ws-anderson-interview-for-resident-evil-afterlife/>)
-

George Karreman

George Karreman	
Born	November 4, 1921Rotterdam, Netherlands
Died	January 23, 1997 (aged 75)Philadelphia, United States
Residence	Netherlands, United States
Nationality	US and Netherlands
Fields	Physics, mathematical biology, Quantum biology
Institutions	University of Pennsylvania
Alma mater	Leiden University, University of Chicago
Doctoral advisor	Nicolas Rashevsky
Other academic advisors	Hendrik Anthony Kramers
Known for	Quantum biology

George Karreman (b. 4 November 1921 in Rotterdam, Netherlands – 23 January 1997 in Philadelphia, USA) was a Dutch-born US physicist, mathematical biophysicist and mathematical/theoretical biologist. He was the first President of the Society for Mathematical Biology (SMB).^[1]

Biography

Karreman's father was Chief Engineer for the Dutch Merchant Marine. George Karreman studied physics and mathematics at Leiden University. In August 1948 Karreman emigrated to Chicago, USA, where he contacted Nicolas Rashevsky at the University of Chicago. He became one of Rashevsky's best PhD students in Mathematical Biophysics.^[2] In 1950 Karreman underwent heart surgery at the University of Chicago and from then on had to carry an implanted pacemaker. He married Anneke Halbertsma in 1953, and they moved to Cape Cod, Massachusetts, where their daughter, Grace, was born in 1954. In slower succession, his first son, Frank Karreman was born in 1958, and then in 1962 his second son, Hubert-Jan. Later, his three children received advanced degrees from the University of Pennsylvania in several fields.

Karreman was an exceptionally devoted educator, who was always supportive of his research associates, family, and friends; he was a generous man, obviously having not forgotten Rashevsky's help in his early life at the University of Chicago.^[3] He had a wide range of interests in his readings, a keen interest in the fine arts—such as paintings, was an advanced chess player, and a most devoted husband and father.^[4] Between 1987 and 1997, he frequently travelled to the Pacific Northwest where his son and daughter had their homes and children.

Academic career

Karreman earned his B.S. in Physics and Mathematics in 1939. He completed in 1941 his M.S. in Theoretical Physics under the supervision of Hendrik Anthony Kramers at Leiden University. For the remainder of World War II he survived by tutoring students in physics and mathematics. He was awarded a University of Chicago Fellowship that supported him to complete a Ph.D. in Mathematical Biology in 1951 under the supervision of the founder of Mathematical Biophysics and Mathematical Biology, Nicolas Rashevsky.^[5] Karreman was then selected as a consultant to Albert Szent-Györgyi at the Institute for Muscle Research, Marine Biological Laboratory, Woods Hole.^[6] To follow his interest in mathematics applied to biology, physiology, and medicine he went to the University of Pennsylvania in Philadelphia in 1957, to take up the position of Senior Medical Research Scientist at the Eastern Research Center.^[7] He was appointed associate professor of physiology at the School of Medicine in the

University of Pennsylvania, where he also worked at the Bockus Research Institute at the Graduate Hospital. He was promoted to Full Professor of Physiology at the same university in 1970, where he held this position until 1983, when he became the first Professor Emeritus of Mathematical Biology.^[8] His main research interests were many, but all were in related fields that included: mathematical biology and mathematical biophysics, membrane biophysics, photosynthetic mechanisms, quantum biochemistry and quantum biophysics,^[9] biological energy transfer, quantum biology,^[10] physiological irritability, mathematical and systems analysis of cardiovascular and other biosystems, cooperativity, threshold phenomena in biomembranes, adsorption mechanisms at membrane surfaces and ion binding to biomembranes.

After Rashevsky's passing away in 1972, Karreman was a co-founder of the Society for Mathematical Biology (SMB) in 1974—together with H. Landahl and A. Bartholomay, and in 1975 he became its first president. Karreman was also a member of several prestigious scientific societies, including the American Physiological Society, the New York Academy of Sciences, the Franklin Institute, the Society for Supramolecular Biology, Sigma Xi, the Physiological Society of Philadelphia, and the Society for Vascular System Dynamics.

Honours

- 1974 – First President of the The Society for Mathematical Biology.^[1]

See also

- Society for Mathematical Biology
- Mathematical and theoretical biology
- Quantum biology
- Nicolas Rashevsky
- Herbert Daniel Landahl
- Hendrik Anthony Kramers

Selected publications

- George Karreman, R.H. Steele, and Albert Szent-Gyorgyi. "On resonance transfer of excitation energy between aromatic aminoacids in proteins.", *Proc. Natl. Acad. Sci.* (1958) **44**: 140-143.
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- [11] <http://www.springerlink.com/content/847x186ju2307t7m/>
- [12] <http://www.springerlink.com/content/066gkj167w842141/>

External links

- The Society for Mathematical Biology (<http://www.smb.org>)
- Biographies of physicists on PlanetPhysics.org (<http://planetphysics.org/encyclopedia/>)

Alberte Pullman

Alberte Pullman (née Bucher) was born in France in 1920. She is a theoretical and quantum chemist. She studied at the Sorbonne starting in 1938. During her studies she worked on calculations at Centre National de la Recherche Scientifique (CNRS). From 1943 she worked with Raymond Daudel. She completed her doctorate in 1946. On his return from war service in 1946, she married Bernard Pullman. She and her husband worked together until his death in 1996. Together they wrote several books including *Quantum Biochemistry*, Interscience Publishers, 1963. Their work in the 1950s and 1960s was the beginning of the new field of Quantum Biochemistry. They pioneered the application of quantum chemistry to predicting the carcinogenic properties of aromatic hydrocarbons.

She is a member of the International Academy of Quantum Molecular Science and a member and former President of The International Society of Quantum Biology and Pharmacology.^[1]

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External links

- An interview with Mme Prof. Dr. Alberte Pullman (<http://www.quantum-chemistry-history.com/Pull1.htm>)
-

Richard Feynman

Richard Feynman	
 Feynman around 1943–1945	
Born	May 11, 1918Far Rockaway, Queens, New York, USA
Died	February 15, 1988 (aged 69)Los Angeles, California, USA
Residence	United States
Nationality	American
Fields	Physics
Institutions	Manhattan Project Cornell University California Institute of Technology
Alma mater	Massachusetts Institute of Technology (B.S.), Princeton University (Ph.D.)
Doctoral advisor	John Archibald Wheeler
Other academic advisors	Manuel Sandoval Vallarta
Doctoral students	F. L. Vernon, Jr. ^[1] Willard H. Wells ^[1] Al Hibbs ^[1] George Zweig ^[1] Giovanni Rossi Lomanitz ^[1] Thomas Curtright ^[1]
Other notable students	Douglas D. Osheroff Robert Barro
Known for	Feynman diagrams Feynman point Feynman–Kac formula Wheeler–Feynman absorber theory Feynman sprinkler Feynman Long Division Puzzles Hellmann–Feynman theorem Feynman slash notation Feynman parametrization Sticky bead argument One-electron universe Quantum cellular automata
Influences	Paul Dirac

Notable awards	Albert Einstein Award (1954) E. O. Lawrence Award (1962) Nobel Prize in Physics (1965) Oersted Medal (1972) National Medal of Science (1979)
Signature 	
Notes He was the father of Carl Feynman and Michelle Feynman. He was the brother of Joan Feynman.	

Richard Phillips Feynman (pronounced /ˈfaɪnmən/, May 11, 1918 – February 15, 1988) was an American physicist known for his work in the path integral formulation of quantum mechanics, the theory of quantum electrodynamics and the physics of the superfluidity of supercooled liquid helium, as well as in particle physics (he proposed the parton model). For his contributions to the development of quantum electrodynamics, Feynman, jointly with Julian Schwinger and Sin-Itiro Tomonaga, received the Nobel Prize in Physics in 1965. He developed a widely used pictorial representation scheme for the mathematical expressions governing the behavior of subatomic particles, which later became known as Feynman diagrams. During his lifetime, Feynman became one of the best-known scientists in the world.

He assisted in the development of the atomic bomb and was a member of the panel that investigated the Space Shuttle Challenger disaster. In addition to his work in theoretical physics, Feynman has been credited with pioneering the field of quantum computing,^[2] and introducing the concept of nanotechnology.^[3] He held the Richard Chace Tolman professorship in theoretical physics at the California Institute of Technology.

Feynman was a keen popularizer of physics through both books and lectures, notably a 1959 talk on top-down nanotechnology called *There's Plenty of Room at the Bottom* and *The Feynman Lectures on Physics*. Feynman also became known through his semi-autobiographical books (*Surely You're Joking, Mr. Feynman!* and *What Do You Care What Other People Think?*) and books written about him, such as *Tuva or Bust!*

Feynman also had a deep interest in biology, and was a friend of the geneticist and microbiologist Esther Lederberg, who developed replica plating and discovered bacteriophage lambda.^[4] They had several mutual physicist friends who, after beginning their careers in nuclear research, moved for moral reasons into genetics, among them Leó Szilárd, Guido Pontecorvo, and Aaron Novick.

Biography

Early life

Richard Phillips Feynman was born on May 11, 1918,^[5] in Far Rockaway, Queens, New York.^[6] His family originated from Russia and Poland; both of his parents were Jewish,^[7] but they were not devout. In fact, by his early youth, Feynman described himself as an "avowed atheist".^[8] Feynman (in common with the famous physicists Edward Teller and Albert Einstein) was a late talker; by his third birthday he had yet to utter a single word. The young Feynman was heavily influenced by his father, Melville, who encouraged him to ask questions to challenge orthodox thinking. From his mother, Lucille, he gained the sense of humor that he had throughout his life. As a child, he delighted in repairing radios and had a talent for engineering. His younger sister Joan also became a professional physicist.^[9] ^[10]

Education

In high school, his IQ was determined to be 125 (on an unspecified test, making this statement dubious): high, but "merely respectable" according to biographer Gleick.^[11] Feynman later scoffed at psychometric testing. By 15, he had learned differential and integral calculus. Before entering college, he was experimenting with and re-creating mathematical topics, such as the half-derivative, utilizing his own notation. In high school, he was developing the mathematical intuition behind his Taylor series of mathematical operators.^[12]

His habit of direct characterization sometimes rattled more conventional thinkers; for example, one of his questions when learning feline anatomy was "Do you have a map of the cat?" (referring to an anatomical chart).

Feynman attended Far Rockaway High School, a school that also produced fellow laureates Burton Richter and Baruch Samuel Blumberg.^[13] A member of the Arista Honor Society, in his last year in high school, Feynman won the New York University Math Championship; the large difference between his score and those of his closest competitors shocked the judges.^[14]

He applied to Columbia University, but was not accepted, because of the "Jewish quota" (a discriminatory practice of limiting the number of places available to students of Jewish background).^[15] ^[16] Instead he attended the Massachusetts Institute of Technology, where he received a bachelor's degree in 1939, and in the same year was named a Putnam Fellow. While there, Feynman took every physics course offered, including a graduate course on theoretical physics while only in his second year.

He obtained a perfect score on the graduate school entrance exams to Princeton University in mathematics and physics — an unprecedented feat — but did rather poorly on the history and English portions.^[14] Attendees at Feynman's first seminar included Albert Einstein, Wolfgang Pauli, and John von Neumann. He received a Ph.D. from Princeton in 1942; his thesis advisor was John Archibald Wheeler. Feynman's thesis applied the principle of stationary action to problems of quantum mechanics, laying the groundwork for the "path integral" approach and Feynman diagrams, and was entitled "The Principle of Least Action in Quantum Mechanics".

This was Richard Feynman nearing the crest of his powers. At twenty-three ... there was no physicist on earth who could match his exuberant command over the native materials of theoretical science. It was not just a facility at mathematics (though it had become clear ... that the mathematical machinery emerging from the Wheeler-Feynman collaboration was beyond Wheeler's own ability). Feynman seemed to possess a frightening ease with the substance behind the equations, like Albert Einstein at the same age, like the Soviet physicist Lev Landau—but few others.

— James Gleick, *Genius: The Life and Science of Richard Feynman*

The Manhattan Project



Feynman (center) with Robert Oppenheimer (right) relaxing at a Los Alamos social function during the Manhattan Project.

At Princeton, the physicist Robert R. Wilson encouraged Feynman to participate in the Manhattan Project—the wartime U.S. Army project at Los Alamos developing the atomic bomb. Feynman said he was persuaded to join this effort to build it before Nazi Germany developed their own bomb.

He was assigned to Hans Bethe's theoretical division, and impressed Bethe enough to be made a group leader. He and Bethe developed the Bethe-Feynman formula (which might still be classified^[17]) for calculating the yield of a fission bomb, which built upon previous work by Robert Serber.

He immersed himself in work on the project, and was present at the Trinity bomb test. Feynman claimed to be the only person to see the explosion without the very dark glasses or welder's lenses provided, reasoning that it was safe to look through a truck windshield, as it would screen out the harmful ultraviolet radiation.

As a junior physicist, he was not central to the project. The greater part of his work was administering the computation group of human computers in the Theoretical division (one of his students there, John G. Kemeny, later went on to co-write the computer language BASIC). Later, with Nicholas Metropolis, he assisted in establishing the system for using IBM punched cards for computation. Feynman succeeded in solving one of the equations for the project that were posted on the blackboards. However, they did not "do the physics right" and Feynman's solution was not used.

Feynman's other work at Los Alamos included calculating neutron equations for the Los Alamos "Water Boiler", a small nuclear reactor, to measure how close an assembly of fissile material was to criticality. On completing this work he was transferred to the Oak Ridge facility, where he aided engineers in devising safety procedures for material storage so that criticality accidents (for example, due to sub-critical amounts of fissile material inadvertently stored in proximity on opposite sides of a wall) could be avoided. He also did theoretical work and calculations on the proposed uranium-hydride bomb, which later proved not to be feasible.

Feynman was sought out by physicist Niels Bohr for one-on-one discussions. He later discovered the reason: most physicists were too in awe of Bohr to argue with him. Feynman had no such inhibitions, vigorously pointing out anything he considered to be flawed in Bohr's thinking. Feynman said he felt as much respect for Bohr as anyone else, but once anyone got him talking about physics, he would become so focused he forgot about social niceties.

Due to the top secret nature of the work, Los Alamos was isolated. In Feynman's own words, "There wasn't anything to *do* there". Bored, he indulged his curiosity by learning to pick the combination locks on cabinets and desks used to secure papers. Feynman played many jokes on colleagues. In one case he found the combination to a locked filing cabinet by trying the numbers a physicist would use (it proved to be 27–18–28 after the base of natural logarithms, $e = 2.71828\dots$), and found that the three filing cabinets where a colleague kept a set of atomic bomb research notes all had the same combination. He left a series of notes as a prank, which initially spooked his colleague, Frederic de Hoffman, into thinking a spy or saboteur had gained access to atomic bomb secrets. On several occasions, Feynman drove to Albuquerque to see his ailing wife in a car borrowed from Klaus Fuchs, who was later discovered to be a real spy for the Soviets, transporting nuclear secrets in his car to Santa Fe.

On occasion, Feynman would find an isolated section of the mesa to drum in the style of American natives; "and maybe I would dance and chant, a little". These antics did not go unnoticed, and rumors spread about a mysterious Indian drummer called "Injun Joe". He also became a friend of laboratory head J. Robert Oppenheimer, who unsuccessfully tried to court him away from his other commitments after the war to work at the University of California, Berkeley.

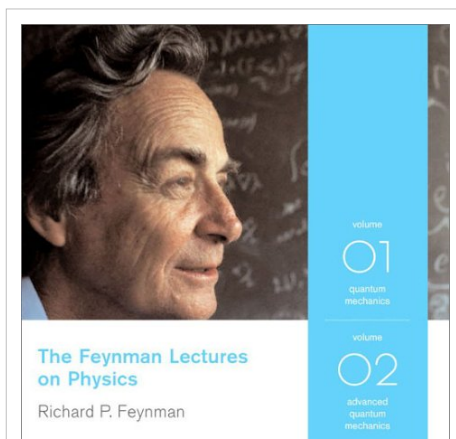
Feynman alludes to his thoughts on the justification for getting involved in the Manhattan project in *The Pleasure of Finding Things Out*. As mentioned earlier, he felt the possibility of Nazi Germany developing the bomb before the Allies was a compelling reason to help with its development for the US. However, he goes on to say that it was an error on his part not to reconsider the situation when Germany was defeated. In the same publication, Feynman also talks about his worries in the atomic bomb age, feeling for some considerable time that there was a high risk that the bomb would be used again soon so that it was pointless to build for the future. Later he describes this period as a "depression."

Early academic career

Following the completion of his Ph.D. in 1942, Feynman held an appointment at the University of Wisconsin-Madison (UW) as an assistant professor of physics. The appointment was spent on leave for his involvement in the Manhattan project. In 1945, he received a letter from Dean Mark Ingraham of the College of Letters and Science requesting his return to UW to teach in the coming academic year. His appointment was not extended when he did not commit to return. In a talk given several years later at UW, Feynman quipped, "It's great to be back at the only university that ever had the good sense to fire me".^[18]

After the war, Feynman declined an offer from the Institute for Advanced Study in Princeton, New Jersey, despite the presence there of such distinguished faculty members as Albert Einstein, Kurt Gödel, and John von Neumann. Feynman followed Hans Bethe, instead, to Cornell University, where Feynman taught theoretical physics from 1945 to 1950.^[12] During a temporary depression following the destruction of Hiroshima by the bomb produced by the Manhattan Project, he focused on complex physics problems, not for utility, but for self-satisfaction. One of these was analyzing the physics of a twirling, nutating dish as it is moving through the air. His work during this period, which used equations of rotation to express various spinning speeds, soon proved important to his Nobel Prize-winning work. Yet because he felt burned out, and had turned his attention to less immediately practical but more entertaining problems, he felt surprised by the offers of professorships from renowned universities.^[12]

Despite yet another offer from the Institute for Advanced Study, which would have included teaching duties (which was not included in the Institute's initial offer, a factor in his rejection of it), Feynman opted for the California Institute of Technology (Caltech) — as he says in his book *Surely You're Joking Mr. Feynman!* — because a desire to live in a mild climate had firmly fixed itself in his mind while installing tire chains on his car in the middle of a snowstorm in Ithaca.



Feynman the "Great Explainer": *The Feynman Lectures on Physics* found an appreciative audience beyond the undergraduate community

Feynman has been called the "Great Explainer".^[19] He gained a reputation for taking great care when giving explanations to his students and for making it a moral duty to make the topic accessible. His guiding principle was that if a topic could not be explained in a freshman lecture, it was not yet fully understood. Feynman gained great pleasure^[20] from coming up with such a "freshman-level" explanation, for example, of the connection between spin and statistics. What he said was that groups of particles with spin $1/2$ "repel", whereas groups with integer spin "clump". This was a brilliantly simplified way of demonstrating how Fermi-Dirac statistics and Bose-Einstein statistics evolved as a consequence of studying how fermions and bosons behave under a rotation of 360° . This was also a question he pondered in his more advanced lectures and to which he demonstrated the solution in the 1986 Dirac memorial lecture.^[21] In the same lecture, he further explained that antiparticles must exist, for if particles only had positive energies, they would not be restricted to a so-called "light cone".

He opposed rote learning or unthinking memorization and other teaching methods that emphasized form over function. He put these opinions into action whenever he could, from a conference on education in Brazil to a State Commission on school textbook selection. *Clear thinking* and *clear presentation* were fundamental prerequisites for his attention. It could be perilous even to approach him when unprepared, and he did not forget the fools or pretenders.^[22]

During one sabbatical year, he returned to Newton's *Principia Mathematica* to study it anew; what he learned from Newton, he passed along to his students, such as Newton's attempted explanation of diffraction.

Caltech years

Feynman did significant work while at Caltech, including research in:

- Quantum electrodynamics. The theory for which Feynman won his Nobel Prize is known for its accurate predictions.^[23] ^[24] This theory was developed in the earlier years during Feynman's work at Cornell. He helped develop a functional integral formulation of quantum mechanics, in which every possible path from one state to the next is considered, the final path being a *sum* over the possibilities (also referred to as sum-over-paths or sum over histories).^[25]
- Physics of the superfluidity of supercooled liquid helium, where helium seems to display a complete lack of viscosity when flowing. Applying the Schrödinger equation to the question showed that the superfluid was displaying quantum mechanical behavior observable on a macroscopic scale. This helped with the problem of superconductivity; however, the solution eluded Feynman.^[26] It was solved with the BCS theory of superconductivity, proposed by John Bardeen, Leon Neil Cooper, and John Robert Schrieffer.
- A model of weak decay, which showed that the current coupling in the process is a combination of vector and axial (an example of weak decay is the decay of a neutron into an electron, a proton, and an anti-neutrino). Although E. C. George Sudarshan and Robert Marshak developed the theory nearly simultaneously, Feynman's collaboration with Murray Gell-Mann was seen as seminal because the weak interaction was neatly described by the vector and axial currents. It thus combined the 1933 beta decay theory of Enrico Fermi with an explanation of parity violation.

He also developed Feynman diagrams, a bookkeeping device which helps in conceptualizing and calculating interactions between particles in spacetime, notably the interactions between electrons and their antimatter counterparts, positrons. This device allowed him, and later others, to approach time reversibility and other fundamental processes. Feynman's mental picture for these diagrams started with the *hard sphere* approximation, and the interactions could be thought of as *collisions* at first. It was not until decades later that physicists thought of analyzing the nodes of the Feynman diagrams more closely. Feynman famously painted Feynman diagrams on the exterior of his van.^[27]

From his diagrams of a small number of particles interacting in spacetime, Feynman could then model *all of physics* in terms of those particles' spins and the range of coupling of the fundamental forces.^[28] Feynman attempted an explanation of the strong interactions governing nucleons scattering called the parton model. The parton model emerged as a complement to the quark model developed by his Caltech colleague Murray Gell-Mann. The relationship between the two models was murky; Gell-Mann referred to Feynman's partons derisively as "put-ons". In the mid 1960s, physicists believed that quarks were just a bookkeeping device for symmetry numbers, not real particles, as the statistics of the Omega-minus particle, if it were interpreted as three identical strange quarks bound together, seemed impossible if quarks were real. The Stanford linear accelerator deep inelastic scattering experiments of the late 1960s showed, analogously to Ernest Rutherford's experiment of scattering alpha particles on gold nuclei in 1911, that nucleons (protons and neutrons) contained point-like particles which scattered electrons. It was natural to identify these with quarks, but Feynman's parton model attempted to interpret the experimental data in a way which did not introduce additional hypotheses. For example, the data showed that some 45% of the energy momentum was carried by electrically neutral particles in the nucleon. These electrically neutral particles are now seen to be the gluons which carry the forces between the quarks and carry also the three-valued color quantum number which solves the Omega-minus problem. Feynman did not dispute the quark model; for example, when the fifth quark was discovered in 1977, Feynman immediately pointed out to his students that the discovery implied the

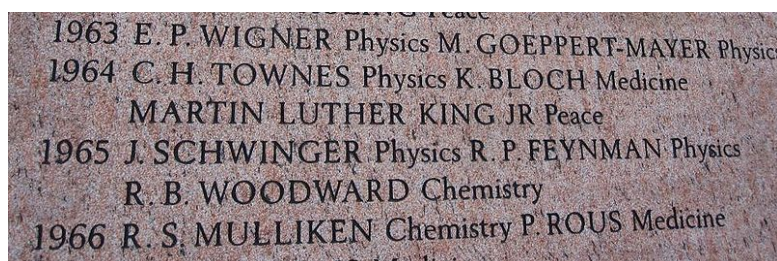


The Feynman section at the Caltech bookstore

existence of a sixth quark, which was duly discovered in the decade after his death.

After the success of quantum electrodynamics, Feynman turned to quantum gravity. By analogy with the photon, which has spin 1, he investigated the consequences of a free massless spin 2 field, and was able to derive the Einstein field equation of general relativity, but little more.^[29] However, the computational device that Feynman discovered then for gravity, "ghosts", which are "particles" in the interior of his diagrams which have the "wrong" connection between spin and statistics, have proved invaluable in explaining the quantum particle behavior of the Yang-Mills theories, for example QCD and the electro-weak theory.

In 1965, Feynman was appointed a foreign member of the Royal Society.^[30] At this time in the early 1960s, Feynman exhausted himself by working on multiple major projects at the same time, including a request, while at Caltech, to "spruce up" the teaching of undergraduates. After three years devoted to the task, he produced a series of lectures that eventually became the *Feynman Lectures on*



Mention of Feynman's prize on the monument at the American Museum of Natural History in New York City. Because the monument is dedicated to American Laureates, Tomonaga is not mentioned.

Physics, one reason that Feynman is still regarded as one of the greatest teachers of physics. He wanted a picture of a drumhead sprinkled with powder to show the modes of vibration at the beginning of the book. Outraged by many rock and roll and drug connections that could be made from the image, the publishers changed the cover to plain red, though they included a picture of him playing drums in the foreword. Feynman later won the Oersted Medal for teaching, of which he seemed especially proud.^[31]

His students competed keenly for his attention; he was once awakened when a student solved a problem and dropped it in his mailbox; glimpsing the student sneaking across his lawn, he could not go back to sleep, and he read the student's solution. The next morning his breakfast was interrupted by another triumphant student, but Feynman informed him that he was too late.

Partly as a way to bring publicity to progress in physics, Feynman offered \$1000 prizes for two of his challenges in nanotechnology, claimed by William McLellan and Tom Newman, respectively.^[32] He was also one of the first scientists to conceive the possibility of quantum computers.

Many of his lectures and other miscellaneous talks were turned into books, including *The Character of Physical Law* and *QED: The Strange Theory of Light and Matter*. He gave lectures which his students annotated into books, such as *Statistical Mechanics* and *Lectures on Gravity*. *The Feynman Lectures on Physics*^[33] occupied two physicists, Robert B. Leighton and Matthew Sands as part-time co-authors for several years. Even though they were not adopted by most universities as textbooks, the books continue to be bestsellers because they provide a deep understanding of physics. As of 2005, *The Feynman Lectures on Physics* has sold over 1.5 million copies in English, an estimated 1 million copies in Russian, and an estimated half million copies in other languages.

In 1974, Feynman delivered the Caltech commencement address on the topic of cargo cult science, which has the semblance of science but is only pseudoscience due to a lack of "a kind of scientific integrity, a principle of scientific thought that corresponds to a kind of utter honesty" on the part of the scientist. He instructed the graduating class that "The first principle is that you must not fool yourself—and you are the easiest person to fool. So you have to be very careful about that. After you've not fooled yourself, it's easy not to fool other scientists. You just have to be honest in a conventional way after that."^[34]

In 1984–86, he developed a variational method for the approximate calculation of path integrals which has led to a powerful method of converting divergent perturbation expansions into convergent strong-coupling expansions

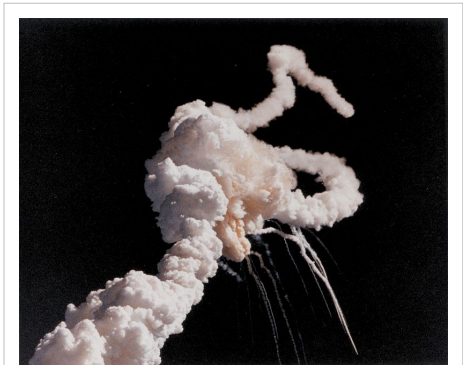
(variational perturbation theory) and, as a consequence, to the most accurate determination^[35] of critical exponents measured in satellite experiments.^[36]

In the late 1980s, according to "Richard Feynman and the Connection Machine", Feynman played a crucial role in developing the first massively parallel computer, and in finding innovative uses for it in numerical computations, in building neural networks, as well as physical simulations using cellular automata (such as turbulent fluid flow), working with Stephen Wolfram at Caltech.^[37] His son Carl also played a role in the development of the original Connection Machine engineering; Feynman influencing the interconnects while his son worked on the software.

Feynman diagrams are now fundamental for string theory and M-theory, and have even been extended topologically. The *world-lines* of the diagrams have developed to become *tubes* to allow better modeling of more complicated objects such as *strings* and *membranes*. However, shortly before his death, Feynman criticized string theory in an interview: "I don't like that they're not calculating anything," he said. "I don't like that they don't check their ideas. I don't like that for anything that disagrees with an experiment, they cook up an explanation—a fix-up to say, 'Well, it still might be true.'" These words have since been much-quoted by opponents of the string-theoretic direction for particle physics.^[38]

Challenger disaster

Feynman played an important role on the Presidential Rogers Commission, which investigated the *Challenger* disaster. Feynman devoted the latter half of his book *What Do You Care What Other People Think?* to his experience on the Rogers Commission, straying from his usual convention of brief, light-hearted anecdotes to deliver an extended and sober narrative. Feynman's account reveals a disconnect between NASA's engineers and executives that was far more striking than he expected. His interviews of NASA's high-ranking managers revealed startling misunderstandings of elementary concepts. He concluded that the NASA management's space shuttle reliability estimate was fantastically unrealistic. He warned in his appendix to the commission's report, "For a successful technology, reality must take precedence over public relations, for Nature cannot be fooled."^[39]



The 1986 Space Shuttle *Challenger* disaster.

Personal life

While researching for his Ph.D., Feynman married his first wife, Arline Greenbaum (often spelled *Arlene*). She was diagnosed with tuberculosis, but she and Feynman were careful, and he never contracted it. She succumbed to the disease in 1945. This portion of Feynman's life was portrayed in the 1996 film *Infinity*, which featured Feynman's daughter Michelle in a cameo role.

He was married a second time in June 1952, to Mary Louise Bell of Neodesha, Kansas; this marriage was brief and unsuccessful. He later married Gweneth Howarth from Ripponden, Yorkshire, who shared his enthusiasm for life and spirited adventure.^[40] Besides their home in Altadena, California, they had a beach house in Baja California, purchased with the prize money from Feynman's Nobel Prize, his one third share of \$55,000. They remained married until Feynman's death. They had a son, Carl, in 1962, and adopted a daughter, Michelle, in 1968.^[40]

Feynman had a great deal of success teaching Carl, using discussions about ants and Martians as a device for gaining perspective on problems and issues; he was surprised to learn that the same teaching devices were not useful with Michelle.^[41] Mathematics was a common interest for father and son; they both entered the computer field as consultants and were involved in advancing a new method of using multiple computers to solve complex problems—later known as parallel computing. The Jet Propulsion Laboratory retained Feynman as a computational

consultant during critical missions. One co-worker characterized Feynman as akin to Don Quixote at his desk, rather than at a computer workstation, ready to do battle with the windmills.

Feynman traveled a great deal, notably to Brazil, and near the end of his life schemed to visit the Russian land of Tuva, a dream that, because of Cold War bureaucratic problems, never became reality.^[42] The day after he died, a letter arrived for him from the Soviet government giving him authorization to travel to Tuva. During this period, he discovered that he had a form of cancer, but, with surgery, managed to forestall it. Out of his enthusiastic interest in reaching Tuva came the phrase "Tuva or Bust" (also the title of a book about his efforts to get there), which was tossed about frequently amongst his circle of friends in hope that they, one day, could see it firsthand. The documentary movie *Genghis Blues* mentions some of his attempts to communicate with Tuva, and chronicles the successful journey there by his friends.

Responding to Hubert Humphrey's congratulation for his Nobel Prize, Feynman admitted to a long admiration for the then vice president.^[43] In a letter to an MIT professor dated December 6, 1966, Feynman expressed interest in running for the governor of California.^[44]

Feynman took up drawing at one time and enjoyed some success under the pseudonym "Ofey", culminating in an exhibition dedicated to his work. He learned to play a metal percussion instrument (*frigideira*) in a samba style in Brazil, and participated in a samba school.

In addition, he had some degree of synesthesia for equations, explaining that the letters in certain mathematical functions appeared in color for him, even though invariably printed in standard black-and-white.^[45]

According to *Genius*, the James Gleick-authored biography, Feynman experimented with LSD during his professorship at Caltech.^[14] Somewhat embarrassed by his actions, Feynman largely sidestepped the issue when dictating his anecdotes; he mentions it in passing in the "O Americano, Outra Vez" section, while the "Altered States" chapter in *Surely You're Joking, Mr. Feynman!* describes only marijuana and ketamine experiences at John Lilly's famed sensory deprivation tanks, as a way of studying consciousness.^[12] Feynman gave up alcohol when he began to show early signs of alcoholism, as he did not want to do anything that could damage his brain—the same reason given in "O Americano, Outra Vez" for his reluctance to experiment with LSD.^[12]

In *Surely You're Joking, Mr. Feynman!*, he gives advice on the best way to pick up a girl in a hostess bar. At Caltech, he used a nude/topless bar as an office away from his usual office, making sketches or writing physics equations on paper placemats. When the county officials tried to close the place, all visitors except Feynman refused to testify in favor of the bar, fearing that their families or patrons would learn about their visits. Only Feynman accepted, and in court, he affirmed that the bar was a public need, stating that craftsmen, technicians, engineers, common workers "and a physics professor" frequented the establishment. While the bar lost the court case, it was allowed to remain open as a similar case was pending appeal.^[12]

Feynman developed two rare forms of cancer, Liposarcoma and Waldenström's macroglobulinemia, dying shortly after a final attempt at surgery for the former on February 15, 1988, aged 69.^[14] His last recorded words are noted as "I'd hate to die twice. It's so boring."^[14] ^[46]

Popular legacy

On May 4, 2005, the United States Postal Service issued the *American Scientists* commemorative set of four 37-cent self-adhesive stamps in several configurations. The scientists depicted were Richard Feynman, John von Neumann, Barbara McClintock and Josiah Willard Gibbs. Feynman's stamp, sepia-toned, features a photograph of a 30-something Feynman and eight small Feynman diagrams. The stamps were designed by artist Victor Stabin under the direction of U.S. Postal Service art director Carl T. Herrman.

The main building for the Computing Division at Fermilab is named the "Feynman Computing Center" in his honor.^[47]

Real Time Opera premiered its opera *Feynman* at the Norfolk (CT) Chamber Music Festival in June 2005.^[48]

In the television series *Star Trek: The Next Generation*, the shuttlecraft *Feynman* is named after him.

On the 20th anniversary of Feynman's death, composer Edward Manukyan dedicated a piece for solo clarinet to his memory.^[49] It was premiered by Doug Storey, the principal clarinetist of the Amarillo Symphony.

In 2009 and 2010, respectively, clips of an interview with Feynman were used by composer John Boswell as part of the Symphony of Science project in his second science education video, *We Are All Connected*, and in his fifth installment called *The Poetry of Reality*.^[50]

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Textbooks and lecture notes

The Feynman Lectures on Physics is perhaps his most accessible work for anyone with an interest in physics, compiled from lectures to Caltech undergraduates in 1961–64. As news of the lectures' lucidity grew, a number of professional physicists and graduate students began to drop in to listen. Co-authors Robert B. Leighton and Matthew Sands, colleagues of Feynman, edited and illustrated them into book form. The work has endured, and is useful to this day. They were edited and supplemented in 2005 with "Feynman's Tips on Physics: A Problem-Solving Supplement to the Feynman Lectures on Physics" by Michael Gottlieb and Ralph Leighton (Robert Leighton's son), with support from Kip Thorne and other physicists.

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- *No Ordinary Genius: The Illustrated Richard Feynman* ^[58], with Christopher Sykes, W. W. Norton & Co, 1996, ISBN 039331393X.
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- *Six Not So Easy Pieces: Einstein's Relativity, Symmetry and Space-Time*, Addison Wesley, 1997, ISBN 0-201-15026-3.
- *The Meaning of It All: Thoughts of a Citizen Scientist*, Perseus Publishing, 1999, ISBN 0738201669.
- *The Pleasure of Finding Things Out: The Best Short Works of Richard P. Feynman*, edited by Jeffrey Robbins, Perseus Books, 1999, ISBN 0738201081.
- *Classic Feynman: All the Adventures of a Curious Character*, edited by Ralph Leighton, W. W. Norton & Co, 2005, ISBN 0-393-06132-9. Chronologically reordered omnibus volume of *Surely You're Joking, Mr. Feynman!* and *What Do You Care What Other People Think?*, with a bundled CD containing one of Feynman's signature lectures.

Audio and video recordings

- *Safecracker Suite* (a collection of drum pieces interspersed with Feynman telling anecdotes)
- *Los Alamos From Below* (talk given by Feynman at Santa Barbara on February 6, 1975)
- *Six Easy Pieces* (original lectures upon which the book is based)
- *Six Not So Easy Pieces* (original lectures upon which the book is based)
- The Feynman Lectures on Physics: The Complete Audio Collection
- Samples of Feynman's drumming, chanting and speech are included in the songs "Tuva Groove (Bolur Daa-Bol, Bolbas Daa-Bol)" and "Kargyraa Rap (Dürgen Chugaa)" on the album *Back Tuva Future, The Adventure Continues* by Kongar-ool Ondar. The hidden track on this album also includes excerpts from lectures without musical background.
- The Messenger Lectures, given at Cornell in 1964, in which he explains basic topics in physics. Available ^[59] on Project Tuva for free (See also the book *The Character of Physical Law*)
- Take the world from another point of view [videorecording] / with Richard Feynman; Films for the Hu (1972)
- The Douglas Robb Memorial Lectures ^[60] Four public lectures of which the four chapters of the book QED: The Strange Theory of Light and Matter are transcripts. (1979)
- The Pleasure of Finding Things Out ^[61] at Google Video (Adobe Flash video) (1981) (not to be confused with the later published book of same title)
- Richard Feynman: Fun to Imagine Collection ^[62], BBC Archive of 6 short films of Feynman talking in an accessible to all about the physics behind common to all experiences. (1983)
- *Elementary Particles and the Laws of Physics* (1986)
- The Last journey of a genius ^[63], a BBC TV production in association with WGBH Boston (1989)

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Further reading

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Books

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- Dyson, Freeman (1979) *Disturbing the Universe*. Harper and Row. ISBN 0-06-011108-9. Dyson's autobiography. The chapters "A Scientific Apprenticeship" and "A Ride to Albuquerque" describe his impressions of Feynman in the period 1947–48 when Dyson was a graduate student at Cornell
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- Levine, Harry, III (2009) *The Great Explainer: The Story of Richard Feynman (Profiles in Science series)* Morgan Reynolds, Greensboro, North Carolina, ISBN 978-1-59935-113-1; for high school readers
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Films and plays

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- Parnell, Peter (2002) "QED" Applause Books, ISBN 978-1557835925, (play).
- Whittell, Crispin (2006) "Clever Dick" Oberon Books, (play)
- "The Pleasure of Finding Things Out" A film documentary autobiography of Richard Feynman, Nobel laureate and theoretical physicist extraordinary. 1982, BBC TV 'Horizon' and PBS 'Nova' (50 mins film). See Christopher Sykes Productions <http://www.sykes.easynet.co.uk/>
- "The Quest for Tannu Tuva", with Richard Feynman and Ralph Leighton. 1987, BBC TV 'Horizon' and PBS 'Nova' (under the title "Last Journey of a Genius") (50 mins film)
- "No Ordinary Genius" A two-part documentary about Feynman's life and work, with contributions from colleagues, friends and family. 1993, BBC TV 'Horizon' and PBS 'Nova' (a one-hour version, under the title "The Best Mind Since Einstein") (2 x 50 mins films)

External links

- Annotated Bibliography for Richard Feynman from the Alsos Digital Library for Nuclear Issues (<http://alsos.wlu.edu/qsearch.aspx?browse=people/Feynman,+Richard>)
- Seven free lectures presented by Feynman, and filmed by the BBC, in 1964 at Cornell University. —Based on some of Feynman's 1961–62 freshman physics lectures at Caltech, which formed the basis of *The Feynman Lectures on Physics*, Volume I (<http://research.microsoft.com/apps/tools/tuva/>)
- Guide to the Papers of Richard Phillips Feynman, 1933–1988 (<http://content.cdlib.org/view?docId=kt5n39p6k0&chunk.id=dsc-1.7.7&query=FEYNMAN&brand=oac>)
- Feynman at The Caltech Institute Archives (http://archives.caltech.edu/search_catalog.cfm?search_field=Feynman)
- The Feynman Lectures Website (<http://www.feynmanlectures.info/>)
- About Richard Feynman (http://www.nobel-winners.com/Physics/richard_phillips_feynman.html)
- Feynman's Scientific Publications (<http://www.physik.fu-berlin.de/~kleinert/feynman/feynmanpub.htm>)
- A Biography of R. P. Feynman as a mathematician (<http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Feynman.html>)
- Gallery of Drawings by Richard P. Feynman (<http://www.museumsyndicate.com/artist.php?artist=380>)
- Photos of Richard Feynman at the Emilio Segrè Visual Archives, American Institute of Physics (<http://photos.aip.org/veritySearch1.jsp?page=1&chapter=0&collection=storeTest&name=Feynman,+Richard+Phillips&desc1=&search=SEARCH+>)
- The Pleasure of Finding Things Out (<http://video.google.com/videoplay?docid=7136440703094429927>): A 49-minute BBC Horizon TV programme from the 1980s in which Feynman reminisces about his life and work.
- Edge-video: Murray Gell-mann reminisces about Feynman's personal eccentricities (<http://www.edge.org/video/dsl/gell-mann.html>)
- Vega Trust Series videos Feynman gives his Auckland, NZ lectures that ended up the basis for his book, QED. (<http://vega.org.uk/video/subseries/8>)
- Biography and Bibliographic Resources (<http://www.osti.gov/accomplishments/feynman.html>), from the Office of Scientific and Technical Information, United States Department of Energy
- Feynman discussing determining the rules of chess as an analogy for determining laws of physics. (<http://www.youtube.com/watch?v=o1dgrvIWML4>)
- Feynman talks about confusion (<http://www.youtube.com/watch?v=lytxafTXg6c>)
- Feynman's BBC interview on uncertainty (http://www.youtube.com/watch?v=_MmpUWEW6Is)
- Feynman on disrespect of authority (<http://www.youtube.com/watch?v=yhD0MxacnIE>)
- Works by Richard Feynman on Open Library at the Internet Archive

Murray Gell-Mann

Murray Gell-Mann	
 Murray Gell-Mann lecturing at TED in 2007	
Born	September 15, 1929Manhattan, New York City, U.S.
Residence	United States
Nationality	American
Fields	Physics
Institutions	University of Southern California Santa Fe Institute California Institute of Technology University of New Mexico
Alma mater	Yale University, MIT
Doctoral advisor	Victor Weisskopf
Doctoral students	Kenneth G. Wilson Sidney Coleman Rod Crewther James Hartle Christopher T. Hill H. Jay Melosh Barton Zwiebach Kenneth Young Todd Brun ^[1]
Known for	Elementary particles Gell-Mann matrices Gell-Mann–Nishijima formula Gell-Mann–Okubo mass formula Effective complexity
Notable awards	Nobel Prize in Physics (1969)

Murray Gell-Mann (born September 15, 1929, pronounced /ˈmʌriː ˈɡɛl ˈmæn/) is an American physicist who received the 1969 Nobel Prize in physics for his work on the theory of elementary particles. He is currently the Presidential Professor of Physics and Medicine at the University of Southern California.^[2]

He formulated the quark model of hadronic resonances, and identified the SU(3) flavor symmetry of the light quarks, extending isospin to include strangeness, which he also discovered. He discovered the V-A theory of chiral neutrinos in collaboration with Richard Feynman. He created current algebra in the 1960s as a way of extracting predictions from quark models when the fundamental theory was still murky, which led to model-independent sum rules confirmed by experiment.

Gell-Mann, along with Maurice Levy, discovered the sigma model of pions, which describes low energy pion interactions. Modifying the integer-charged quark model of Han and Nambu, Fritzsch and Gell-Mann were the first to write down the modern accepted theory of quantum chromodynamics although they did not anticipate asymptotic freedom.

Gell-Mann is responsible for the see-saw theory of neutrino masses, that produces masses at the inverse-GUT scale in any theory with a right-handed neutrino, like the SO(10) model. He is also known to have played a large role in keeping string theory alive through the 1970s, supporting that line of research at a time when it was unpopular.

Biography

Born in lower Manhattan into a family of Jewish immigrants, his father was a Romanian Jew from Czernowitz, Austro-Hungarian Empire,^{[3] [4]} Gell-Mann quickly revealed himself as a child prodigy. Propelled by an intense boyhood curiosity and love for nature, he entered Yale as a member of Jonathan Edwards College at fifteen after graduating valedictorian from the Columbia Grammar & Preparatory School.

Gell-Mann's work in the 1950s involved recently discovered cosmic ray particles that came to be called kaons and hyperons. Classifying these particles led him to propose a new quantum number called strangeness. Another of Gell-Mann's ideas is the Gell-Mann–Nishijima formula, which was, initially, a formula based on empirical results, but was later explained by the quark model. Gell-Mann and Abraham Pais were involved in explaining many puzzling aspects of the physics of these particles.

In 1961, this led him (and Kazuhiko Nishijima) to introduce a classification of elementary particles called hadrons (also independently proposed by Yuval Ne'eman six months earlier, although Gell-Mann alone got the Nobel Prize). This scheme is now explained by the quark model. Gell-Mann's own name for the classification scheme was the *Eightfold Way*, because of the *octets* of particles in the classification. The term is a reference to the eightfold way of Buddhism.

Gell-Mann, and, independently, George Zweig, went on, in 1964, to postulate the existence of quarks, the particles from which the hadrons are composed. The name was coined by Gell-Mann and is a reference to the novel *Finnegans Wake*, by James Joyce ("Three quarks for Muster Mark!" book 2, episode 4). Zweig had referred to the particles as "aces",^[5] but Gell-Mann's name caught on. Quarks were soon accepted as the underlying elementary objects in the study of the structure of hadrons. In 1972 he introduced with Harald Fritzsch the quantum number 'color' and later, in a joint paper with Heinrich Leutwyler, the full theory of quantum chromodynamics (QCD) was released as the gauge theory of strong interactions (cf. references). The quark model is part of QCD, and has been robust enough to survive the discovery of other flavours of quarks. Gell-Mann and Richard Feynman, working together, and a rival group of George Sudarshan and Robert Marshak, were the first to discover the vector structure of the weak interaction in physics. This work followed the discovery of parity violation by Chien-Shiung Wu, as suggested by Chen Ning Yang and Tsung-Dao Lee.

In the 1990s his interest turned to the emerging study of complexity, where he was closely associated with the Santa Fe Institute. He wrote a popular science book about these matters, *The Quark and the Jaguar: Adventures in the Simple and the Complex*. The title of the book is taken from a line of an Arthur Sze poem: "The world of the quark has everything to do with a jaguar circling in the night." Gell-Mann is also a collector of East Asian antiquities and a keen linguist.

George Johnson wrote a biography of Gell-Mann, which is entitled *Strange Beauty: Murray Gell-Mann and the Revolution in 20th-Century Physics*.

Timeline

Gell-Mann earned a bachelor's degree in physics from Yale University in 1948, and a PhD in physics from MIT in 1951. He was a postdoctoral research associate in 1951, and a visiting research professor at University of Illinois at Urbana–Champaign from 1952 to 1953. After serving as Visiting Associate Professor at Columbia University in 1954–55, he became a professor at the University of Chicago before moving to the California Institute of Technology, where he taught from 1955 until 1993. He was awarded a Nobel Prize in physics in 1969 for his discovery of a system for classifying subatomic particles.

He is currently the Robert Andrews Millikan Professor of Theoretical Physics Emeritus at Caltech as well as a University Professor in the Physics and Astronomy Department of the University of New Mexico in Albuquerque, New Mexico. He is a member of the editorial board of the *Encyclopædia Britannica*. In 1984 Gell-Mann co-founded the Santa Fe Institute—a non-profit research institute in Santa Fe, New Mexico—to study complex systems and disseminate the notion of a separate interdisciplinary study of complexity theory. There he met and befriended Pulitzer Prize-winning novelist Cormac McCarthy.^[6]

Personal life

Gell-Mann married Marcia Southwick in 1992, after the death of his first wife, J. Margaret Dow (d. 1981), whom he married in 1955. His children are Elizabeth Sarah Gell-Mann (b. 1956) and Nicholas Webster Gell-Mann (b. 1963); and he has a stepson, Nicholas Southwick Levis (b. 1978).

Awards

- Dannie Heineman Prize (1959)
- Erice Prize (1990)
- Ernest O. Lawrence Award (1966)
- Franklin Medal (1967)
- John J. Carty Medal (1968)
- Nobel Prize in Physics (1969)
- Humanist of the Year (2005)
- Albert Einstein Medal (2005)

Awards and honors

- Yale University — D.Sc (h.c.), 1959
 - American Physical Society — Dannie Heineman Prize, 1959
 - Ernest O. Lawrence Award, 1966
 - University of Chicago — Sc.D.(h.c.), 1967
 - Franklin Institute of Philadelphia — Franklin Medal, 1967
 - National Academy of Sciences — John J. Carty Medal, 1968
 - University of Illinois — Sc.D.(h.c.), 1968
 - Wesleyan University — Sc.D.(h.c.), 1968
 - Research Corporation Award, 1969
 - University of Turin, Italy — Honorary Doctorate, 1969
 - Nobel Prize in Physics, 1969
 - University of Utah — Sc.D.(h.c.), 1970
 - Columbia University — Sc.D.(h.c.), 1977
 - University of Cambridge, England — Sc.D.(h.c.), 1980
 - United Nations Environment Programme Roll of Honor for Environmental Achievement (The Global 500), 1988
-

- World Federation of Scientists — Erice Prize, 1990
- University of Oxford, England — D.Sc.(h.c.), 1992
- Southern Illinois University — Sc.D.(h.c.), 1993
- University of Florida — Sc.D.(h.c.), Doctorate of Natural Resources, 1994
- Southern Methodist University — Sc.D.(h.c.), 1999
- Telluride Tech Festival Award of Technology, Telluride, Colorado, 2002
- American Humanist Association - Humanist of the Year, 2005

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- Murray Gell-Mann Home page at Santa Fe Institute (<http://www.santafe.edu/~mgm/>)
- Murray Gell-Mann (<http://webofstories.com/gl/murray.gell-mann>) video at Web of Stories (<http://webofstories.com>)
- Strange Beauty home page (<http://talaya.net/strangebeauty.html>)
- TedTalks March 2007: Beauty and truth in physics (<http://www.ted.com/index.php/talks/view/id/194>)
- The Making of a Physicist: A Talk With Murray Gell-Mann (http://www.edge.org/3rd_culture/gell-mann03/gell-mann_print.html)
- The Man Who Knows Everything (<http://query.nytimes.com/gst/fullpage.html?res=9504E6DB1530F93BA35756C0A962958260>), David Berreby, New York Times, May 8, 1994
- The Man With Five Brains (http://www.webcitation.org/query?url=http://www.geocities.com/omegaman_uk/gellmann.html&date=2009-10-26+00:00:53)
- The many worlds of Murray Gell-Mann (<http://physicsweb.org/articles/world/16/6/2/1>)
- The Simple and the Complex, Part I: The Quantum and the Quasi-Classical with Murray Gell-Mann, Ph.D. (<http://www.williamjames.com/transcripts/gell1.htm>)
- Nobel Prize Biography (http://nobelprize.org/nobel_prizes/physics/laureates/1969/gell-mann-bio.html)

External links

- Biography and Bibliographic Resources (<http://www.osti.gov/accomplishments/gellmann.html>), from the Department of Energy, Office of Scientific & Technical Information
 - Gell-Mann's Home Page at SFI (<http://www.santafe.edu/~mgm/>)
 - TED Talks: Murray Gell-Mann on beauty and truth in physics (<http://www.ted.com/talks/view/id/194>) at TED in 2007
 - TED Talks: Murray Gell-Mann on the ancestor of language (<http://www.ted.com/talks/view/id/276>) at TED
-

Stephen Weinberg

Steven Weinberg	
 Steven Weinberg at the 2010 Texas Book Festival.	
Born	May 3, 1933New York City, New York, USA
Residence	United States
Nationality	United States
Fields	Physics
Institutions	University of California, Berkeley MIT Harvard University University of Texas at Austin
Alma mater	Cornell University Princeton University
Doctoral advisor	Sam Treiman
Doctoral students	Orlando Alvarez Claude Bernard Lay Nam Chang Bob Holdom Ubirajara van Kolck Rafael Lopez-Mobilia John Preskill Fernando Quevedo Mark G. Raizen Scott Willenbrock
Known for	Electromagnetism and Weak Force unification Weinberg-Witten theorem
Influenced	Alan Guth
Notable awards	Nobel Prize in Physics (1979)
Notes	He is married to the professor of law, Louise Weinberg.

Steven Weinberg (born May 3, 1933) is an American physicist and Nobel laureate in Physics for his contributions with Abdus Salam and Sheldon Glashow to the unification of the weak force and electromagnetic interaction between elementary particles.

Biography

Steven Weinberg was born in 1933 in New York City to Jewish immigrants Frederick and Eva Weinberg, but is an atheist^[1]. He graduated from Bronx High School of Science in 1950 and received his bachelor's degree from Cornell University in 1954, living at the Cornell branch of the Telluride Association. He left Cornell and went to the Niels Bohr Institute in Copenhagen where he started his graduate studies and research. After one year, Weinberg returned to Princeton University where he earned his Ph.D. degree in Physics in 1957, studying under Sam Treiman.

Academic career

After completing his Ph.D., Weinberg worked as a post-doctoral researcher at Columbia University (1957–1959) and University of California, Berkeley (1959) and then he was promoted to faculty at Berkeley (1960–1966). He did research in a variety of topics of particle physics, such as the high energy behavior of quantum field theory, symmetry breaking, pion scattering, infrared photons and quantum gravity.^[2] It was also during this time that he developed the approach to quantum field theory that is described in the first chapters of his book *The Quantum Theory of Fields*^[3] and started to write his textbook *Gravitation and Cosmology*. Both textbooks, perhaps especially the second, are among the most influential texts in the scientific community in their subjects.

In 1966, Weinberg left Berkeley and accepted a lecturer position at Harvard. In 1967 he was a visiting professor at MIT. It was in that year at MIT that Weinberg proposed his model of unification of electromagnetism and of nuclear weak forces (such as those involved in beta-decay and kaon-decay)^[4]. This model is now known as the electroweak unification theory. An important feature of this model is the prediction of the existence of another interaction mechanism between leptons, known as neutral current and mediated by the Z boson. The 1973 experimental discovery of this Z boson was one verification of the electroweak unification. The paper by Weinberg in which he presented this theory was one of the highest cited theoretical works ever in high energy physics as of 2009^[5].

After his 1967 seminal work on the unification of weak and electromagnetic interactions, Steven Weinberg continued his work in many aspects of particle physics, quantum field theory, gravity, supersymmetry, superstrings and cosmology, as well as a theory called Technicolor.

In the years after 1967, the full Standard Model of elementary particle theory was developed through the work of many contributors. In it, the weak and electromagnetic interactions already unified by the work of Weinberg, Abdus Salam and Sheldon Glashow, are further unified with the strong interactions, in one overarching theory. One of its fundamental aspects was the prediction of the existence of the Higgs boson. In 1973 Weinberg proposed a modification of the Standard Model which did not contain that model's fundamental Higgs boson.

Weinberg became Higgins Professor of Physics at Harvard University in 1973.

It is of special importance that in 1979 he pioneered the modern view on the renormalization aspect of quantum field theory that considers all quantum field theories as effective field theories and changed completely the viewpoint of previous work (including his own) that a sensible quantum field theory must be renormalizable^[6]. This approach allowed the development of effective theory of quantum gravity^[7], low energy QCD, heavy quark effective field theory and other developments, and it is a topic of considerable interest in current research.

In 1979, after the experimental discovery of the neutral currents — i.e. the discovery of the inferred existence of the Z boson —, Steven Weinberg was awarded the Nobel Prize in Physics together with Abdus Salam and Sheldon Glashow for developing their theory of electroweak unification.

In 1982 Weinberg moved to the University of Texas at Austin as the Jack S. Josey-Welch Foundation Regents Chair in Science and founded the *Theory Group* of the Physics Department.

There is current (2008) interest in Weinberg's 1976 proposal of the existence of new strong interactions^[8] -- a proposal dubbed "Technicolor" by Leonard Susskind -- because of its chance of being observed in the LHC as an explanation of the hierarchy problem.

Steven Weinberg's influence and importance are confirmed by the fact that he is frequently among the top scientists with highest research effect indices, such as the h-index and the creativity index.^[9]

Other intellectual legacy

Besides his scientific research, Steven Weinberg has been a prominent public spokesman for science, testifying before Congress in support of the Superconducting Super Collider, writing articles for the *New York Review of Books*^[10], and giving various lectures on the larger meaning of science. His books on science written for the public combine the typical scientific popularization with what is traditionally considered history and philosophy of science and atheism.

Weinberg was a major participant in what is known as the Science Wars, standing with Paul R. Gross, Norman Levitt, Alan Sokal, Lewis Wolpert, and Richard Dawkins, on the side arguing for the hard realism of science and scientific knowledge and against the constructionism proposed by such social scientists as Stanley Aronowitz, Barry Barnes, David Bloor, David Edge, Harry Collins, Steve Fuller, and Bruno Latour.

Weinberg is also known for his support of Israel. While this is not extraordinary in itself, he, like many American Jews, supports Israel from a liberal point of view. He wrote an essay titled "Zionism and Its Cultural Adversaries" to explain his views on the issue.

Weinberg has canceled trips to universities in the United Kingdom because of British boycotts directed towards Israel. He has explained:

"Given the history of the attacks on Israel and the oppressiveness and aggressiveness of other countries in the Middle East and elsewhere, boycotting Israel indicated a moral blindness for which it is hard to find any explanation other than antisemitism."^[11]

His views on religion were expressed in a speech from 1999 in Washington, D.C.:

"With or without religion, good people can behave well and bad people can do evil; but for good people to do evil—that takes religion."^[12]

He has also said:

"The more the universe seems comprehensible, the more it seems pointless."^[13]

He attended and was a speaker at the Beyond Belief symposium in November 2006.

Personal

He is married to Louise Weinberg and has one daughter, Elizabeth.

Honors and awards

The honors and awards that Professor Weinberg received include:

- Honorary Doctor of Science degrees from a dozen institutions: University of Chicago, Knox College, University of Rochester, Yale University, City University of New York, Dartmouth College, Weizmann Institute, Clark University, Washington College, Columbia University, Bates College.
 - American Academy of Arts and Sciences, elected 1968
 - National Academy of Sciences, elected 1972
 - J. R. Oppenheimer Prize, 1973
 - Dannie Heineman Prize for Mathematical Physics, 1977
 - Steel Foundation Science Writing Award, 1977, for authorship of *The First Three Minutes* (1977)
 - Elliott Cresson Medal (Franklin Institute), 1979
 - Nobel Prize in Physics, 1979
-

- Elected to American Philosophical Society, Royal Society of London (Foreign Honorary Member), Philosophical Society of Texas
- James Madison Medal of Princeton University, 1991
- National Medal of Science, 1991
- Lewis Thomas Prize for Writing about Science, 1999.
- 2002 Humanist of the Year, American Humanist Association
- James Joyce Award, University College Dublin, 2009

Selected publications

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- *The First Three Minutes: A Modern View of the Origin of the Universe* (1977, updated with new afterword in 1993, ISBN 0-465-02437-8)
- *The Discovery of Subatomic Particles* (1983)
- *Elementary Particles and the Laws of Physics: The 1986 Dirac Memorial Lectures* (1987; with Richard Feynman)
- *Dreams of a Final Theory: The Search for the Fundamental Laws of Nature* (1993), ISBN 0-09-922391-0
- *The Quantum Theory of Fields* (three volumes: 1995, 1996, 2003)
- *Facing Up: Science and Its Cultural Adversaries* (2001, 2003, HUP)
- *Glory and Terror: The Coming Nuclear Danger* (2004, NYRB)
- *Cosmology* (2008, OUP)
- *Lake Views: This World and the Universe* (2010), Belknap Press of Harvard University Press, ISBN 0674035151.

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External links

- Biography and Bibliographic Resources (<http://www.osti.gov/accomplishments/weinberg.html>), from the Office of Scientific and Technical Information, United States Department of Energy
- Home Page of Steven Weinberg at University of Texas at Austin (<http://www.ph.utexas.edu/~weintech/weinberg.html>)
- Steven Weinberg on LHC (<http://www.youtube.com/watch?v=Zl4W3DYTIKw>)
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- Weinberg author page and archive (<http://www.nybooks.com/authors/201>) from *The New York Review of Books*
- Publications (http://arxiv.org/find/hep-th/1/au:+Weinberg_S/0/1/0/all/0/1) on ArXiv

Anatole Abragam

Anatole Abragam	
No free image  Anatole Abragam	
Born	December 15, 1914
Nationality	French
Fields	physics
Alma mater	University of Paris
Known for	<i>The Principles of Nuclear Magnetism</i> nuclear magnetic resonance
Notable awards	Lorentz Medal in 1982

Anatole Abragam (born December 15, 1914) is a French physicist who wrote *The Principles of Nuclear Magnetism* and has made significant contributions to the field of nuclear magnetic resonance.^[1] Originally from Russia, Abragam and his family emigrated to France in 1925.

After being educated at the University of Paris, (1933–1936), he served in the Second World War. After the war, he resumed his studies at the École Supérieure d'Électricité and subsequently obtained his Ph.D. from Oxford University in 1950 under the supervision of Maurice Pryce. In 1976, he was made an Honorary Fellow of both Merton, Magdalen, and Jesus Colleges, Oxford.^[2]

From 1960 to 1985, he worked as a professor at the Collège de France.^[3] He was awarded the Lorentz Medal in 1982.

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Ștefan Procopiu

Ștefan Procopiu	
 ȘTEFAN PROCOPIU 1890-1972 5^L POȘTA ROMÂNĂ Ștefan Procopiu	
Born	January 18, 1890Bârlad
Died	February 22, 1972Iași
Residence	Iași
Citizenship	Romanian
Nationality	Romanian
Fields	Physics
Alma mater	Alexandru Ioan Cuza University of Iași
Known for	Bohr-Procopiu magneton Procopiu effect Procopiu phenomenon
Notable awards	Romanian State Prize (1964)

Ștefan Procopiu (b. January 19, 1890 in Bârlad, Romania, d. August 22, 1972 in Iași, Romania) was a Romanian physicist.

Biography

Ștefan Procopiu was born on January 19, 1890 in Bârlad. His father, Emanoil Procopiu, was employed at the Bârlad courthouse. His mother, Ecaterina Tașcă was the daughter of Gheorghe I. Tașcă (see Tașcă family) ^[1] He attended the Gheorghe Roșca Codreanu High School in Bârlad from 1901 to 1908, continuing his studies at the Faculty of Sciences of the "Alexandru Ioan" Cuza University of Iași from 1908 to 1912. After graduation he became assistant to professor Dragomir Hurmuzescu.^[2]

In 1919 he obtained a scholarship to continue his studies in Paris, attending courses of famous scientists, such as Gabriel Lippmann, Marie Curie, Paul Langevin, Aimé Cotton. On 5 March 1924, Procopiu obtained the title of doctor in physics with the thesis "On the electric birefringence of suspensions" presented to a commission including professor Aimé Cotton as coordinator and Charles Fabry and Henri Mouton as cross-examiners. ^[3]

After his return to Romania on January 15, 1925 professor of the gravitation, heat and electricity department of the "Alexandru Ioan Cuza" University of Iași, replacing his former teacher Dragomir Hurmuzescu, who had retired., Procopiu coordinated the department until his retirement in 1962.^[4] At the same time he was appointed professor at the "Gheorghe Asachi" Polytechnic Institute of Iași^[3] In 1939 Ștefan Procopiu published his treatise on "Electricity and Magnetism", followed in 1948 by his monography on "Thermodynamics".

On June, 1948 he was appointed corresponding member of the Romanian Academy, being promoted to full membership on July 2, 1955.^[3] In 1964 he was awarded the Romanian State Prize^[4] He was also decorated with the Order of Work (Ordinul Muncii), Order of the Star of Romania and the Order of Scientific Merit. Procopiu was also selected twice as member in the Commission for the award of the Nobel Prize,^[2]

Ștefan Procopiu was also deeply involved in the cultural life of the city of Iași. He was an active member of the Board of Directors of the National Theatre "Vasile Alecsandri" of Iași^[4]

Ștefan Procopiu died on August 22, 1972 in Iași age 82.^[5]

Scientific activity

Ștefan Procopiu started scientific research even before graduating. He continued this activity while he was assistant professor.

The magnetic moment of electrons

The first important paper by Ștefan Procopiu is "Determining the Molecular Magnetic Moment by M. Planck's Quantum Theory". After studying Planck's quantum theory and Langevin's magnetism theory, established the magnetic moment and determined the physical constant of magnetic moment, named magneton.^[6] Ștefan Procopiu published his results two years before Niels Bohr made the same discovery independently.^[7] The magneton is now known as Bohr-Procopiu magneton.

Continuing his studies, in 1954 he established a method for the experimental determination of the magneton, which he improved in 1963^[8].

Other research before and during World War I

Ștefan Procopiu also worked on wireless communications and in 1913 published a paper on "Experimental Research on Wireless Telegraphy". In 1916 he invented a device for locating and establishing the depth of bullets in the bodies of the wounded soldiers.^[7]

Longitudinal depolarization of light

In 1921, Procopiu discovered and analyzed in the Physics Laboratory of Sorbonne University a new optical phenomenon which consisted in the longitudinal depolarization of light by suspensions and colloids.^[8] In 1930, the occurrence was designated as "Procopiu Phenomenon" by prof. Augustin Boutaric. Part of this research was included in Procopiu's doctoral thesis.

Electromotive force of galvanic elements

Thus, in 1930, studying the Barkhausen effect, Ștefan Procopiu discovered a circular effect of magnetic discontinuity. In 1951, this effect was named "Procopiu Effect".^[4] This discovery had important applications in the development of the memory of computers^[2]

Studies of the earth magnetism

Earth's magnetism was a continuous concern of Ștefan Procopiu, For 25 years he studied this phenomenon in Romania and developed the magnetic maps of the country. He also identified the magnetic anomaly located on the Iași-Botoșani line.

In 1947, Procopiu identified a variation of the earth's magnetic field, with a periodicity of approximately 500 years, indicating that, starting 1932 earth's magnetic moment increases from the Ecuator to the poles.^{[2] [3]}

1980 Invited Visiting Professor at Tokyo University □ 1981-1985 Founder and Scientific Director of SOLEMS Company □ 1987 President of the Scientific Council of PHOTOTRONICS (a French-German Company for photovoltaic products) □ 1988, June 22, Elected Member of the Physics Institute of the French Academy of Sciences[6] □ Laboratory of Condensed Matter Physics

Awards and prizes

□ 1958 Grand Prix for Research (with Anatole Abragam and J. Combrisson) □ 1963 The CNRS Silver Medal □ 1969 Robin Prize of the French Physics Society (Société Française de Physique) □ 1972 Holweck Prize of the Institute of Physics and S.F.P (French Physics Society) □ 1981 The Y. Peyches Prize of the Academy of Sciences

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